Schwarz–Pick lemma for harmonic maps which are conformal at a point

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Abstract

- We shall prove a sharp estimate on the norm of the differential of a harmonic map from the unit disc $\mathbb{D}$ in $\mathbb{C}$ to the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, $n \geq 2$, at any point where the map is conformal.
- In dimension $n = 2$, this generalizes the classical Schwarz–Pick lemma to harmonic maps $\mathbb{D} \to \mathbb{D}$ which are conformal (only) at the reference point.
- In dimensions $n \geq 3$ it gives the optimal Schwarz–Pick lemma for conformal minimal discs $\mathbb{D} \to \mathbb{B}^n$.
- We shall then give a differential-geometric interpretation, showing that every conformal harmonic immersion $M \to \mathbb{B}^n$ from a hyperbolic conformal surface is distance-decreasing in the Poincaré metric on M and the Cayley–Klein metric $\mathcal{CK}$ on the ball $\mathbb{B}^n$, and the extremal maps are the conformal embeddings of the disc $\mathbb{D}$ onto affine discs in $\mathbb{B}^n$.
- Using these results, we lay foundations of the hyperbolicity theory for domains in $\mathbb{R}^n$ based on minimal surfaces.

F.F. & D.Kalaj, Hyperbolicity theory for conformal minimal surfaces in $\mathbb{R}^n$.
https://arxiv.org/abs/2102.12403

The classical Schwarz–Pick Lemma

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc.

The following result is due to **Karl Hermann Amandus Schwarz, 1869**, with an improvement by **Georg Alexander Pick, 1915**.

Theorem (Schwarz–Pick lemma for holomorphic maps)

If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map then for every $z \in \mathbb{D}$ we have that

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \tag{1}$$

with equality at one point if and only if f is a biholomorphism of the disc $\mathbb{D}$.

By using precompositions and postcompositions by holomorphic automorphisms of $\mathbb{D}$, the proof reduces to the case $z = 0$ and $f(0) = 0$. In this special case, it follows from the maximum principle applied to the holomorphic function $g(z) = f(z)/z$ on $\mathbb{D}$.

This is the most fundamental rigidity result in complex analysis which leads to the theory of **Kobayashi hyperbolic manifolds.**

Differential-theoretic interpretation

Let $\mathcal{P}$ denote the **Poincaré metric** on the disc $\mathbb{D} = \{|z| < 1\}$:

$$\mathcal{P}(z,\xi) = \frac{|\xi|}{1-|z|^2}, \qquad z \in \mathbb{D},\ \xi \in T_z\mathbb{C} \cong \mathbb{C}.$$

Then, the Schwarz–Pick lemma is equivalent to the statement that for any holomorphic map $f : \mathbb{D} \to \mathbb{D}$ we have

$$\mathcal{P}(f(z), df_z(\xi)) \leq \mathcal{P}(z,\xi), \quad z \in \mathbb{D},\ \xi \in \mathbb{C},$$

with equality at one point if and only if f is an automorphism of $\mathbb{D}$,

$$f(z) = e^{\mathrm{i}t}\frac{z+a}{1+\bar{a}z}, \qquad z, a \in \mathbb{D},\ t \in \mathbb{R}.$$

That is, **holomorphic maps $\mathbb{D} \to \mathbb{D}$ are distance-decreasing in the Poincaré metric, and orientation-preserving isometries are precisely the elements of $\mathrm{Aut}(\mathbb{D})$.** The analogus conclusion holds for the Poincaré distance function

$$\mathrm{dist}_{\mathcal{P}}(z,w) = \frac{1}{2}\log\left(\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}\right), \quad z, w \in \mathbb{D}.$$

Schwarz–Pick Lemma for harmonic maps which are conformal at a point

The following special case of our main result gives the same conclusion at a given point for a much bigger class of maps.

Corollary

Let $f : \mathbb{D} \to \mathbb{D}$ be a harmonic map. If f is conformal at a point $z \in \mathbb{D}$, then at this point we have that

$$|f'(z)| = \|df_z\| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \tag{2}$$

with equality if and only if f is a conformal diffeomorphism of the disc $\mathbb{D}$.

By using precompositions by holomorphic automorphisms of $\mathbb{D}$, the proof reduces to the case $z = 0$. On the other hand, postcompositions of harmonic maps $\mathbb{D} \to \mathbb{D}$ by holomorphic automorphism of $\mathbb{D}$ need not be harmonic, so we cannot exchange $f(0)$ and 0. Also, $f(z)/z$ need not be harmonic. The standard proof of the classical Schwarz–Pick lemma breaks down at this point.

Without conformality, this fails for some harmonic diffeomorphisms of $\mathbb{D}$.

Schwarz-Pick lemma for harmonic maps into balls

Theorem (1)

Let $f : \mathbb{D} \to \mathbb{B}^n$ is a harmonic map for some $n \geq 2$ which is conformal at a point $z \in \mathbb{D}$. Denote by $\theta \in [0, \pi/2]$ the angle between the vector $f(z)$ and the plane $df_z(\mathbb{R}^2)$. Then at this point we have that

$$\|df_z\| \ \leq \ \frac{1 - |f(z)|^2}{1 - |z|^2} \frac{1}{\sqrt{1 - |f(z)|^2 \sin^2 \theta}},$$

with equality if and only if f is a conformal diffeomorphism onto the affine disc $\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n$.

This is a **precise Schwarz–Pick lemma for conformal harmonic maps** $\mathbb{D} \to \mathbb{B}^n$ for any $n \geq 2$, saying that the extremal maps are precisely the conformal parameterizations of affine linear discs in the ball.

In dimension $n = 2$ we have $\theta = 0$, so the previous corollary is a special case.

An estimate without conformality

Note that for a fixed value of $|f(z)| \in [0,1)$, the maximum of the right hand side over angles $\theta \in [0, \pi/2]$ equals $\frac{\sqrt{1-|f(z)|^2}}{1-|z|^2}$ and is reached precisely at $\theta = \pi/2$, i.e, when the vector $f(z)$ is orthogonal to $\Lambda = df_z(\mathbb{R}^2)$, unless $f(z) = 0$ in which case it equals $\frac{1}{1-|z|^2}$ for all θ. We show that this weaker estimate holds for all harmonic maps $\mathbb{D} \to \mathbb{B}^n$.

Theorem (2)

For every harmonic map $f : \mathbb{D} \to \mathbb{B}^n$ $(n \geq 2)$ we have that

$$\frac{1}{\sqrt{2}}|\nabla f(z)| \ \leq \ \frac{\sqrt{1-|f(z)|^2}}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{3}$$

Equality holds for some $z_0 \in \mathbb{D}$ if $f(z_0)$ is orthogonal to the 2-plane $\Lambda = df_{z_0}(\mathbb{R}^2)$ and f is a conformal diffeomorphism onto the affine disc $(f(z_0) + \Lambda) \cap \mathbb{B}^n$.
In particular, if $f(z_0) = 0$ then $|\nabla f(z_0)| \leq \frac{\sqrt{2}}{1-|z_0|^2}$, with equality if and only if f is a conformal diffeomorphism onto the linear disc $\Lambda \cap \mathbb{B}^n$.

Quantitative Calabi–Yau problem

Theorem 2 implies a bound on the area of the image of f. It is classical that for a $\mathscr{C}^1$ map $f : D \to \mathbb{R}^n$ $(n \geq 2)$ from a domain $D \subset \mathbb{R}^2$ we have

$$\mathrm{Area}\, f(D) \;\leq\; \frac{1}{2}\int_D |\nabla f|^2 dxdy,$$

with equality if and only if f is conformal. Hence, if $f : \mathbb{D} \to \mathbb{B}^n$ is a harmonic map then Theorem 2 implies for every $0 < r < 1$ that

$$\mathrm{Area}\, f(r\mathbb{D}) \;\leq\; \int_{|z|<r} \frac{1-|f(z)|^2}{(1-|z|^2)^2}\, dxdy \;\leq\; \frac{\pi r^2}{1-r^2}, \qquad z = x + \mathrm{i}y.$$

The second inequality is obtained by deleting the term $-|f(z)|^2$ in the numerator, so it may be far from optimal. However, assuming that $|f| \leq c$ for some $0 < c < 1$, it is optimal up to a multiplicative constant.

Problem (Quantitative Calabi–Yau problem for minimal discs)

Are there complete minimal discs in the ball achieving the asymptotic rate of growth of the area in the above inequality up to a multiplicative constant?

Discussion

The precise upper bound on the size of the gradient of harmonic maps $f : \mathbb{D} \to \mathbb{B}^n$ with a given centre $f(0) = \mathbf{x} \in \mathbb{B}^n \setminus \{0\}$ is not known, except for $n = 1$. The **harmonic Schwarz lemma** says that any harmonic function $f : \mathbb{B}^m \to (-1, +1)$ for $m \geq 2$ satisfies the estimate

$$|\nabla f(0)| \;\leq\; 2\frac{\mathrm{Vol}(\mathbb{B}^{m-1})}{\mathrm{Vol}(\mathbb{B}^m)},$$

with equality if and only if $f = U \circ R$ where R is an orthogonal rotation on $\mathbb{R}^m$ and U is the harmonic function on $\mathbb{B}^m$ whose boundary values equal $+1$ on the upper hemisphere S^+ and -1 on the lower hemisphere S^-. In this case, $f(0) = 0$. For $m = 2$ the inequality reads $|\nabla f(0)| \;\leq\; \frac{4}{\pi}$. A simple proof was given by Kalaj and Vuorinen (2012) who showed more generally that any harmonic function $f : \mathbb{D} \to (-1, +1)$ satisfies the sharp estimate

$$|\nabla f(z)| \leq \frac{4}{\pi}\frac{1 - |f(z)|^2}{1 - |z|^2} \quad \text{for every } z \in \mathbb{D}.$$

This follows from the classical Schwarz–Pick lemma applied to the holomorphic function $\phi \circ F : \mathbb{D} \to \mathbb{D}$, where $F = f + ig : \mathbb{D} \to \Omega = (-1, +1) \times i\mathbb{R}$ is a holomorphic extension of f and $\phi : \Omega \to \mathbb{D}$ is a biholomorphism.

Proof of the main theorem, 1

It suffices to prove Theorem 1 for $z = 0$. Indeed, with f and z as in the theorem, let $\phi_z \in \mathrm{Aut}(\mathbb{D})$ be such that $\phi_z(0) = z$. The harmonic map $g = f \circ \phi_z : \mathbb{D} \to \mathbb{B}^n$ is then conformal at the origin. Since $|\phi_z'(0)| = 1 - |z|^2$, the estimate follows from the same estimate for the map g applied at $z = 0$.

We now give an explicit conformal parameterization of affine discs in $\mathbb{B}^n$. We may use postcomposition of maps $\mathbb{D} \to \mathbb{B}^n$ by orthogonal rotations of $\mathbb{R}^n$. Fix a point $\mathbf{q} \in \mathbb{B}^n$ and a 2-plane $0 \in \Lambda \subset \mathbb{R}^n$, and consider the affine disc $\Sigma = (\mathbf{q} + \Lambda) \cap \mathbb{B}^n$. Let $\mathbf{p} \in \Sigma$ be the closest point to the origin.

If $n = 2$ then $\mathbf{p} = 0$ and $\Sigma = \mathbb{D}$. Suppose now that $n = 3$; the case $n > 3$ will be the same. By an orthogonal rotation we may assume that

$$\mathbf{p} = (0, 0, p) \quad \text{and} \quad \Sigma = \left\{ (x, y, p) : x^2 + y^2 < 1 - p^2 \right\}.$$

Let $\mathbf{q} = (b_1, b_2, p) \in \Sigma$, and let θ denote the angle between $\mathbf{q}$ and Σ. Set

$$c = \sqrt{1 - p^2} = \sqrt{1 - |\mathbf{q}|^2 \sin^2 \theta}, \qquad a = \frac{b_1 + \imath b_2}{c} \in \mathbb{D}, \qquad |a| = \frac{|\mathbf{q}| \cos \theta}{c}.$$

We orient Σ by the tangent vectors ∂_x, ∂_y.

Proof, 2

Every orientation preserving conformal parameterization $f : \mathbb{D} \to \Sigma$ with $f(0) = \mathbf{q}$ is then of the form

$$f(z) = \left(c\,\Re \frac{e^{it}z + a}{1 + \bar{a}e^{it}z}, c\,\Im \frac{e^{it}z + a}{1 + \bar{a}e^{it}z}, p \right) = \left(c\,\frac{e^{it}z + a}{1 + \bar{a}e^{it}z}, p \right)$$

for $z \in \mathbb{D}$ and some $t \in \mathbb{R}$. If $n = 2$ then $p = 0$, $c = 1$, and we drop the last coordinate. We have

$$\begin{aligned} \|df_0\| &= c\,(1 - |a|^2) = \frac{c^2 - c^2|a|^2}{c} = \frac{1 - |\mathbf{q}|^2 \sin^2\theta - |\mathbf{q}|^2 \cos^2\theta}{c} \\ &= \frac{1 - |\mathbf{q}|^2}{\sqrt{1 - |\mathbf{q}|^2 \sin^2\theta}} = \frac{1 - |f(0)|^2}{\sqrt{1 - |f(0)|^2 \sin^2\theta}}. \end{aligned}$$

This shows that the conformal parameterizations of affine discs satisfy the equality in the theorem at every point.

Proof, 3

Let $f : \mathbb{D} \to \mathbb{B}^3$ be as above, where we may assume that $t = 0$.

Suppose that $g : \mathbb{D} \to \mathbb{B}^3$ is a harmonic map such that $g(0) = f(0)$, g is conformal at 0, and $dg_0(\mathbb{R}^2) = df_0(\mathbb{R}^2)$. Up to replacing g by $g(e^{\mathrm{i}t}z)$ or $g(e^{\mathrm{i}t}\bar{z})$ for some $t \in \mathbb{R}$, we may assume that

$$dg_0 = r\, df_0 \quad \text{for some } r > 0.$$

We must prove that $r \le 1$, and that $r = 1$ if and only if $g = f$.

Consider the holomorphic map $F : \mathbb{D} \to \Omega = \mathbb{B}^3 \times \mathrm{i}\mathbb{R}^3$ given by

$$F(z) = \left(c\, \frac{z+a}{1+\bar{a}z}, -c\,\mathrm{i}\frac{z+a}{1+\bar{a}z}, p \right), \quad z \in \mathbb{D}.$$

Then, $f = \Re F$. Let $G : \mathbb{D} \to \Omega$ be the holomorphic map with $\Re G = g$ and $G(0) = F(0)$. By the Cauchy–Riemann equations, condition $dg_0 = r\, df_0$ implies

$$G'(0) = r\, F'(0).$$

Proof, 4

It follows that the map $(F(z) - G(z))/z$ is holomorphic on $\mathbb{D}$ and

$$\lim_{z \to 0} \frac{F(z) - G(z)}{z} = F'(0) - G'(0) = (1 - r)F'(0).$$

Since $g : \mathbb{D} \to \mathbb{B}^3$ is a bounded harmonic map, it has a nontangential boundary value at almost every point of the circle $\mathbb{T} = b\mathbb{D}$. Since the Hilbert transform is an isometry on the Hilbert space $L^2(\mathbb{T})$, the same is true for G.

Denote by $\langle \cdot, \cdot \rangle$ the complex bilinear form on $\mathbb{C}^n$ given by

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i$$

for $z, w \in \mathbb{C}^n$. Note that on vectors in $\mathbb{R}^n$ this is the Euclidean inner product.

Proof, 5

For each $z = e^{\mathrm{i}t} \in b\mathbb{D}$ the vector $f(z) \in b\mathbb{B}^3$ is the unit normal vector to the sphere $b\mathbb{B}^3$ at the point $f(z)$. Since $\mathbb{B}^3$ is strongly convex, we have that

$$\Re \left\langle F(z) - G(z), f(z) \right\rangle = \left\langle f(z) - g(z), f(z) \right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D},$$

and the value is positive for almost every $z \in b\mathbb{D}$ if and only if $g \neq f$.

Consider the function $\tilde{f}$ on the circle $b\mathbb{D}$ given by

$$\tilde{f}(z) = z|1 + \bar{a}z|^2 f(z), \qquad |z| = 1.$$

Explicit calculation, taking into account $z\bar{z} = 1$, shows that

$$\tilde{f}(z) = \begin{pmatrix} \frac{c}{2}\left(1 + a^2 + 4(\Re a)z + (1 + \bar{a}^2)z^2\right) \\ \frac{c}{2}\left(\mathrm{i}(1 - a^2) + 4(\Im a)z + \mathrm{i}(\bar{a}^2 - 1)z^2\right) \\ p\,(z + a)(1 + \bar{a}z) \end{pmatrix}.$$

Conclusion of the proof

We extend $\tilde{f}$ to all $z \in \mathbb{C}$ by letting it equal the holomorphic polynomial map on the right hand side above. Since $|1+\bar{a}z|^2 > 0$ for $z \in \overline{\mathbb{D}}$, we have

$$\begin{aligned} h(z) &:= \Re \left\langle F(z) - G(z), |1+\bar{a}z|^2 f(z) \right\rangle \\ &= \left\langle f(z) - g(z), |1+\bar{a}z|^2 f(z) \right\rangle \geq 0 \quad a.e.\ z \in b\mathbb{D}, \end{aligned}$$

and $h > 0$ almost everywhere on $b\mathbb{D}$ if and only if $g \neq f$. From the definition of $\tilde{f}$ we see that

$$h(z) = \Re \left\langle \frac{F(z) - G(z)}{z}, \tilde{f}(z) \right\rangle \quad a.e.\ z \in b\mathbb{D}$$

Since the maps $(F(z) - G(z))/z$ and $\tilde{f}(z)$ are holomorphic on $\mathbb{D}$, h extends to a nonnegative harmonic function on $\mathbb{D}$ which is positive on $\mathbb{D}$ unless $f = g$. At $z = 0$ we have

$$h(0) = \Re \left\langle F'(0) - G'(0), \tilde{f}(0) \right\rangle = (1-r) \Re \left\langle F'(0), \tilde{f}(0) \right\rangle \geq 0,$$

with equality if and only if $f = g$. Applying this argument to the linear map $g(z) = f(0) + r\, df_0(z)$ $(z \in \mathbb{D})$ for a small $r > 0$ we get $\Re \left\langle F'(0), \tilde{f}(0) \right\rangle > 0$. It follows that $r \leq 1$, with equality if and only if $g = f$.

Discussion

The above proof is motivated by the seminal work of **Lempert (1981)** on Kobayashi extremal holomorphic discs in bounded strongly convex domains $\Omega \subset \mathbb{C}^n$ with smooth boundaries.

In Lempert's terminology, a proper holomorphic disc $F : \mathbb{D} \to \Omega$ extending continuously to $\overline{\mathbb{D}}$ is a **stationary disc** if, denoting by $\nu(z)$ the unit normal to $b\Omega$ along the boundary circle $F(b\mathbb{D})$, there is a positive function $q > 0$ on $b\mathbb{D}$ such that the function $z\, q(z)\overline{\nu(z)}$ extends from the circle $|z| = 1$ to a holomorphic function $\tilde{f}(z)$ on $\mathbb{D}$. The use of such a function, along with the convexity of the domain, enables the arguments used above to show that a stationary disc F is the unique Kobayashi extremal disc in Ω through the point $F(a)$ in the tangent direction $F'(a)$ for every $a \in \mathbb{D}$.

In our case $\nu(z) = f(z)$, which is real-valued, and a suitable function $\tilde{f}$ is

$$\tilde{f}(z) = z|1 + \bar{a}z|^2 f(z), \qquad |z| = 1.$$

The fact that $\Omega = \mathbb{B}^n \times i\mathbb{R}^n$ is an unbounded tube does not matter since the affine conformal discs in $\mathbb{B}^n$ lift to proper holomorphic discs in Ω without any boundary points at infinity.

The Cayley–Klein metric on the ball

We can interpret Theorem 1 as the distance-decreasing property of conformal harmonic maps $\mathbb{D} \to \mathbb{B}^n$ with respect to the following Riemannian metric $\mathcal{CK}$ on $\mathbb{B}^n$, called the **Cayley–Klein metric**:

$$\mathcal{CK}(\mathbf{x}, \mathbf{v}) = \frac{\sqrt{1 - |\mathbf{x}|^2 \sin^2 \phi}}{1 - |\mathbf{x}|^2} |\mathbf{v}|, \qquad \mathbf{x} \in \mathbb{B}^n, \ \mathbf{v} \in \mathbb{R}^n,$$

where $\phi \in [0, \pi/2]$ is the angle between the vector $\mathbf{x}$ and the line $\mathbb{R}\mathbf{v} \subset \mathbb{R}^n$. Equivalently,

$$\mathcal{CK}(\mathbf{x}, \mathbf{v})^2 = \frac{(1 - |\mathbf{x}|^2)|\mathbf{v}|^2 + |\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2} = \frac{|\mathbf{v}|^2}{1 - |\mathbf{x}|^2} + \frac{|\mathbf{x} \cdot \mathbf{v}|^2}{(1 - |\mathbf{x}|^2)^2}.$$

The **Cayley–Klein model**, also called the **Beltrami–Klein model** of hyperbolic geometry was introduced by Arthur Cayley (1859) and Eugenio Beltrami (1968), and it was developed by Felix Klein (1871, 1873). The underlying space is the n-dimensional unit ball, and geodesics are straight line segments with ideal endpoints on the boundary sphere. This is a special case of the Hilbert metric on convex domains in $\mathbb{R}^n$, introduced by David Hilbert in 1895.

Comments on the Cayley–Klein metric

The Cayley–Klein metric $\mathcal{CK}$ is the restriction of the Kobayashi metric on the unit ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ to points $\mathbf{x} \in \mathbb{B}^n = \mathbb{B}^n_{\mathbb{C}} \cap \mathbb{R}^n$ and tangent vectors in $T_{\mathbf{x}}\mathbb{R}^n \cong \mathbb{R}^n$.

It also equals $1/\sqrt{n+1}$ times the Bergman metric on $\mathbb{B}^n_{\mathbb{C}}$ restricted to $\mathbb{B}^n$ and real tangent vectors. (On the ball of $\mathbb{C}^n$, most holomorphically invariant metrics coincide up to scalar factors.)

The metric $\mathcal{CK}$ is not conformally equivalent to the Euclidean metric on $\mathbb{B}^n$. It coincide with the Poincaré metric on $\mathbb{B}^n$, given by $\frac{|\mathbf{v}|}{1-|\mathbf{x}|^2}$, in the radial direction parallel to the base point $\mathbf{x} \in \mathbb{B}^n$, but is strictly smaller in the direction perpendicular to $\mathbf{x}$. We have that

$$\frac{|\mathbf{v}|}{\sqrt{1-|\mathbf{x}|^2}} \;\le\; \mathcal{CK}(\mathbf{x},\mathbf{v}) \;\le\; \frac{|\mathbf{v}|}{1-|\mathbf{x}|^2},$$

with the upper bound reached for $\phi = 0$ and the lower bound for $\phi = \pi/2$.

Comparison with a Finsler metric

The inequality in Theorem 1 can be rewritten as

$$\frac{\sqrt{1-|f(z)|^2\sin^2\theta}}{1-|f(z)|^2}\,|df_z(\xi)| \;\le\; \frac{|\xi|}{1-|z|^2}, \qquad \xi \in T_z\mathbb{D} = \mathbb{R}^2,$$

where $\theta \in [0, \pi/2]$ is the angle between $f(z)$ and the 2-plane $\Lambda = df_z(\mathbb{R}^2)$.

Let $G_2(\mathbb{R}^n)$ denote the Grassmann manifold of all 2-planes in $\mathbb{R}^n$, and define

$$\mathcal{M}(\mathbf{x}, \Lambda) \;=\; \frac{\sqrt{1-|\mathbf{x}|^2\sin^2\theta}}{1-|\mathbf{x}|^2}, \qquad \mathbf{x} \in \mathbb{B}^n,\ \Lambda \in G_2(\mathbb{R}^n),$$

where $\theta \in [0, \pi/2]$ is the angle between $\mathbf{x}$ and Λ. Note that

$$\mathcal{M}(\mathbf{x}, \Lambda) = \inf\{1/\|df_0\| : f \in \mathrm{CH}(\mathbb{D}, \mathbb{B}^n),\ f(0) = \mathbf{x},\ df_0(\mathbb{R}^2) = \Lambda\}.$$

Comparison with a Finsler metric

If $\mathbf{v} \neq \mathbf{0}$ is a vector having angle $\phi \in [0, \pi/2]$ with the line $\mathbb{R}\mathbf{x}$, then every 2-plane Λ containing $\mathbf{v}$ makes an angle $\theta \in [0, \phi]$ with $\mathbf{x}$, and the maximum of θ over all such Λ equals ϕ. Hence,

$$\begin{aligned} \mathcal{CK}(\mathbf{x}, \mathbf{v})/|\mathbf{v}| &= \min\{\mathcal{M}(\mathbf{x}, \Lambda) : \Lambda \in G_2(\mathbb{R}^n),\ \mathbf{0} \neq \mathbf{v} \in \Lambda\}, \\ \mathcal{M}(\mathbf{x}, \Lambda) &= \max\{\mathcal{CK}(\mathbf{x}, \mathbf{v})/|\mathbf{v}| : \mathbf{v} \in \Lambda\} \ \text{ for all } \Lambda \in G_1(\mathbb{R}^n). \end{aligned}$$

Applying this with $\mathbf{x} = f(z)$ and $\mathbf{v} = df_z(\xi) \in \Lambda = df_z(\mathbb{R}^2)$ gives

$$\begin{aligned} \mathcal{CK}\big(f(z), df_z(\xi)\big) &\leq \mathcal{M}(f(z), df_z(\mathbb{R}^2)) \cdot |df_z(\xi)| \\ &= \frac{\sqrt{1 - |f(z)|^2 \sin^2 \theta}}{1 - |f(z)|^2} |df_z(\xi)| \\ &\leq \frac{|\xi|}{1 - |z|^2} = \mathcal{P}_{\mathbb{D}}(z, \xi). \end{aligned}$$

The first inequality is equality if and only if the angle ϕ between the line $f(z)\mathbb{R}$ and the vector $df_z(\xi) \in \Lambda$ equals θ.

The second inequality is equality of and only if f is a conformal diffeomorphism onto the linear disc $(f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n$.

Conformal harmonic maps are distance-decreasing

Corollary

If $f : \mathbb{D} \to \mathbb{B}^n$ *is a conformal harmonic map then*

$$\mathcal{CK}\big(f(z), df_z(\xi)\big) \le \frac{|\xi|}{1-|z|^2} = \mathcal{P}_{\mathbb{D}}(z,\xi), \qquad z \in \mathbb{D},\ \xi \in \mathbb{R}^2, \tag{4}$$

with equality for some $z \in \mathbb{D}$ *and* $\xi \in \mathbb{R}^2 \setminus \{0\}$ *if and only if* f *is a conformal diffeomorphism onto the affine disc* $\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n$ *and the vector* $df_z(\xi)$ *is tangent to the diameter of* Σ *through the point* $f(z)$.

The analogous conclusion holds if $\mathbb{D}$ *is replaced by any hyperbolic conformal surface* M *with the Poincaré metric* $\mathcal{P}_M$. *Equality can only occur if* $M = \mathbb{D}$.

A hyperbolic conformal surface is one whose universal conformal covering is the disc. One introduces the Poincaré metric on such surface M by asking that the universal covering projection $h : \mathbb{D} \to M$ be a local isometry. The result is trivial unless M is of hyperbolic type, i.e., it admits bounded harmonic functions.

A pseudodistance on a domain in $\mathbb{R}^n$

There is a natural procedure to define a pseudodistance function $\rho = \rho_D$ on any domain $D \subset \mathbb{R}^n$ using conformal minimal discs $\mathbb{D} \to D$. It is motivated by Kobayashi's construction of his pseudometric on complex manifolds.

Fix a pair of points $\mathbf{x}, \mathbf{y} \in D$ and consider finite chains of conformal harmonic discs $f_i : \mathbb{D} \to D$ and points $a_i \in \mathbb{D}$ $(i = 1, \ldots, k)$ such that

$$f_1(0) = \mathbf{x}, \quad f_{i+1}(0) = f_i(a_i) \text{ for } i = 1, \ldots, k-1, \quad f_k(a_k) = \mathbf{y}.$$

To any such chain we associate the number

$$\sum_{i=1}^{k} \frac{1}{2} \log \frac{1 + |a_i|}{1 - |a_i|} \geq 0.$$

The i-th term in the sum is the Poincaré distance from 0 to a_i in $\mathbb{D}$.

The pseudodistance $\rho_D : D \times D \to \mathbb{R}_+$ is the infimum of the numbers obtained in this way. Clearly it satisfies the triangle inequality.

If D is a domain in $\mathbb{C}^n$ and we use only holomorphic discs, then the corresponding pseudodistance ρ is precisely the one of Kobayashi.

Distance-decreasing property

Lemma

(A) Conformal harmonic maps $M \to D$ from any hyperbolic conformal surface are distance-decreasing in the Poincaré distance function on M and the distance ρ_D on D.

(B) ρ_D is the largest pseudodistance function on D for which this holds.

Proof of (A) For $M = \mathbb{D}$, this follows from the definition since every conformal harmonic map $f : \mathbb{D} \to D$ is a candidate for computing ρ_D and we are taking the infimum. For general M, the result follows by precomposing f by a universal conformal covering $h : \mathbb{D} \to M$.

Proof of (B) Suppose that τ is a pseudodistance on D such that every conformal harmonic map $\mathbb{D} \to D$ is distance-decreasing. Let $f_i : \mathbb{D} \to D$ and $a_i \in \mathbb{D}$ for $i = 1, \ldots, k$ be a chain connecting the points $\mathbf{x}, \mathbf{y} \in D$. Then,

$$\tau(\mathbf{x}, \mathbf{y}) \;\le\; \sum_{i=1}^{k} \tau(f_i(0), f_i(a_i)) \;\le\; \sum_{i=1}^{k} \frac{1}{2} \log \frac{1 + |a_i|}{1 - |a_i|}.$$

Taking the infimum over all such chains gives $\tau(\mathbf{x}, \mathbf{y}) \le \rho_D(\mathbf{x}, \mathbf{y})$.

$\rho_{\mathbb{B}^n} = \mathrm{dist}_{\mathcal{CK}}$

Theorem

On the ball $\mathbb{B}^n$, we have $\rho_{\mathbb{B}^n} = \mathrm{dist}_{\mathcal{CK}}$.

Proof Fix a pair of distinct points $\mathbf{x}, \mathbf{y} \in \mathbb{B}^n$. Let $\mathbf{p}$ be the point on the affine line L through $\mathbf{x}$ and $\mathbf{y}$ which is closest to the origin.

Let $\Lambda \subset \mathbb{R}^n$ be the affine 2-plane containing L and such that $\mathbf{p}$ is orthogonal to Λ (such Λ is unique unless $\mathbf{p} = \mathbf{0}$). Then, $\Sigma := \Lambda \cap \mathbb{B}^n$ is an affine disc, and the points $\mathbf{x}$ and $\mathbf{y}$ lie on the diameter $L \cap \mathbb{B}^n$ of Σ.

These diameters are geodesics (length minimizers) for the Cayley–Klein metric on $\mathbb{B}^n$, and $\mathrm{dist}_{\mathcal{CK}}(\mathbf{x}, \mathbf{y})$ equals the Poincaré distance between $\mathbf{x}$ and $\mathbf{y}$ within the affine disc Σ.

By the previous lemma applied with $\tau = \mathrm{dist}_{\mathcal{CK}}$, we have $\mathrm{dist}_{\mathcal{CK}}(\mathbf{x}, \mathbf{y}) \leq \rho_{\mathbb{B}^n}(\mathbf{x}, \mathbf{y})$. Since the affine disc Σ is a candidate for computing $\rho_{\mathbb{B}^n}(\mathbf{x}, \mathbf{y})$, equality follows.

Hyperbolic domains

Definition (Hyperbolic domains in $\mathbb{R}^n$)

A domain $D \subset \mathbb{R}^n$ $(n \geq 3)$ is *hyperbolic* if the pseudodistance ρ_D is a distance function on D, and is *complete hyperbolic* if (D, ρ_D) is a complete metric space (i.e., Cauchy sequences converge).

Example

(A) The ball $\mathbb{B}^n \subset \mathbb{R}^n$ $(n \geq 3)$ is complete hyperbolic since the Cayley–Klein metric is complete.

(B) Every bounded domain $D \subset \mathbb{R}^n$ is hyperbolic since it is contained in a ball. However, it need not be complete hyperbolic.

(C) Every bounded strongly convex domain in $\mathbb{R}^n$ is complete hyperbolic.

(D) The half-space $\mathbb{H}^n = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ is not hyperbolic since the pseudodistance $\rho_{\mathbb{H}^n}$ vanishes on planes $x_n = const$.

Problems

Problem

(A) Is the complement of a catenoid in $\mathbb{R}^3$ *hyperbolic?*

(B) Is every bounded strongly mean-convex domain in $\mathbb{R}^3$ *complete hyperbolic?*

$\sim$ **Thank you for your attention** $\sim$

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