Every meromorphic function is the Gauss map of a conformal minimal surface

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Abstract

Let M be an open Riemann surface.

- We prove that every meromorphic function on M is the complex Gauss map of a conformal minimal immersion $M \to \mathbb{R}^3$ which may furthermore be chosen as the real part of a holomorphic null curve $M \to \mathbb{C}^3$.

- Analogous results are proved for conformal minimal immersions $M \to \mathbb{R}^n$ and null curves $M \to \mathbb{C}^n$ for any $n > 3$.

- We also show that every conformal minimal immersion $M \to \mathbb{R}^n$ is isotopic (through conformal minimal immersions) to a flat one, and we identify the path connected components of the space of all conformal minimal immersions $M \to \mathbb{R}^n$ for any $n \geq 3$.

Based on joint work with **Antonio Alarcón and Francisco J. López, University of Granada**; http://arxiv.org/abs/1604.00514

Conformal minimal surfaces in $\mathbb{R}^n$

Let M be an open Riemann surface and $n \geq 3$. The following are equivalent for a **conformal immersion** $X = (X_1, \ldots, X_n) : M \to \mathbb{R}^n$:

- X parametrizes a minimal surface.
- X has identically vanishing mean curvature vector.
- X is harmonic: $\triangle X = 0$.
- $\Phi = \partial X = (\phi_1, \ldots, \phi_n)$ is a nowhere vanishing holomorphic 1-form satisfying the following nullity condition:

$$(\phi_1)^2 + (\phi_2)^2 + \cdots + (\phi_n)^2 = 0.$$

Conversely, if $\Phi = (\phi_1, \ldots, \phi_n)$ satisfies the nullity condition and

$$\int_\gamma \Re(\Phi) = 0 \quad \text{for all} \quad \gamma \in H_1(M; \mathbb{Z}),$$

then

$$X(p) = X(p_0) + \int_{p_0}^{p} 2\Re\Phi, \quad p_0, p \in M$$

is a conformal minimal immersion $M \to \mathbb{R}^n$.

Weierstrass representation of minimal surfaces

Fix a nowhere vanishing holomorphic 1-form θ on M. The above shows that every conformal minimal immersion $X\colon M \to \mathbb{R}^n$ is of the form

$$X(p) = X(p_0) + \int_{p_0}^{p} \Re(f\theta), \quad p, p_0 \in M,$$

where $f\colon M \to A_*^{n-1} = A^{n-1} \setminus \{0\}$ is a holomorphic map with values in the **null quadric**

$$A^{n-1} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \colon z_1^2 + z_2^2 + \cdots + z_n^2 = 0 \right\}$$

such that the $\mathbb{C}^n$-valued 1-form $f\theta$ has **vanishing real periods**.

The **flux** of a conformal minimal immersion $X\colon M \to \mathbb{R}^n$ is the group homomorphism $\mathrm{Flux}_X\colon H_1(M; \mathbb{Z}) \to \mathbb{R}^n$ given by

$$\mathrm{Flux}_X(\gamma) = \int_{\gamma} \Im(\partial X) = -\mathrm{i} \int_{\gamma} \partial X \quad \text{for every closed curve } \gamma \subset M.$$

Construction of null curves and CMI's

We have $\mathrm{Flux}_X = 0$ iff X admits a harmonic conjugate surface $Y \colon M \to \mathbb{R}^n$. In this case, $Z = X + iY \colon M \to \mathbb{C}^n$ is a **holomorphic null curve**, i.e., a holomorphic immersion satisfying the nullity condition

$$(dZ_1)^2 + (dZ_2)^2 + \cdots + (dZ_n)^2 = 0.$$

Thus, we have bijective correspondences (up to constants):

$$\{Z \colon M \to \mathbb{C}^n \text{ null curve}\} \longleftrightarrow \{f \colon M \to A_* \text{ holomorphic}, \ f\theta \text{ exact}\}$$

$$Z(p) = Z(p_0) + \int_{p_0}^{p} f\theta; \qquad p \in M.$$

$$\{X \colon M \to \mathbb{R}^n \text{ conformal minimal}\} \longleftrightarrow \{f \colon M \to A_* \text{ holo.}, \ \Re(f\theta) \text{ exact}\}$$

$$X(p) = X(p_0) + \int_{p_0}^{p} \Re(f\theta); \qquad p \in M.$$

The generalized Gauss map

We now come to the main subject of this talk.

Let $X = (X_1, \ldots, X_n) \colon M \to \mathbb{R}^n$ be a conformal minimal immersion.

Since the 1-form $\partial X = (\partial X_1, \ldots, \partial X_n)$ is holomorphic and nowhere vanishing, it determines the Kodaira type holomorphic map

$$G_X \colon M \to \mathbb{CP}^{n-1}, \quad G_X(p) = [\partial X_1(p) \colon \cdots \colon \partial X_n(p)] \quad (p \in M).$$

The map G_X is known as the **generalized Gauss map of X** and is of great importance in the theory of minimal surfaces.

Since $\sum_{j=1}^{n} (\partial X_j)^2 = 0$, G_X assumes values in the hyperquadric

$$Q^{n-2} = \{[z_1 : \ldots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \cdots + z_n^2 = 0\} = \pi(A_*^{n-1}),$$

where $\pi \colon \mathbb{C}_*^n = \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ denotes the canonical projection.

The main theme

Theorem

Let M be an open Riemann surface and let $n \geq 3$ be an integer.

- For every holomorphic map $\mathscr{G} \colon M \to Q^{n-2} \subset \mathbb{CP}^{n-1}$ there exists a conformal minimal immersion $X \colon M \to \mathbb{R}^n$ with the generalized Gauss map $G_X = \mathscr{G}$ and with vanishing flux (hence, X is the real part of a holomorphic null curve $Z \colon M \to \mathbb{C}^n$).

- If in addition the map $\mathscr{G}$ is full (i.e., its image is not contained in any proper projective subspace of $\mathbb{CP}^{n-1}$), then X can be chosen to have arbitrary flux and to be an embedding if $n \geq 5$ and an immersion with simple double points if $n = 4$.

Problem

Suppose that M and $\mathscr{G}$ are algebraic (so $M = \overline{M} \setminus \{p_1, \ldots, p_m\}$ with $\overline{M}$ a compact Riemann surface); when can X be chosen algebraic?

The Weierstrass representation for $n = 3$

The quadric $Q^1 \subset \mathbb{CP}^2$ is the image of a quadratically embedded Riemann sphere $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$, and the **complex Gauss map** of a conformal minimal immersion $X = (X_1, X_2, X_3) \colon M \to \mathbb{R}^3$ is defined to be the holomorphic map

$$g_X = \frac{\partial X_3}{\partial X_1 - i\,\partial X_2} = \frac{\partial X_2 - i\,\partial X_1}{i\,\partial X_3} : M \longrightarrow \mathbb{CP}^1.$$

The function g_X is the stereographic projection of the real Gauss map $N = (N_1, N_2, N_3) \colon M \to \mathbb{S}^2 \subset \mathbb{R}^3$ to the Riemann sphere $\mathbb{CP}^1$:

$$g_X = \frac{N_1 + iN_2}{1 - N_3} : M \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1.$$

We can recover the differential $\partial X = (\partial X_1, \partial X_2, \partial X_3)$ from the pair (g_X, ϕ_3), with $\phi_3 = \partial X_3$, by the classical **Weierstrass formula**:

$$\partial X = \Phi = (\phi_1, \phi_2, \phi_3) = \left(\frac{1}{2}\left(\frac{1}{g_X} - g_X \right), \frac{i}{2}\left(\frac{1}{g_X} + g_X \right), 1 \right) \phi_3.$$

The case $n = 3$

Conversely, given a pair (g, ϕ_3) consisting of a holomorphic map $g \colon M \to \mathbb{CP}^1$ and a meromorphic 1-form ϕ_3 on M, the meromorphic 1-form $\Phi = (\phi_1, \phi_2, \phi_3)$ defined by the Weierstrass formula satisfies

$$(\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 = 0.$$

Φ is the differential ∂X of a conformal minimal immersion $X \colon M \to \mathbb{R}^3$ iff it is holomorphic and nowhere vanishing, and **its real periods vanish**.

Corollary

Let M be an open Riemann surface.

Every holomorphic map $g \colon M \to \mathbb{CP}^1$ is the complex Gauss map of a holomorphic null curve $Z = X + iY \colon M \to \mathbb{C}^3$, and hence of conformal minimal immersion $X \colon M \to \mathbb{R}^3$.

If g is nonconstant, then we can find X with $g_X = g$ and arbitrary flux.

Example: the catenoid and the helicoid

Example

Consider the null curve

$$Z(\zeta) = (\cos\zeta, \sin\zeta, -\mathrm{i}\zeta) \in \mathbb{C}^3, \qquad \zeta = u + \mathrm{i}v \in \mathbb{C}$$

and the associated family of minimal surfaces in $\mathbb{R}^3$ for $t \in \mathbb{R}$:

$$
\begin{aligned}
X_t(\zeta) &= \Re\left(e^{\mathrm{i}t} Z(\zeta)\right) \\
&= \cos t \begin{pmatrix} \cos u \cdot \cosh v \\ \sin u \cdot \cosh v \\ v \end{pmatrix} + \sin t \begin{pmatrix} \sin u \cdot \sinh v \\ -\cos u \cdot \sinh v \\ u \end{pmatrix}
\end{aligned}
$$

At $t = 0$ we have a **catenoid** and at $t = \pm\pi/2$ a **helicoid**.

$$\partial Z = (-\sin\zeta, \cos\zeta, -\mathrm{i})d\zeta$$

$$g_Z(\zeta) = \frac{\partial Z_2 - \mathrm{i}\,\partial Z_1}{\mathrm{i}\,\partial Z_3} = \cos\zeta + \mathrm{i}\sin\zeta.$$

Stable minimal surfaces in $\mathbb{R}^3$

A minimal surface $S \subset \mathbb{R}^3$ is said to be **stable** if any relatively compact smoothly bounded domain $D \subset S$ has minimal area among all small variations of $\overline{D}$ which keep the boundary bD fixed.

Barbosa and do Carmo, 1976
Let $X \colon M \to \mathbb{R}^3$ be a conformal minimal immersion. The minimal surface $X(M) \subset \mathbb{R}^3$ is stable if the spherical image $g_X(M) \subset \mathbb{CP}^1$ of its Gauss map has area less than 2π.

This holds in particular if $g_X(M)$ lies in the unit disk $\mathbb{D} \subset \mathbb{C}$.

Corollary

If M is an open Riemann surface and $g \colon M \to \mathbb{CP}^1$ is a holomorphic map whose image $g(M)$ has area less than 2π, then there is a stable conformal minimal immersion $M \to \mathbb{R}^3$ with the complex Gauss map g.

Proof of the main theorem

An open Riemann surface M is homotopy equivalent to a wedge of circles.

The projection $\pi \colon \mathbb{C}_*^n \to \mathbb{C}\mathbb{P}^{n-1}$ is a $\mathbb{C}_*$ fiber bundle.

Hence, every holomorphic map $\mathscr{G} \colon M \to \mathbb{C}\mathbb{P}^{n-1}$ lifts to a holomorphic map $f = (f_1, \ldots, f_n) \colon M \to \mathbb{C}_*^n$ such that

$$\mathscr{G} = \pi \circ f = [f_1 \colon \cdots \colon f_n] \colon M \to \mathbb{C}\mathbb{P}^{n-1}.$$

Clearly, $\mathscr{G}(M) \subset Q^{n-2}$ if and only if $f(M) \subset A_*$.

To prove the theorem, we find a **holomorphic multiplier** $h \colon M \to \mathbb{C}_*$ such that the $\mathbb{C}_*^n$-valued holomorphic 1-form

$$\Phi = hf\theta = h(f_1, \ldots, f_n)\theta$$

has vanishing periods, so it integrates to a holomorphic null curve

$$Z(p) = X(p) + \mathrm{i}Y(p) = \int_*^p \Phi, \qquad p \in M.$$

Then, $\partial Z = 2\partial X = \Phi$ and hence $G_X = G_Z = \mathscr{G}$.

The main theorem (complex analysis)

Theorem

Let M be an open Riemann surface, $n \in \mathbb{N}$, let

$$\Phi_t = (\phi_{t,1}, \ldots, \phi_{t,n}), \quad t \in [0,1]$$

be a continuous family of full holomorphic 1-forms on M with values in $\mathbb{C}_*^n$, and let

$$\mathfrak{q}_t \colon H_1(M;\mathbb{Z}) \to \mathbb{C}^n, \quad t \in [0,1]$$

be a continuous family of group homomorphisms.

Then there exists a continuous family of holomorphic multipliers $h_t \colon M \to \mathbb{C}_*$ $(t \in [0,1])$ such that

$$\int_\gamma h_t \, \Phi_t = \mathfrak{q}_t(\gamma) \quad \text{for every closed curve } \gamma \subset M \text{ and } t \in [0,1].$$

If the above condition holds at $t = 0$ with the constant function $h_0 = 1$, then the homotopy $h_t \colon M \to \mathbb{C}_*$ can be chosen with $h_0 = 1$.

Comments on the main theorem

This result shows that we can arbitrarily change the period map in an isotopy of $\mathbb{C}_*^n$-valued 1-forms by a suitable isotopy of multipliers. In particular, we can always arrange that $h_1\Phi$ is exact, so it integrates to a holomorphic immersion $Z\colon M \to \mathbb{C}^n$ with $G_Z = [\Phi]$.

If $\Re\Phi$ is exact, then $\Re\Phi$ integrates to an immersion $M \to \mathbb{R}^n$ with the Gauss map Φ. Hence, we obtain

Corollary

Every conformal minimal immersion $X_0\colon M \to \mathbb{R}^n$ ($n \geq 3$) is isotopic through conformal minimal immersions $X_t\colon M \to \mathbb{R}^3$ ($t \in [0,1]$) to the real part $X_1 = \Re Z_1$ of a holomorphic null curve $Z_1\colon M \to \mathbb{C}^n$ such that $G_{X_t} = G_{X_0}$ for all $t \in [0,1]$.

All immersions in the isotopy can have the same Gauss map!
This improves an earlier theorem of Alarcón & F., Crelle 2015, in press.

Proof of the main theorem

A desired family of holomorphic multipliers $h_t \colon M \to \mathbb{C}_*$ ($t \in I = [0, 1]$) is constructed by induction with respect to an exhaustion of M by an increasing sequence of smoothly bounded Runge domains

$$D_1 \subset D_2 \subset \cdots \subset \cup_{j=1}^{\infty} D_j = M.$$

Three main ingredients are employed at every step, combining complex analysis with Gromov's convex integration theory:

- construction of multipliers on an arc (or a loop) in M that give approximately correct values of periods;

- construction of a period dominating spray of multipliers (to make arbitrary small corrections of periods);

- Mergelyan's approximation theorem — to approximate continuous families of continuous multipliers on curves in M by continuous families of holomorphic multipliers.

Lemma 1: Multipliers giving prescribed integrals

Lemma (1)

Let $I = [0, 1] \to \mathbb{C}_*$, and let

$$f \colon I^2 = I \times I \to \mathbb{C}^n, \qquad \alpha \colon I \to \mathbb{C}^n$$

be continuous maps. Assume that for every $t \in I$ the path $f_t := f(t, \cdot) \colon I \to \mathbb{C}^n$ is nowhere flat. Then there exists a continuous function $h \colon I^2 \to \mathbb{C}_*$ such that

$$h(t, s) = 1 \quad \text{for } t \in I \text{ and } s = 0, 1$$

and

$$\int_0^1 h(t, s) f(t, s) \, ds = \alpha(t), \quad t \in I.$$

If $\int_0^1 f(0, s) \, ds = \alpha(0)$, then we can take $h(0, s) = 1$ for $s \in [0, 1]$.

Proof of Lemma 1, part 1

It suffices to prove that for any $\epsilon > 0$ there exists $h\colon I^2 \to \mathbb{C}_*$ such that

$$\left| \int_0^1 h(t,s) f(t,s) \, ds - \alpha(t) \right| < \epsilon, \quad t \in I.$$

The exact result is obtained by splitting $I = I_1 \cup I_2 = [0, 1/2] \cup [1/2, 1]$, applying the approximate result with a small $\epsilon > 0$ on I_1, and a period dominating argument on I_2 (see Lemma 2) to correct the error.

Since f_t is nowhere flat and hence full for each fixed $t \in [0, 1]$, there is a division $0 = s_0 < s_1 < \cdots < s_N = 1$ of I such that

$$\operatorname{span}\{f_t(s_1), \ldots, f_t(s_N)\} = \mathbb{C}^n \quad \text{for all } t \in I.$$

For every $j = 1, \ldots, N$ we set

$$V_j(t) = \int_{s_{j-1}}^{s_j} f_t(s) \, ds \approx f_t(s_j)(s_j - s_{j-1}) \in \mathbb{C}^n.$$

By passing to a finer division, we may therefore assume that

$$\operatorname{span}\{V_1(t), \ldots, V_N(t)\} = \mathbb{C}^n, \quad t \in I.$$

For each $t \in I$, let $\Sigma_t \subset \mathbb{C}^N$ denote the affine complex hyperplane

$$(*) \qquad \Sigma_t = \left\{ (g_1, \ldots, g_N) \in \mathbb{C}^N : \sum_{j=1}^{N} g_j V_j(t) = \alpha(t) \right\}.$$

Clearly, there exists a continuous map $g = (g_1, \ldots, g_N) \colon I \to \mathbb{C}^N$ such that $g(t) \in \Sigma_t$ for every $t \in I$. (We may view g as a section of the affine bundle over $I = [0, 1]$ whose fiber over the point $t \in I$ equals Σ_t.) Hence

$$\sum_{j=1}^{N} \int_{s_{j-1}}^{s_j} g_j(t) f_t(s) \, ds = \sum_{j=1}^{N} g_j(t) V_j(t) \overset{(*)}{=} \alpha(t), \quad t \in I.$$

Note that

$$\sum_{j=1}^{N} V_j(t) = \sum_{j=1}^{N} \int_{s_{j-1}}^{s_j} f_t(s) \, ds = \int_0^1 f_t(s) \, ds.$$

Hence, if $\int_0^1 f(0, s) \, ds = \alpha(0)$ then we can take $g(0) = (1, \ldots, 1) \in \mathbb{C}^N$.

Proof of Lemma 1, part 3

By a small perturbation, we may assume that $g_j(t) \in \mathbb{C}_*$ for every $t \in I$ and all j. This changes the exact condition to the approximate condition

$$\left| \sum_{j=1}^{N} \int_{s_{j-1}}^{s_j} g_j(t) f_t(s) \, ds - \alpha(t) \right| < \frac{\epsilon}{2}, \quad t \in I.$$

View the vector $g(t) = (g_j(t))_{j=1,\ldots,N} \in \mathbb{C}^N$ for every fixed $t \in I$ as a step function of the variable $s \in I$ which equals the constant $g_j(t) \in \mathbb{C}_*$ on the j-segment $s \in [s_{j-1}, s_j]$.

Next, approximate this step function $g(t)$ by a continuous function $h_t = h(t, \cdot) \colon I \to \mathbb{C}_*$ ($t \in I$) which agrees with the step function, except very near the division points $s_1, \ldots, s_{N-1}$. This yields

$$\left| \int_0^1 h(t,s) f(t,s) \, ds - \alpha(t) \right| < \epsilon, \quad t \in I.$$

This proves Lemma 1, subject to Lemma 2.

Lemma 2: Period dominating sprays of multipliers

Lemma (2)

Let I' be a nontrivial closed subinterval of $I = [0,1]$, let Q be a compact Hausdorff parameter space (e.g. $Q = [0,1]$), and let $n \in \mathbb{N}$.

Given a continuous map $f \colon Q \times I \to \mathbb{C}^n$ such that $f(q, \cdot)$ is full on I' for every $q \in Q$, there exist continuous functions $g_1, \ldots, g_N \colon I \to \mathbb{C}$ ($N \geq n$) supported on I' such that the function $h \colon \mathbb{C}^N \times I \to \mathbb{C}$ given by

$$h(\zeta, s) = 1 + \sum_{i=1}^{N} \zeta_i g_i(s), \quad \zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N, \ s \in I$$

*is a **period dominating multiplier** of f, meaning that the map*

$$\frac{\partial}{\partial \zeta}\bigg|_{\zeta=0} \int_0^1 h(\zeta, s) f(q, s) \, ds \colon T_0 \mathbb{C}^N \cong \mathbb{C}^N \to \mathbb{C}^n$$

is surjective for every $q \in Q$.

Proof of Lemma 2

Assume that

$$h(\zeta, s) := 1 + \sum_{i=1}^{N} \zeta_i g_i(s), \quad (\zeta, s) \in \mathbb{C}^N \times I.$$

Let $\mathcal{P} \colon Q \times \mathbb{C}^N \to \mathbb{C}^n$ be the map

$$\mathcal{P}(q, \zeta) = \int_0^1 h(\zeta, s) f(q, s) \, ds, \quad (q, \zeta) \in Q \times \mathbb{C}^N.$$

Then,

$$\left. \frac{\partial \mathcal{P}(q, \zeta)}{\partial \zeta_i} \right|_{\zeta=0} = \int_0^1 \left. \frac{\partial h(\zeta, s)}{\partial \zeta_i} \right|_{\zeta=0} f(q, s) \, ds = \int_0^1 g_i(s) f(q, s) \, ds.$$

Since $f(q, \cdot)$ is full on I' for every $q \in Q$, there are distinct points $s_1, \ldots, s_N \in \mathring{I}'$ for a big $N \in \mathbb{N}$ such that

$$\mathrm{span}\{f(q, s_1), \ldots, f(q, s_N)\} = \mathbb{C}^n \quad \text{for all } q \in Q.$$

Proof of Lemma 2

Let $\epsilon > 0$ be small enough such that the intervals $[s_i - \epsilon, s_i + \epsilon]$ $(i = 1, \ldots, N)$ are pairwise disjoint and contained in I'. Let $g_i \colon I \to \mathbb{C}$ be continuous function supported on $(s_i - \epsilon, s_i + \epsilon) \subset I'$ and satisfying

$$\int_0^1 g_i(s)\, ds = \int_{s_i - \epsilon}^{s_i + \epsilon} g_i(s)\, ds = 1.$$

We then have

$$\left. \frac{\partial \mathcal{P}(q, \zeta)}{\partial \zeta_i} \right|_{\zeta = 0} = \int_0^1 g_i(s) f(q, s)\, ds \approx f(q, s_i)$$

for all $q \in Q$ and $i \in \{1, \ldots, N\}$. Therefore, if $\epsilon > 0$ is small enough,

$$\mathrm{span} \left\{ \left. \frac{\partial \mathcal{P}(q, \zeta)}{\partial \zeta_1} \right|_{\zeta = 0}, \ldots, \left. \frac{\partial \mathcal{P}(q, \zeta)}{\partial \zeta_N} \right|_{\zeta = 0} \right\} = \mathbb{C}^n \quad \text{for all } q \in Q.$$

This proves Lemma 2.

Sketch of proof of the main theorem

Fix a nowhere vanishing holomorphic 1-form θ on M.

We are given a homotopy

$$\Phi_t = (\phi_{t,1}, \ldots, \phi_{t,n}) = f_t \theta, \quad t \in [0,1],$$

where $f_t \colon M \to \mathbb{C}_*^n$ is a family of full holomorphic maps.

By using lemmas 1 and 2 and an induction over an exhaustion of M by smoothly bounded Runge domains, we construct a family of holomorphic multipliers $h_t \colon M \to \mathbb{C}_*$ such that

$$\int_\gamma h_t \, \Phi_t = \int_\gamma h_t \, f_t \theta = \mathfrak{q}_t(\gamma) \in \mathbb{C}^n$$

holds for every closed curve $\gamma \subset M$ and $t \in [0,1]$.

Spaces of conformal minimal immersions $M \to \mathbb{R}^n$

Assume that M is an open Riemann surface and $n \geq 3$ is an integer.

A conformal minimal immersion $X \colon M \to \mathbb{R}^n$ is said to be **flat** if its image $X(M)$ lies in an affine 2-plane of $\mathbb{R}^n$; otherwise it is **nonflat**. Let

$$\mathfrak{M}(M, \mathbb{R}^n)$$

denote the space of all conformal minimal immersions $M \to \mathbb{R}^n$ endowed with the compact-open topology, and let $\mathfrak{M}_*(M, \mathbb{R}^n)$ denote the open subset of $\mathfrak{M}(M, \mathbb{R}^n)$ consisting of all nonflat immersions.

Fix a nowhere vanishing holomorphic 1-form θ on M and consider the maps

$$\mathfrak{M}(M, \mathbb{R}^n) \longrightarrow \mathscr{O}(M, A_*) \hookrightarrow \mathscr{C}(M, A_*),$$

where $A_* = A_*^{n-1} \subset \mathbb{C}^n$ is the punctured null quadric.

The first map is given by $X \mapsto \partial X / \theta$, and the second map is the natural inclusion of the space of all holomorphic maps $M \to A_*$ into the space of continuous maps.

Path components of the space $\mathfrak{M}_*(M, \mathbb{R}^n)$

Since A_* is complex homogeneous and hence an Oka manifold, the inclusion $\mathscr{O}(M, A_*) \hookrightarrow \mathscr{C}(M, A_*)$ is a weak homotopy equivalence.

Finnur Lárusson and myself recently proved that the restricted map

$$\mathfrak{M}_*(M, \mathbb{R}^n) \to \mathscr{O}(M, A_*), \quad X \mapsto \partial X/\theta$$

is also a weak homotopy equivalence. If $H_1(M; \mathbb{Z})$ is finitely generated then both these maps are actually homotopy equivalences.

It follows that the path components of $\mathfrak{M}_*(M, \mathbb{R}^n)$ are in bijective correspondence with the path components of the space $\mathscr{C}(M, A_*^{n-1})$.

Since M is homotopy equivalent to a bouquet of circles, we have

$$H_1(M; \mathbb{Z}) \cong \mathbb{Z}^\ell, \quad \ell \in \mathbb{Z}_+ \cup \{\infty\},$$

$$\pi_1(A_*^2) = H_1(A_*; \mathbb{Z}) = \mathbb{Z}_2, \quad \pi_1(A_*^{n-1}) = 0 \text{ if } n > 3.$$

Thus, the path components of $\mathfrak{M}_*(M, \mathbb{R}^3)$ equal $(\mathbb{Z}_2)^\ell$, and $\mathfrak{M}_*(M, \mathbb{R}^n)$ is path connected if $n > 3$.

Path components of the space $\mathfrak{M}(M, \mathbb{R}^n)$

Theorem (Alarcón, López and F.)

Let M be an open connected Riemann surface. The natural inclusion

$$\mathfrak{M}_*(M, \mathbb{R}^n) \hookrightarrow \mathfrak{M}(M, \mathbb{R}^n)$$

induces a bijection of path components of the two spaces. In particular,

$$\pi_0(\mathfrak{M}(M, \mathbb{R}^3)) = (\mathbb{Z}_2)^\ell \quad \text{where} \quad H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell$$

and $\mathfrak{M}(M, \mathbb{R}^n)$ is path connected if $n > 3$.

The case $n > 3$ trivially follows from the following deformation result.

Theorem (Alarcón, López and F.)

Let M be a connected open Riemann surface. Given a flat conformal minimal immersion $X \colon M \to \mathbb{R}^n$ ($n \geq 3$), there exists an isotopy $X_t \colon M \to \mathbb{R}^n$ ($t \in [0,1]$) of conformal minimal immersions such that $X_0 = X$ and X_1 is nonflat.

Proof of the deformation theorem

Clearly it suffices to prove the theorem for $n = 3$. Let $X: M \to \mathbb{R}^3$ be a flat conformal minimal immersion. We may assume that $\partial X = (1, i, 0)\phi_3$ where ϕ_3 is an exact holomorphic 1-form without zeros on M.

The gist of the proof is show that there is nonconstant holomorphic function $g: M \to \mathbb{C}_*$ such that $g\phi_3$ **and** $g^2\phi_3$ **are exact** 1-**forms**. This is similar to the proof of Lemma 1. Then,

$$\Phi_t = \left(1 - t^2 g^2, i(1 + t^2 g^2), 2tg\right)\phi_3, \quad t \in \mathbb{C}$$

is an exact holomorphic 1-form and the map Φ_t/ϕ_3 assumes values in $A_* \subset \mathbb{C}^3$ for every $t \in \mathbb{C}$. Thus, Φ_t provides a conformal minimal immersion $X_t: M \to \mathbb{R}^3$ by

$$X_t(p) = X(p_0) + 2\int_{p_0}^{p} \Re(\Phi_t), \quad p \in M.$$

Since $\Phi_0 = \partial X$, we have that $X_0 = X$. Furthermore, since g is nonconstant, X_1 is nonflat, and we are done.

Path components of the space $\mathfrak{M}(M, \mathbb{R}^3)$

In dimension $n = 3$, we obtain the above theorem to the effect that

$$\pi_0(\mathfrak{M}(M, \mathbb{R}^3)) = \mathbb{Z}_2^\ell$$

by using the corresponding result for nonflat immersions,

$$\pi_0(\mathfrak{M}_*(M, \mathbb{R}^3)) = \mathbb{Z}_2^\ell,$$

along with the deformation theorem and the following result.

Theorem

Let M be a connected open Riemann surface and let θ be a nowhere vanishing holomorphic 1-form on M. For every group homomorphism

$$\mathfrak{p} \colon H_1(M; \mathbb{Z}) \cong \mathbb{Z}^\ell \to \mathbb{Z}_2$$

*there exists a **flat conformal minimal immersion** $X \colon M \to \mathbb{R}^3$ satisfying $H_1(\partial X/\theta) = \mathfrak{p}$.*

The Gauss map can omit two points

We also prove the following result concerning isotopies of conformal minimal immersions into $\mathbb{R}^3$.

Theorem

Given an open Riemann surface M and a conformal minimal immersion $X \colon M \to \mathbb{R}^3$, there exists an isotopy

$$X_t \colon M \to \mathbb{R}^3, \quad t \in [0, 1]$$

of conformal minimal immersions such that $X_0 = X$ and the complex Gauss map of X_1 is nonconstant and avoids any two given points of the Riemann sphere.

There also exists an isotopy X_t as above such that $X_0 = X$ and X_1 is flat.

If M is covered by $\mathbb{C}$, then the Gauss map cannot omit three point of $\mathbb{CP}^1$ by Picard's theorem, unless it is constant and the immersion is flat.

THANK YOU FOR YOUR ATTENTION