The h-principle for minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$

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Abstract

Let M be an open Riemann surface.

Alarcón and Forstnerič, 2015 (Crelle's Journal, in press):
Every conformal minimal immersion $M \to \mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve $M \to \mathbb{C}^3$.

This is a basic h-principle. We upgrade it to a parametric h-principle:

Theorem (F. Lárusson & F. Forstnerič, 2016)

For any $n \geq 3$, the inclusion $\iota : \Re\mathfrak{N}_*(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M, \mathbb{R}^n)$ of the space of real parts of nonflat null holomorphic immersions $M \to \mathbb{C}^n$ into the space of nonflat conformal minimal immersions $M \to \mathbb{R}^n$ satisfies the parametric h-principle with approximation; in particular, it is a weak homotopy equivalence.

If M has finitely generated homology group $H_1(M; \mathbb{Z})$, then $\Re\mathfrak{N}_*(M, \mathbb{C}^n)$ is a strong deformation retract of $\mathfrak{M}_*(M, \mathbb{R}^n)$

Areas of mathematics involved in this result

Analogous results hold for several other related spaces of maps to be introduced.

Based on collaboration with **Finnur Lárusson, University of Adelaide**.

The proof brings together four diverse topics from differential geometry, holomorphic geometry, and topology; namely

- the theory of minimal surfaces in $\mathbb{R}^n$,
- modern Oka theory (a branch of complex analysis),
- Gromov's convex integration theory, and
- the theory of absolute neighborhood retracts (to get strong homotopy equivalences for surfaces M of finite topological type).

Weierstrass representation of minimal surfaces

Let M be an open Riemann surface and $n \geq 3$. The following are equivalent for a **conformal** immersion $u = (u_1, \ldots, u_n) : M \to \mathbb{R}^n$:

- u parametrizes a minimal surface.
- u has identically vanishing mean curvature vector.
- u is harmonic: $\triangle u = 0$.
- $\Phi = \partial u = (\phi_1, \ldots, \phi_n)$ is a nowhere vanishing holomorphic 1-form satisfying the nullity condition

$$(\phi_1)^2 + (\phi_2)^2 + \cdots + (\phi_n)^2 = 0.$$

Conversely, if $\Phi = (\phi_1, \ldots, \phi_n)$ is as above and

$$\int_\gamma \Re(\Phi) = 0 \quad \text{for all} \quad \gamma \in H_1(M;\mathbb{Z})$$

then

$$u(p) = u(p_0) + \int_{p_0}^{p} \Re\Phi, \quad p_0, p \in M,$$

is a conformal minimal immersion $M \to \mathbb{R}^n$.

The null quadric

$$\mathfrak{A} = \mathfrak{A}^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}.$$

Every conformal minimal immersion $M \to \mathbb{R}^n$ ($n \geq 3$) is of the form

$$u(p) = u(p_0) + \int_{p_0}^{p} \Re\,(f\theta)\,, \quad p_0, p \in M$$

where θ is a nowhere vanishing holomorphic 1-form on M,

$$f = 2\partial u/\theta = (f_1, \ldots, f_n) \colon M \to \mathfrak{A}_* = \mathfrak{A} \setminus \{0\} \subset \mathbb{C}^n$$

is a holomorphic map and the real periods of $f\theta$ vanish.

If the complex periods of $f\theta$ vanish, then

$$F(p) = F(p_0) + \int_{p_0}^{p} f\theta \in \mathbb{C}^n, \quad p_0, p \in M$$

is a **holomorphic null curve** in $\mathbb{C}^n$ with $u = \Re F$. Equivalently:

$$\mathrm{Flux}(u)(\gamma) := \int_\gamma \Im(f\theta) = 0 \quad \forall \gamma \in H_1(M; \mathbb{Z}).$$

A diagram of spaces and maps

$\mathfrak{N}_*(M,\mathbb{C}^n)$: nonflat holomorphic null curves $M \to \mathbb{C}^n$

$\mathfrak{M}_*(M,\mathbb{R}^n)$: nonflat conformal minimal immersions $M \to \mathbb{R}^n$

$$\begin{array}{ccccc}
\mathfrak{N}_*(M,\mathbb{C}^n) & \xrightarrow{\ \phi\ } & \mathscr{O}(M,\mathfrak{A}_*) & \hookrightarrow & \mathscr{C}(M,\mathfrak{A}_*) \\
\Big\downarrow \Re & & \Big\uparrow \psi & & \\
\Re\mathfrak{N}_*(M,\mathbb{C}^n) & \xhookrightarrow{\ \iota\ } & \mathfrak{M}_*(M,\mathbb{R}^n) & &
\end{array}$$

- Our main theorem: ι **is a weak homotopy equivalence (WHE).**
- The projection $\Re : F \mapsto \Re F$ of a null curve to its real part is clearly a homotopy equivalence.
- The map ϕ is given by $F \mapsto \partial F/\theta$, and ψ is given by $u \mapsto 2\partial u/\theta$. Hence, if one of the maps ϕ, ψ is a WHE, then so is the other one. **We prove that both of them are WHEs.**
- The inclusion $\mathscr{O}(M,\mathfrak{A}_*) \hookrightarrow \mathscr{C}(M,\mathfrak{A}_*)$ is a WHE by the **Oka-Grauert principle** (since $\mathfrak{A}_*$ is an **Oka manifold**).

Connected components of the space $\mathfrak{M}_*(M, \mathbb{R}^n)$

Recall that $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^l$ with $l \in \mathbb{Z}_+ \cup \{\infty\}$.

The punctured null quadric $\mathfrak{A}_*^{n-1} \subset \mathbb{C}^n$ is simply connected when $n \geq 4$, while $\pi_1(\mathfrak{A}_*^2) \cong \mathbb{Z}_2$ in view of the two-sheeted universal covering

$$\pi \colon \mathbb{C}_*^2 = \mathbb{C}^2 \setminus \{0\} \to \mathfrak{A}_*^2, \quad \pi(u,v) = \big(u^2 - v^2, \mathrm{i}(u^2+v^2), 2uv\big).$$

Hence, the path components of the space $\mathscr{C}(M, \mathfrak{A}_*^2)$ are in one-to-one correspondence with the elements of $(\mathbb{Z}_2)^l$, and $\mathscr{C}(M, \mathfrak{A}_*^{n-1})$ is path connected if $n \geq 4$.

Corollary

Let M be a connected open Riemann surface with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^l$. Then the path connected components of $\mathfrak{M}_*(M, \mathbb{R}^3)$ and $\mathfrak{N}_*(M, \mathbb{C}^3)$ are in one-to-one correspondence with the elements of $(\mathbb{Z}_2)^l$.
If $n \geq 4$ then $\mathfrak{M}_*(M, \mathbb{R}^n)$ and $\mathfrak{N}_*(M, \mathbb{C}^n)$ are path connected.

Weierstrass representation in dimension $n = 3$

We illustrate the Corollary by a few examples in dimension $n = 3$.

Let M be $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ or an annulus, with $\theta = dz$.

Since $\pi_1(\mathfrak{A}_*) = \mathbb{Z}_2$, there are precisely two homotopy classes of holomorphic maps $f \colon M \to \mathfrak{A}_*$.

Let $\pi \colon \mathbb{C}^2_* \to \mathfrak{A}_*$ be the universal covering map as above. Note that f is nullhomotopic if and only if it factors through π.

Consider the **Weierstrass representation:**

$$f_1 = (1 - g^2)\eta, \quad f_2 = \mathfrak{i}(1 + g^2)\eta, \quad f_3 = 2g\eta,$$

where g is meromorphic and η is holomorphic on M. Assume for simplicity that g is holomorphic or, equivalently, that η has no zeros.

Then, f factors through π if and only if η has a square root on M.

Indeed, if η has a square root then $f = \pi(\sqrt{\eta}, g\sqrt{\eta})$; conversely, if $f = \pi(u, v)$ for some holomorphic map $(u, v) : M \to \mathbb{C}^2_*$, then $u^2 = \eta$.

Examples in dimension $n = 3$

1. A flat null curve: $M = \mathbb{C}_* = \mathbb{C} \setminus \{0\}$ and $f \colon \mathbb{C}_* \to \mathfrak{A}_* \subset \mathbb{C}^3$ is the map $f(\zeta) = \zeta(1, \mathrm{i}, 0)$. In this case, $g = 0$ and $\eta(\zeta) = \zeta$ does not have a square root on M. Thus, the flat null curve

$$F(\zeta) = \frac{1}{2}(\zeta^2, \mathrm{i}\zeta^2, 0), \quad \zeta \in \mathbb{C}_*$$

lies in the nontrivial isotopy class.

2. The catenoid: $M = \mathbb{C}_*$, $g(\zeta) = \zeta$, and $\eta(\zeta) = 1/\zeta^2$. Since η has a square root on M, we are in the trivial isotopy class.
The same holds for the **helicoid** which is parameterized by $\mathbb{C}$.

3. Henneberg's surface:

$$M = \mathbb{C} \setminus \{0, 1, -1, \mathrm{i}, -\mathrm{i}\}, \quad g(\zeta) = \zeta, \quad \eta(\zeta) = 1 - \zeta^{-4}.$$

On a small punctured disc centered at one of the points 1, -1, i, or $-\mathrm{i}$, η does not have a square root, so we are in the nontrivial isotopy class.

On the punctured disc $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$, the function η has a square root, so we are in the trivial isotopy class.

4. Double sheeted covering of Meeks's minimal Möbius strip:

$$M = \mathbb{C}_*, \quad g(\zeta) = \zeta^2 \frac{\zeta+1}{\zeta-1}, \quad \eta(\zeta) = \mathfrak{i}\, \frac{(\zeta-1)^2}{\zeta^4}.$$

Note that η has a square root on M. Despite the pole of g at 1, we get a factorization through π and we are in the trivial isotopy class.

Let $F = u + \mathfrak{i} v \colon \mathbb{C}_* \to \mathbb{C}^3$ be the null curve with this Weierstrass data. Then u is invariant with respect to the fixed-point-free antiholomorphic involution

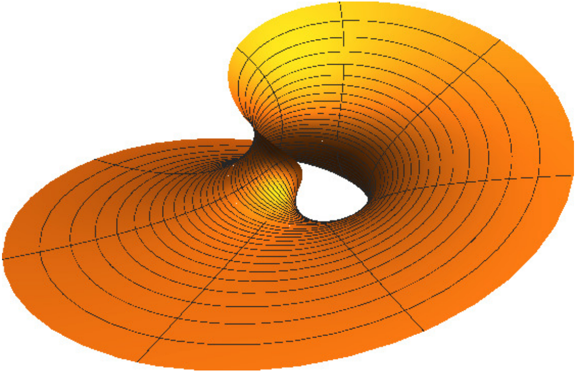
$$\mathfrak{I}(\zeta) = -1/\bar{\zeta} \quad \text{on} \quad \zeta \in \mathbb{C}_*,$$

and hence it induces a conformal minimal immersion $\mathbb{C}_*/\mathfrak{I} \to \mathbb{R}^3$.

This is Meeks's complete (proper) minimal Möbius strip in $\mathbb{R}^3$ with finite total curvature -6π.

Meeks's minimal Möbius strip

W.H. Meeks: The classification of complete minimal surfaces in $\mathbb{R}^3$ with total curvature greater than -8π. Duke Math. J. 48 (1981) 523–535



The illustration: © **Antonio Alarcón**.

Parametric h-principle for $\mathfrak{RN}_*(M,\mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M,\mathbb{R}^n)$

Theorem

Assume that M is an open Riemann surface, $Q \subset P$ are compact Hausdorff spaces, $D \Subset M$ is a smoothly bounded Runge domain, and $u \colon M \times P \to \mathbb{R}^n$ $(n \geq 3)$ is a continuous map satisfying the following:

(a) $u_p = u(\cdot\,, p) \colon M \to \mathbb{R}^n$ *is a nonflat CMI for every* $p \in P$*;*

(b) $u_p|_{\overline{D}} \colon \overline{D} \to \mathbb{R}^n$ *has vanishing flux for every* $p \in P$*;*

(c) $\mathrm{Flux}(u_p) = 0$ *for every* $p \in Q$.

Given $\epsilon > 0$, there exists a homotopy $u^t \colon M \times P \to \mathbb{R}^n$ $(t \in [0,1])$ such that each map $u^t_p := u^t(\cdot\,, p) \colon M \to \mathbb{R}^n$ is a nonflat CMI satisfying

(1) $u^t_p = u_p$ *for every* $(p,t) \in (P \times \{0\}) \cup (Q \times [0,1])$*;*

(2) $|u^t_p(x) - u_p(x)| < \epsilon$ *for all* $x \in \overline{D}$ *and* $(p,t) \in P \times [0,1]$*;*

(3) $u^t_p|_{\overline{D}}$ *has vanishing flux for every* $(p,t) \in P \times [0,1]$*;*

(4) $\mathrm{Flux}(u^1_p) = 0$ *for every* $p \in P$.

The WHE-principle for $\mathfrak{RM}_*(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M, \mathbb{R}^n)$

This is the parametric h-principle with approximation for the inclusion

$$\mathfrak{RM}_*(M, \mathbb{C}^n) = \{u \in \mathfrak{M}_*(M, \mathbb{R}^n) : \mathrm{Flux}(u) = 0\} \longhookrightarrow \mathfrak{M}_*(M, \mathbb{R}^n).$$

Assuming that this result holds, we now give:

Proof of the WHE-principle.

Let $k \in \mathbb{Z}_+$. Applying the h-principle with $P = \mathbb{S}^k$ (the real k-sphere) and $Q = \emptyset$ shows that the inclusion induced map

$$\pi_k(\mathfrak{RM}_*(M, \mathbb{C}^n)) \longrightarrow \pi_k(\mathfrak{M}_*(M, \mathbb{R}^n)),$$

is surjective.

Applying the h-principle with $P = \overline{\mathbb{B}}^{k+1}$ (the closed ball in $\mathbb{R}^{k+1}$) and $Q = \mathbb{S}^k = b\mathbb{B}^{k+1}$ shows that the above map is also injective.

Thus, it is an isomorphism for every $k \in \mathbb{Z}_+$. □

Proof of the h-principle for $\mathfrak{R}\mathfrak{N}_*(M, \mathbb{C}^n) \hookrightarrow \mathfrak{M}_*(M, \mathbb{R}^n)$

Pick a smooth strongly subharmonic Morse exhaustion function $\rho \colon M \to \mathbb{R}$ and exhaust M by sublevel sets

$$D_j = \{x \in M \colon \rho(x) < c_j\}, \quad j \in \mathbb{N}$$

where $c_1 < c_2 < c_3 < \ldots$ is an increasing sequence of regular values of ρ such that $\lim_{j\to\infty} c_j = \infty$ and each interval $[c_j, c_{j+1}]$ contains at most one critical value of the function ρ.

We may assume that $D = D_1$.

Let $\epsilon > 0$ be as in the theorem. Pick a sequence $\epsilon_j > 0$ with $\sum_{j=1}^{\infty} \epsilon_j < \epsilon$. Set

$$u^t_{p,1} := u_p|_{\overline{D}_1}, \quad (p, t) \in P \times [0, 1].$$

The recursive scheme

We recursively construct a sequence of homotopies of CMI's

$$u_{p,j}^t \colon \overline{D}_j \longrightarrow \mathbb{R}^n, \quad (p,t) \in P \times [0,1], \ \ j \in \mathbb{N}$$

satisfying the following conditions for every $j = 2, 3, \ldots$:

(a_j) $u_{p,j}^t = u_p|_{\overline{D}_j}$ for $(p,t) \in (P \times \{0\}) \cup (Q \times [0,1])$;

(b_j) $\|u_{p,j}^t - u_{p,j-1}^t\|_{\overline{D}_{j-1}} < \epsilon_j$ for all $(p,t) \in P \times [0,1]$;

(c_j) $\mathrm{Flux}(u_{p,j}^t|_{\overline{D}_{j-1}}) = \mathrm{Flux}(u_{p,j-1}^t)$ for every $(p,t) \in P \times [0,1]$;

(d_j) $\mathrm{Flux}(u_{p,j}^1) = 0$ on $\overline{D}_j$ for every $p \in P$.

These conditions imply that the limit

$$u_p^t = \lim_{j \to \infty} u_{p,j}^t \colon M \to \mathbb{R}^n$$

exists and satisfies the conclusion of the theorem.
Indeed, Conditions (1)–(4) follow from (a_j)–(d_j), respectively.

(a) The noncritical case: ρ has no critical values in $[c_j, c_{j+1}]$.

Choose a Runge homology basis $\mathcal{B} = \{\gamma_i \colon i = 1, \ldots, l\}$ for $H_1(\overline{D}_j; \mathbb{Z})$, such that $\mathcal{B}' = \{\gamma_1, \ldots, \gamma_m\}$ for some $m \in \{0, \ldots, l\}$ is a homology basis for $H_1(\overline{D}; \mathbb{Z})$.

Then, $\mathcal{B}$ is also a homology basis for $H_1(\overline{D}_{j+1}; \mathbb{Z})$.

Denote by $\mathcal{P}$ the **period map** associated to $\mathcal{B}$:

$$\mathcal{P}(f) = \left(\int_{\gamma_i} f\theta \right)_{i=1,\ldots,l} \in (\mathbb{C}^n)^l, \qquad f \in \mathcal{A}(\overline{D}_j, \mathfrak{A}_*).$$

Also, $\mathcal{P}' \colon \mathcal{A}(\overline{D}, \mathfrak{A}_*) \to (\mathbb{C}^n)^m$ is the period map with respect to $\mathcal{B}'$.

The noncritical case, continued

Consider the continuous family of nonflat holomorphic maps

$$f_p^t := 2\partial u_{p,j}^t/\theta \colon \overline{D}_j \longrightarrow \mathfrak{A}_*, \quad p \in P,\ t \in [0,1].$$

Conditions on $u_{p,j}^t : \overline{D}_j \to \mathbb{R}^n$ imply the following:

$$\begin{aligned} \Re\mathcal{P}(f_p^t) &= 0, \quad (p,t) \in P \times [0,1]; \\ \mathcal{P}'(f_p^t|_{\overline{D}}) &= 0, \quad (p,t) \in P \times [0,1]; \\ \mathcal{P}(f_p^1) &= 0, \quad p \in P. \end{aligned}$$

We embed f_p^t as the core $f_p^t = f_{p,0}^t$ of a **period dominating spray**

$$f_{p,\zeta}^t \colon \overline{D}_j \longrightarrow \mathfrak{A}_*, \quad \zeta \in B \subset \mathbb{C}^N,\ p \in P,\ t \in [0,1],$$

i.e., the period map

$$B \ni \zeta \longmapsto \mathcal{P}(f_{p,\zeta}^t) = \left(\int_{\gamma_i} f_{p,\zeta}^t \theta \right)_{i=1,\ldots,l} \in (\mathbb{C}^n)^l$$

is submersive at $\zeta = 0$ for every $(p,t) \in P \times [0,1]$.

The noncritical case, continued

Since $\mathfrak{A}_*$ is an **Oka manifold** and $\overline{D}_j$ is a deformation retract of $\overline{D}_{j+1}$, the **parametric Oka property** allows us to approximate the spray $f^t_{p,\zeta} \colon \overline{D}_j \to \mathfrak{A}_*$ by a holomorphic spray

$$g^t_{p,\zeta} \colon \overline{D}_{j+1} \longrightarrow \mathfrak{A}_*, \quad (p,t) \in P \times [0,1], \ \zeta \in rB$$

for some $r \in (1/2, 1)$. If the approximation is sufficiently close, the implicit function theorem gives (in view of the period domination property of the spray $f^t_{p,\zeta}$) a continuous map

$$\zeta \colon P \times [0,1] \longrightarrow rB \subset \mathbb{C}^N,$$

vanishing on $(P \times \{0\}) \cup (Q \times [0,1])$, such that the homotopy of holomorphic maps

$$\tilde f^t_p := g^t_{p,\zeta(p,t)} \colon \overline{D}_{j+1} \longrightarrow \mathfrak{A}_*, \quad (p,t) \in P \times [0,1]$$

satisfies the following period conditions:

$$\mathcal{P}(\tilde f^t_p) = \mathcal{P}(f^t_p), \qquad (p,t) \in P \times [0,1].$$

The noncritical case, conclusion

Assume that the set $\overline{D}_j$ (and hence $\overline{D}_{j+1}$) is connected.

Choose a point $x_0 \in \overline{D}_j$ and set for $(p,t) \in P \times [0,1]$:

$$u^t_{p,j+1}(x) := u^t_{p,j}(x_0) + \int_{x_0}^{x} \Re(\tilde{f}^t_p \theta), \quad x \in \overline{D}_{j+1}.$$

Then, $u^t_{p,j+1} \colon \overline{D}_{j+1} \to \mathbb{R}^n$ is a continuous family of conformal minimal immersions satisfying conditions (a_{j+1})–(d_{j+1}).

In particular,

$$\mathcal{P}(\tilde{f}^1_p) = \mathcal{P}(f^1_p) = 0 \text{ for } p \in P$$

and hence $u^1_{p,j+1}$ has vanishing flux.

If $\overline{D}_j$ is disconnected, we apply the same argument on the components.

The critical case

(a) The critical case: ρ contains a unique critical point $x_0 \in D_{j+1} \setminus \overline{D}_j$. Then, $\overline{D}_{j+1}$ deformation retracts onto a compact set $S = \overline{D}_j \cup E$, where $E \subset M \setminus D_j$ is an embedded arc attached with both endpoints to $\overline{D}_j$.

It suffices to construct an isotopy $u^t_{p,j+1}$ satisfying (a_{j+1})–(d_{j+1}) on a neighborhood of S and apply the noncritical case to extend it to $\overline{D}_{j+1}$.

The key to the proof is to find smooth extension of the map $f^t_p = 2\partial u^t_{p,j}/\theta \colon \overline{D}_j \to \mathfrak{A}_*$ across the arc E with the correct integral in order to ensure the required period conditions on $S = \overline{D}_j \cup E$.

This is accomplished by the following lemma whose proof uses a parametric version of Gromov's convex integration lemma.

Gromov, 1973; cf. **D. Spring:** Convex integration theory. Solutions to the h-principle in geometry and topology. Birkhäuser/Springer Basel AG, Basel, 2010.

An application of Gromov's convex integration lemma

Lemma

Let $Q \subset P$ be compact Hausdorff spaces and let $\sigma\colon P \times [0,1] \to \mathfrak{A}_*$ be a continuous map. Consider $\sigma_p = \sigma(p, \cdot\,)\colon [0,1] \to \mathfrak{A}_*$ as a family of paths in $\mathfrak{A}_*$ depending continuously on the parameter $p \in P$. Set

$$\alpha_p = \int_0^1 \sigma_p(s)\, ds \in \mathbb{C}^n, \quad p \in P.$$

Given a continuous family $\alpha_p^t \in \mathbb{C}^n$ $(p \in P,\ t \in [0,1])$ such that

$$\alpha_p^t = \alpha_p \ \text{ for all } (p,t) \in (P \times \{0\}) \cup (Q \times [0,1]),$$

there exists a homotopy of paths $\sigma_p^t\colon [0,1] \to \mathfrak{A}_*$ $(p \in P,\ t \in [0,1])$ satisfying the following conditions:

(i) $\sigma_p^t = \sigma_p$ *for all* $(p,t) \in (P \times \{0\}) \cup (Q \times [0,1])$;

(ii) $\sigma_p^t(0) = \sigma_p(0)$ *and* $\sigma_p^t(1) = \sigma_p(1)$ *for all* $p \in P$ *and* $t \in [0,1]$;

(iii) $\int_0^1 \sigma_p^t(s)\, ds = \alpha_p^t$ *for all* $p \in P$ *and* $t \in [0,1]$.

Main idea of the proof

Gromov's 1-dimensional Convex Integration Lemma (1973):

Let Ω be an open connected set in a Banach space B. Fix a path $\sigma_0 : [0,1] \to \Omega$, and let Γ be the set of all paths $\sigma : [0,1] \to \Omega$ which are homotopic to σ_0 with fixed ends $\sigma(0) = \sigma_0(0)$, $\sigma(1) = \sigma_0(1)$. Then,

$$\mathfrak{I}(\Gamma) := \left\{ \int_0^1 \sigma(s)\, ds : \sigma \in \Gamma \right\} = \mathrm{Co}(\Omega).$$

The main idea is to approximate the integral $\int_0^1 \sigma(s)\, ds$ by Riemann sums $\sum_{i=1}^N \sigma(s_i)\delta_i$, which are convex combinations of points on the path.

We can represent any given vector $\alpha \in \mathrm{Co}(\Omega)$ as $\alpha = \sum_{i=1}^N p_i \delta_i$ with $p_i \in \Omega$ and $\sum_{i=1}^N \delta_i = 1$ for some big N. Construct a path $\sigma \in \Gamma$ which spends approximately the time δ_i at p_i for each i. Then,

$$\int_0^1 \sigma(s)\, ds \approx \sum_{i=1}^N p_i \delta_i = \alpha.$$

This shows that $\mathfrak{I}(\Gamma)$ is open, convex, and dense in $\mathrm{Co}(\Omega)$; hence it equals $\mathrm{Co}(\Omega)$. A similar argument applies in the parametric case.

How is this is lemma used?

We take $B = \mathbb{C}^n$. The convex hull of the null quadric equals $\mathbb{C}^n$: $\mathrm{Co}(\mathfrak{A}) = \mathbb{C}^n$. Hence we can choose numbers $0 < r < R$ such that

$$\{\alpha_p^t \in \mathbb{C}^n : p \in P,\ t \in [0,1]\} \subset \mathrm{Co}(\mathfrak{A}_{r,R})$$

where

$$\mathfrak{A}_{r,R} = \{z \in \mathfrak{A} : r \leq |z| \leq R\}.$$

Let $\Omega \subset \mathbb{C}^n$ be a thin tubular neighborhood of $\mathfrak{A}_{r,R}$. We apply Gromov's lemma with the pair $\Omega \subset \mathrm{Co}(\Omega)$ to get a deformation (σ_p^t) which enjoys properties (i), (ii), and with (iii) replaced by an approximate condition

$$\left| \int_0^1 \sigma_p^t(s)\, ds - \alpha_p^t \right| < \epsilon, \quad p \in P,\ t \in [0,1].$$

The small error is caused by projecting the paths from Ω to $\mathfrak{A}_{r,R}$.

This is applied on a segment $I \subset E$ of the arc E. The error is corrected by using period dominating sprays on another disjoint segment $I' \subset E$.

Absolute neighbourhood retracts (ANR)

Much of basic algebraic topology is developed for CW complexes, for example **Whitehead's theorem:** A weak homotopy equivalence between CW complexes is a homotopy equivalence.

Spaces of maps usually are not CW complexes, but sometimes they can be shown to have the homotopy type of a CW complex by showing that they are ANR. Whitehead's theorem clearly holds for spaces with the homotopy type of a CW complex.

A metric space X is ANR if whenever it is embedded as a closed subspace of a metric space Y, some nbhd of X in Y retracts onto X. There are several other nontrivially equivalent characterisations.

One is useful in practice: the **Dugundji-Lefschetz characterisation**. Assuming that M is a surface with finitely generated $H_1(M;\mathbb{Z})$, the DL-characterisation can be verified for our mapping spaces by using the parametric h-principle with approximation that we proved.

Conclusion

Let M be a connected open Riemann surface and $n \geq 3$.

$$\begin{array}{ccccc}
\mathfrak{N}_*(M, \mathbb{C}^n) & \longrightarrow & \mathscr{O}(M, \mathfrak{A}_*) & \hookrightarrow & \mathscr{C}(M, \mathfrak{A}_*) \\
\downarrow & & \uparrow & & \\
\Re\mathfrak{N}_*(M, \mathbb{C}^n) & \hookrightarrow & \mathfrak{M}_*(M, \mathbb{R}^n) & &
\end{array}$$

These spaces all have the same weak homotopy type, and when M has finite topological type even the same strong homotopy type, as the space $\mathfrak{H}$ of continuous maps from a wedge of circles to $\mathfrak{A}_*$.

$\Re : \mathbb{C}^n \to \mathbb{R}^n$ gives $\mathfrak{A}_*$ the structure of a fibre bundle, whose fibre is S^{n-2}, over $\mathbb{R}^n \setminus \{0\}$, which is homotopy equivalent to S^{n-1}.

Thus the structure of $\mathfrak{H}$ can be understood in terms of spheres and their loop spaces. The homotopy groups of $\mathfrak{H}$ can be calculated in terms of homotopy groups of spheres.