A gravitational collapse singularity theorem that improves Penrose's

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Talk based on

- A gravitational collapse singularity theorem consistent with black hole evaporation arXiv:1909.07348
- Lorentzian causality theory, *Living Reviews in Relativity* **22** (2019) 3

General relativity

In Einstein's general relativity spacetime
is a *differentiable manifold* M endowed with a metric

$$g = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$$

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Dynamics is determined by the Einstein's equations

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where R is the *Ricci tensor* and T is the *stress-energy tensor*.

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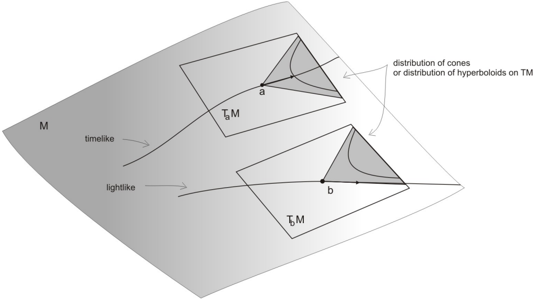
where R is the *Ricci tensor* and T is the *stress-energy tensor*.
The results we are going to obtain really depend only on the energy conditions derived from these equations (null/timelike convergence conditions).

Spacetime is a connected time-oriented Lorentzian manifold. Points on spacetime are called *events*. We have a distribution of causal cones $x \to C_x$, and a distribution of hyperboloids $x \to \mathbb{H}_x$.



A C^1 curve $x\colon t \mapsto x(t)$ is

- Timelike: if $g(\dot{x}, \dot{x}) < 0$, (massive particles),
- Lightlike: if $g(\dot{x}, \dot{x}) = 0$, (massless particles).

The *proper time* of a massive particle/observer is $\tau = \int_{x(t)} \sqrt{-g(\dot{x}, \dot{x})}\mathrm{d}t$.

Spacetime = (conic) causal order + spacetime measure

Simple algebraic lemma

On a vector space of dimension $n \geq 3$ two Lorentzian bilinear forms η_1, η_2 *are proportional* if and only if they have the same causal cone C.

$$C_1 = C_2 \Leftrightarrow \exists a \in \mathbb{R} : \eta_1 = a^2 \eta_2.$$

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Since the volume form induced by the metric is $\sqrt{-\det \eta_{\alpha\beta}}\, \mathrm{d}y^0 \wedge \cdots \wedge \mathrm{d}y^n$ it scales differently under conformal changes so

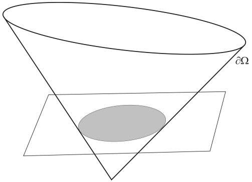
Corollary

Two spacetime metrics g_1 and g_2 *coincide* if and only if they induce the same distribution of causal cones $x \to C_x$ and the same volume form $\mathrm{d}\mu = \sqrt{-\det g}\, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$.

In other words the spacetime (M, g) of general relativity is nothing but a

- **spacetime measure + cone distribution**

where the cones are really *round*: they have ellipsoidal section according to the affine structure of the tangent space $T_x M$.



Causality theory for the most part focuses on the cone distribution, namely on conformal invariant properties.

Causality theory

Causality theory is the study of the global qualitative properties of the solutions $t \mapsto x(t)$, to the differential inclusion

$$\dot{x}(t) \in C_{x(t)},$$

It focuses on the qualitative behavior of *causal curves* with a special attention to *causal geodesics*. It aims to answer questions such as:
According to general relativity

- Can closed timelike curves exist?
- Can they form?
- Is the spacetime singular? (this point requires the conformal factor)
- Do continuous global increasing functions (time functions) exist?

Causality relations and conditions

$$I = \{(p,q) : \text{ there is a timelike curve from } p \text{ to } q\},$$
$$J = \{(p,q) : \text{ there is a causal curve from } p \text{ to } q \text{ or } p = q\}.$$

The *chronology violating set* is the set $\mathscr{C}$ of points through which passes a closed timelike curve.
The weakest causality conditions are

Definition

A spacetime is *non-totally vicious* if $\mathscr{C} \neq M$, and *chronological* if $\mathscr{C} = \emptyset$.

The two strongest causality condition are

Definition

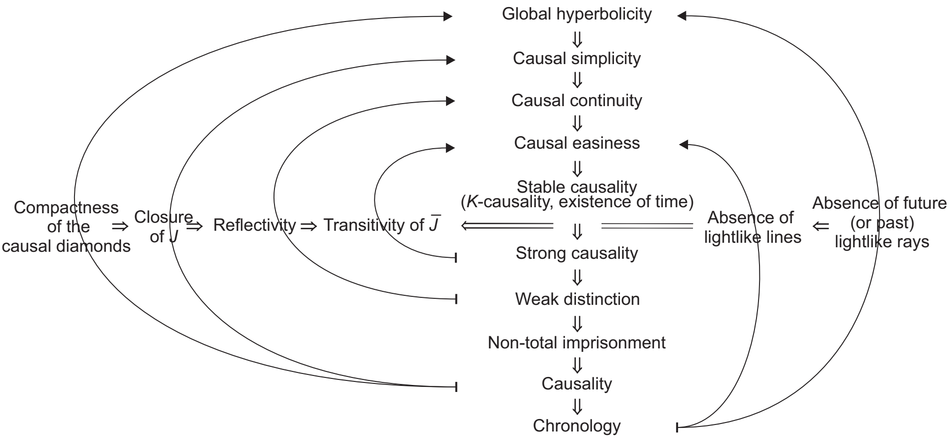
A spacetime is *causally simple* if it is non-totally vicious and J is a closed relation.

Definition

A spacetime is *globally hyperbolic* if the causal diamonds $J^+(p) \cap J^-(q)$ are compact (yes, there is no need to assume causality if the spacetime is non-compact and of dimension larger than two. Recent work with Raymond Hounnonkpe).

The causal ladder

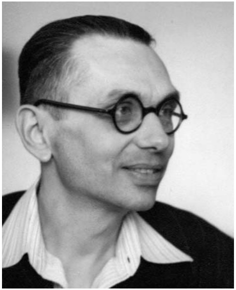
Global hyperbolicity is the strongest causality condition. We are going to present a singularity theorem that does not even need to require chronology.



The key property will be **past reflectivity**: $q \in \overline{J^+(p)} \Rightarrow p \in \overline{J^-(q)}$.

Existence of causal pathologies

Einstein's equations impose very week constraints on causality.



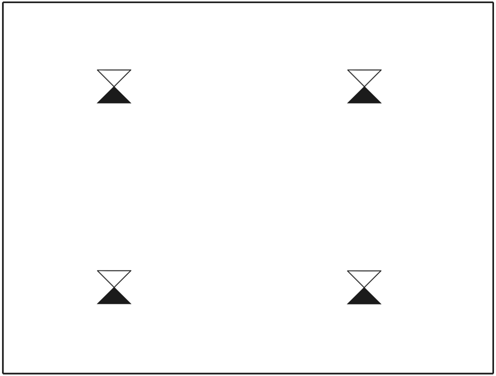
In 1949 Kurt Gödel found the following surprising solution: $M = \mathbb{R}^4$ and

$$g = \frac{1}{2\omega^2}[-(dt + e^x dz)^2 + dx^2 + dy^2 + \tfrac{1}{2}e^{2x}dz^2],$$

which is a solution for $\Lambda = -\omega^2$ and a stress-energy tensor $T_{\mu\nu}$ of dust type.
The problem is that through every point there passes a closed timelike curve.
An observer could go back in time.

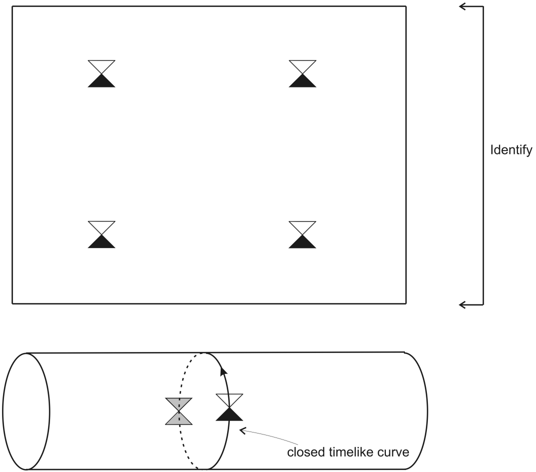
Minkowski spacetime

$M = \mathbb{R}^4$, $g = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.
In pictures we suppress 1 or 2 space dimensions.



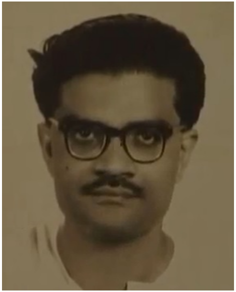
Non-chronological flat example

A spacetime of topology $S^1 \times \mathbb{R}^3$ which satisfies Einstein's equations in which there are closed timelike curves.



Raychaudhuri equation

In 1955 an unknown Indian theoretical physicist, Amal Kumar Raychaudhuri published an equation expressing the evolution of the divergence of a congruence of geodesics.
For a surface-orthogonal lightlike congruence generated by the vector field n it takes the form

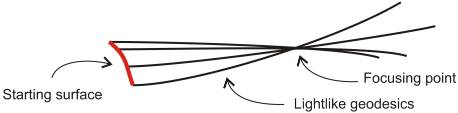


$$\frac{\mathrm{d}}{\mathrm{d}t}\theta = -\frac{1}{2}\theta^2 - 2\sigma^2 - \mathrm{Ric}(n)$$

where θ is the expansion, σ the shear, and Ric the Ricci tensor. By Einstein equations $\mathrm{Ric}(n) = T(n, n)$, and by positivity of energy $T(n, n) \geq 0$, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta \leq -\frac{1}{2}\theta^2, \qquad \theta = \frac{1}{2A}\frac{\mathrm{d}A}{\mathrm{d}t}$$

which if $\theta(t_0) < 0$ implies $\theta \to -\infty$ or *refocusing* within finite affine parameter interval Δt provided the affine parameter extends that far.



Maximization

Causal geodesics locally maximize the *proper time* (length functional)

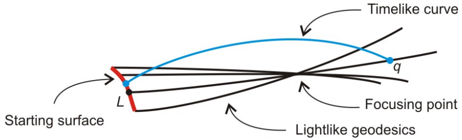
$$\tau(x) = \int_{x(t)} \sqrt{-g(\dot{x},\dot{x})}\mathrm{d}t, \qquad x\colon I \to M$$

among causal curves, but not beyond conjugate points.

Lemma

If two points are connected by a causal curve which is not a maximizing lightlike geodesic then they are connected by a timelike curve.

Physically speaking, any two events p and q are connected by a light ray running from p to q or we can find an ideal observer moving from p to q.



We write $q \in I^+(L)$, where $I^+(L)$ is the *chronological future* of L.

Penrose's theorem

In 1965 Roger Penrose introduced methods of global differential geometry to the study of spacetime singularities.

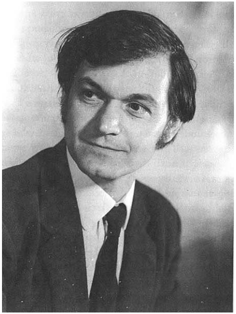
A spacetime which

(a) admits a non-compact Cauchy hypersurface

(b) for which
the null energy condition $T(n,n) = \mathrm{Ric}(n) \geq 0$
holds, and

(c) which admits a *trapped surface* namely a surface for which both ingoing and outgoing lightlike geodesics contract $\theta^{\pm} < 0$,

is future null geodesically incomplete.



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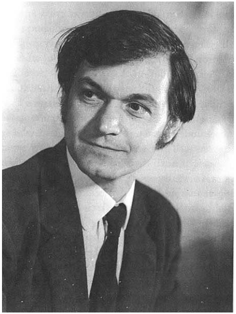

The point was to show that singularities necessarily form when certain conditions are met (e.g. in a gravitational collapse), and are not due to symmetry assumptions used to find exact solutions e.g. Schwarzschild

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2, \quad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$$

A loophole argument

- **Under global hyperbolicity** trapped surfaces lead to the formation of geodesic singularities
- By cosmic censorship the singularity is hidden behind the horizon of a black hole
- The collapsing matter radiates out gravitational energy till the spacetime becomes approximately stationary, that is the black hole converges to a stationary Kerr black hole
- Quantum field theory in curved spacetimes implies that the Kerr black holes evaporates (Hawking's radiation)
- Studies by Kodama (1979), Wald (1984) and Lesourd (2019) show that an evaporating black hole **cannot be globally hyperbolic**

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Our problem

Show that it is possible to remove the global hyperbolicity assumption in Penrose's theorem. In this way the prediction of a singularity becomes consistent with quantum field theory and black hole evaporation.

The Big-Bang unavoidable singularity

In 1965-66 Stephen Hawking immediately realizes that Penrose's argument works for the universe as a whole.

A spacetime which satisfies

(a) it admits a Cauchy hypersurface

(b) the (timelike unit) normals to the Cauchy hypersurface are expanding $\theta > \epsilon > 0$ (universe expansion), and

(c) $\mathrm{Ric}(v) \geq 0$ for every timelike vector,

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The point was to show that the Universe had a singular beginning due to Hubble observational law. Singularities found in exact solutions were not merely due to symmetry assumptions (cosmological principle) used to find them e.g. Friedmann - Lemaître - Robertson - Walker

$$g = -\mathrm{d}t^2 + a(t)^2 (\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2)$$

Removal of causality conditions from Hawking's theorem

Both Penrose's and Hawking's theorem depend on global hyperbolicity, and hence assume chronology. However, Hawking was able to remove all causality assumption from his theorem

Theorem (Hawking 1966,1967)

Let (M, g) be such that

(1) *the timelike convergence condition holds on M (i.e. $\mathrm{Ric}(v) \geq 0$ for all timelike vectors v),*

(2) *M contains a C^2 compact spacelike hypersurface S (hence without edge),*

(3) *S is contracting, i.e. the expansion scalar θ (i.e. the mean curvature of S) is negative.*

Then M is future timelike geodesically incomplete.

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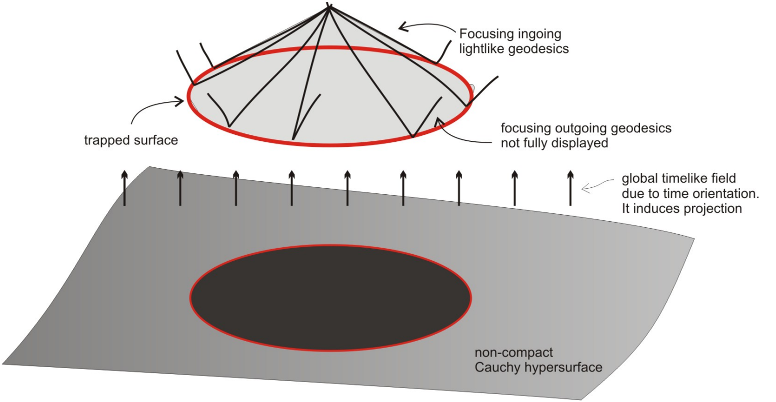
Why is it possible to remove chronology?

Because the hypersurface S is global so conditions on it have, so to say, global character. In general it is easier to remove the causality condition from singularity theorems that are cosmological in scope. Another matter is to do the same for singularity theorems concerned with gravitational collapse, since they are local.

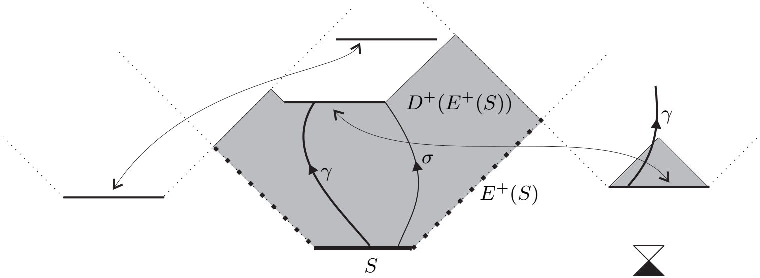
Penrose's theorem

If geodesics are complete the projection of the ingoing lightlike geodesics, taken before the conjugate point, gives a compact set (to be discussed later). The same holds for the outgoing congruence (though the figure does not suggest so), so the Cauchy hypersurface is really the union of two compact sets hence compact: the contradiction proves that the geodesics are incomplete (geodesic singularity).



The edge of the horismos $E^+(S)$

The basic step in Penrose's theorem is to show that $E^+(S) = J^+(S) \backslash I^+(S)$ is compact and has no edge. In this non-globally hyperbolic example it has edge.



One needs causal simplicity to have $\overline{J^+(S)} = J^+(S)$. Then since $\overline{J^+(S)} = \overline{I^+(S)}$ one gets $E^+(S) = \dot{I}^+(S)$, namely $E^+(S)$ is an achronal boundary hence a topological hypersurface (thus with no edge).

Hawking and Ellis commented Penrose's theorem as follows [HE, p. 285]

> *The real weakness of the theorem is the requirement that there be a Cauchy hypersurface H. This was used in two places: first, to show that (M, g) was causally simple which implied that the generators of $\dot{J}^+(S)$ had past endpoints on S, [i.e. $\dot{J}^+(S) = E^+(S)$] and second, to ensure that under the [global timelike vector field flow-projection] map every point of $\dot{J}^+(S)$ was mapped into a point of H.*
> *[Penrose's theorem] does not answer the question of whether singularities occur in physically realistic solutions. To decide this we need a theorem which does not assume the existence of Cauchy hypersurfaces.*

Hawking and Penrose's theorem

Hawking and Penrose's answer this problem with their theorem.

Theorem (Hawking and Penrose 1970)

Let (M, g) be a chronological spacetime which satisfies the causal convergence condition and the causal genericity condition. Suppose that there exists a trapped surface, then (M, g) is causally geodesically incomplete.

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..but it is weaker than Penrose's

The singularity might well be to the past of the future trapped surface so, in the context of a spacetime that had origin through a Big Bang singularity, Hawking and Penrose's theorem does not provide any new information for what concerns the formation of a singularity through gravitational collapse. *The singularity that it signals could just be the Big Bang singularity.*

Moreover, the genericity condition has no physical justification, e.g. think of geodesics imprisoned in compact Cauchy horizons where the condition is known to be violated.

Bardeen spacetime

Bardeen (1968) gave an example of **null geodesically complete** spacetime that satisfies all assumptions of Penrose's theorem but **global hyperbolicity**.

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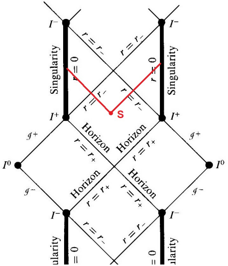

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2),$$

$$f(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} \to f(r) = 1 - \frac{2mr^2}{(r^2+e^2)^{3/2}}$$

The redefinition removes the singularity and preserves the null convergence (energy) condition. Moreover, there are trapped surfaces.

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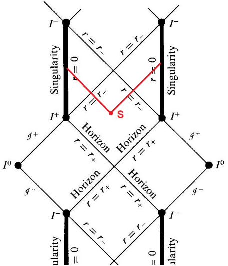

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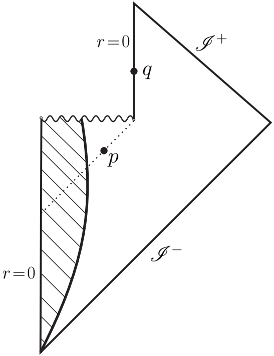
Implications

Hawking and Ellis (1973) concluded that the global hyperbolicity condition in Penrose's theorem is necessary. Borde (1994) suggested that no, the conclusion likely fails because $E^+(S)$ 'swallows' the whole universe. He suggested that Penrose' theorem could hold in all those cases in which there are no closed compact spacelike hypersurfaces (open universes).

Causality in black hole evaporation cannot be good

This is the typical Penrose conformal diagram for an evaporating black hole. Determinism (global hyperbolicity) does not hold because prediction might hold but retrodiction certainly fails.



Notice that $p \in \overline{J^-(q)}$ but $q \notin \overline{J^+(p)}$. That is, future reflectivity is violated (while past reflectivity holds).

Future reflectivity

Definition

The spacetime (M, g) is *future reflecting* if any of the following equivalent properties holds true. For every $p, q \in M$

(i) $(p, q) \in \bar{J} \Rightarrow q \in \overline{J^+(p)}$,

(ii) $p \in \overline{J^-(q)} \Rightarrow q \in \overline{J^+(p)}$,

(iii) $p \in \dot{J}^-(q) \Rightarrow q \in \dot{J}^+(p)$,

(iv) $I^-(p) \subset I^-(q) \Rightarrow I^+(q) \subset I^+(p)$,

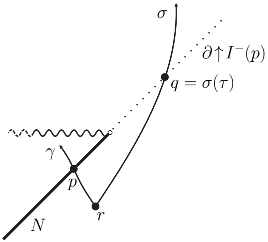
(v) $\uparrow I^-(p) = I^+(p)$,

(vi) $p \mapsto I^+(p)$ is outer continuous,

(vii) the volume function $t^+(p) = -\mu(I^+(p))$ is continuous.

Causality in black hole evaporation cannot be good

Let N be a future C^0 null hypersurface (i.e. the black hole horizon). Let $p \in N$ be a representative point of such a region and let $r \in I^-(p)$. Consider two timelike curves γ and σ, the former curve $\gamma\colon [0,\infty) \to M$, $\gamma(0) = r$, $p = \gamma(a)$, $a > 0$, represents matter that leaves r and crosses the horizon at p, while the latter curve $\sigma\colon [0,\infty) \to M$, $\sigma(0) = r$, $\sigma \cap J^+(N) = \emptyset$, is future inextendible and represents an observer that looks at the infalling matter without being itself causally influenced by the horizon.



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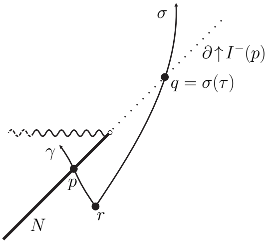

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Definition

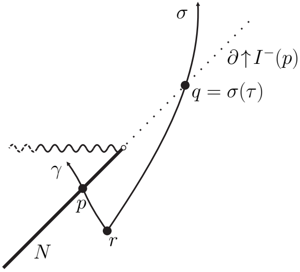
We say that the horizon N *evaporates* at p from the point of view of σ if there is some finite $t > 0$ such that $\gamma([0,a)) \subset J^-\big(\sigma([0,t])\big)$

Waiting further time does not give more information on the infalling matter. In Schwarzschild there is not such evaporation.

Theorem

If an evaporating spacetime (M, g) is past reflective then it is not future reflective, thus reflectivity, global hyperbolicity, causal simplicity, and causal continuity cannot hold.

This is due to the fact that defined $\tau = \inf t$, $q = \sigma(\tau)$, we have under past reflectivity $\gamma([0, a)) \subset J^-(q)$, in particular $p \in \overline{J^-(q)}$ but $q \notin \overline{J^+}(p)$. In particular, the Lorentzian distance cannot be continuous and the spacetime cannot be stationary.



The improvement of Penrose's theorem

Definition

We say that a spacetime is *(spatially) open* if it does not contain a compact spacelike hypersurface.

Theorem

Let (M, g) be a **past reflecting** *spacetime which is* **open** *and satisfies the null convergence condition. Suppose that it admits a future trapped surface S such that $S \cap \mathscr{C} = \emptyset$, then it is future null geodesically incomplete.*

The improvement of Penrose's theorem

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Thus we are weakening the assumptions

(a) global hyperbolicity

(b) the Cauchy hypersurfaces are non-compact

to the much weaker conditions (not even chronology is assumed!)

(a') past reflectivity

(b') the spacetime is open

Borde's intuition was correct

We don't need to assume that the spacetime is open.

Definition

A compact and achronal set S is said to have an *unavoidable* or *swallowing* future horismos if there exists an open neighborhood U of $E^+(S)$ such that $I^-_U(E^+(S)) \subset \mathrm{Int} D^-(E^+(S))$.

In fact under this condition an observer, represented by an inextendible causal curve, that were to pass through a neighborhood of the horismos $E^+(S)$ would be forced to intersect it and hence to fall into its causal influence.

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Theorem

Let (M, g) be a past reflecting spacetime which satisfies the null convergence condition. Suppose that it admits a future trapped surface S such that $S \cap \mathscr{C} = \emptyset$, then it is either future null geodesically incomplete or the horismos $E^+(S)$ is compact, unavoidable and actually coincident with $\dot{I}^+(S)$.

Thus the singularity is avoided only if the horismos swallows the universe.

Ideas of the proof I: araying sets

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Definition

A *future lightlike S-ray* is a future inextendible causal curve which starts from S and does not intersect $I^+(S)$. A set S is a *future null araying set* if there are no future lightlike S-rays.

In short, the set S is araying if observers starting from it catch up with light.

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Theorem

Let S be a non-empty compact set that does not intersect $\mathscr{C}$. If S is a future null araying set then it is a future trapped set.

Under strong causality the converse holds true.

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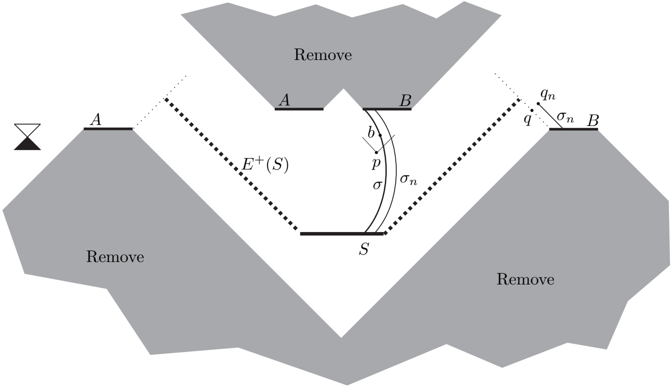
Under strong causality the converse holds true.
The proof of Penrose's theorem, start by assuming null completeness and looking for contradiction. From there one gets focusing and hence absence of lightlike S-rays, that is the araying property and hence the trapped set property but is the former stronger that is important.

Ideas of the proof II: past reflectivity and edge

Theorem

Let (M, g) be past reflecting. If S is a compact and future null araying set that does not intesect $\mathscr{C}$, then $\dot{I}^+(S) = E^+(S)$ and hence $\text{edge}(E^+(S)) = \emptyset$.

Proof by contradiction: suppose there is $q \in \dot{I}^+(S) \setminus E^+(S)$ ..., then past reflectivity is violated, $q \in \overline{J^+(p)}$ but $p \notin \overline{J^-(q)}$.



Conclusions

By dropping global hyperbolicity we have shown that determinism is not required in order to infer geodesic singularities in gravitational collapse. In fact not even chronology is required.

This solves some of the tension between general relativity and quantum field theory (information loss), by showing that *retrodiction* is not necessary already at the classical level, and by showing that both accomodate coherent descriptions of black hole formation and evaporation in non-globally hyperbolic spacetimes.

Thank you for the attention!