Gluing constructions for complete minimal surfaces with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$

(joint work with R. Mazzeo and M. Rodríguez)

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- The first is based on the **method of Anderson for the analogous problem in $\mathbb{H}^3$**.
- The second approach generalizes the classical **method of Jenkins and Serrin** for minimal graphs in $\mathbb{R}^3$. This involves finding a minimal graph over domains of $\mathbb{H}^2$ with prescribed boundary data, possibly $\pm\infty$.
- The third approach is by an **analytic gluing construction**, and this is the method we follow here.

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Horizontal catenoids K_η.

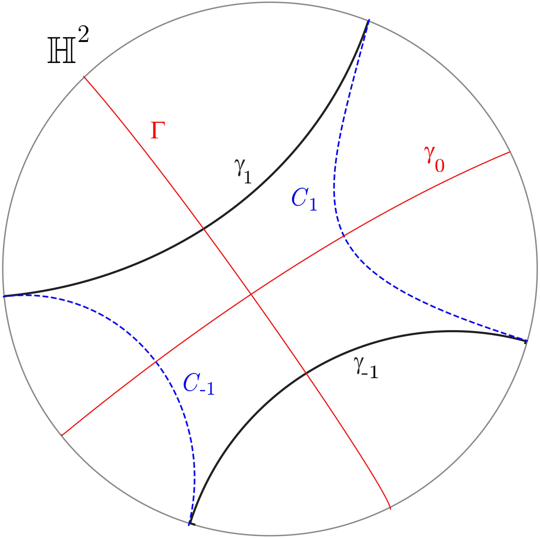

Figure: The boundary of the region Ω_η.

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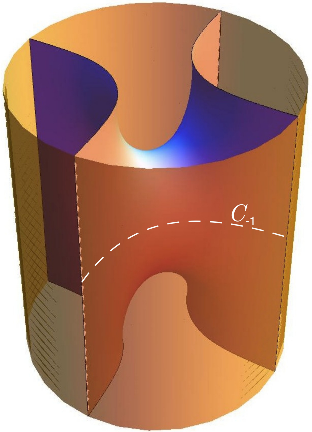

Figure: A horizontal catenoid K_η.

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3. Hauswirth-Nelli-Sa Earp-Toubiana: K_η are the unique complete minimal surfaces with finite total curvature and two ends.
4. $\mathrm{Isom}(K_\eta) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Main Results

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- The horizontal catenoid components have **"necksize" bounded away from zero** are simpler to handle; the minimal surfaces obtained using only this type of component have **a very large number of ends relative to the genus.**
- Alternatively, one may glue together horizontal catenoids with **very small necksizes**, which allows one to obtain viable configurations with relatively few ends for a given genus.

Main Results

Theorem A (—, Mazzeo, Rodríguez)

For each $g \geq 0$, there is a $k_0 = k_0(g)$ such that if $k \geq k_0$, then there exists a properly embedded minimal surface with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$, with genus g and k ends, **each asymptotic to a vertical plane**.

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Theorem (Meeks, Pérez, Ros)

There is **an upper bound, depending only on the genus,** *for the number of ends of a properly embedded minimal surface of finite topology in* $\mathbb{R}^3$.

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Every minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with each end asymptotic to a vertical plane is degenerate since vertical translation (i.e. in the $\mathbb{R}$ direction) always generates such a Jacobi field.

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Theorem B (—, Mazzeo, Rodríguez)

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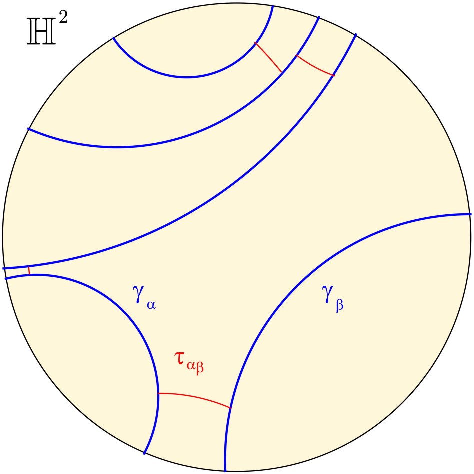
Theorem C (—, Mazzeo, Rodríguez)

Let $\mathcal{M}_k$ denote the space of all complete, properly embedded minimal surfaces with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ with each end asymptotic to an entire vertical plane. If $\Sigma \in \mathcal{M}_k$ is horizontally nondegenerate, then the component of this moduli space containing Σ is a **real analytic space of dimension $2k$**, and Σ is a smooth point in this moduli space. In any case, even without this nondegeneracy assumption, $\mathcal{M}_k$ is a real analytic space of virtual dimension $2k$.

An admissible geodesic network $\mathcal{F}$ consists of a finite set of (complete) geodesic lines $\Gamma = \{\gamma_\alpha\}_{\alpha \in A}$ and geodesic segments $\mathcal{T} = \{\tau_{\alpha\beta}\}_{(\alpha,\beta)\in A'}$ connecting various pairs of elements in Γ.



Geodesic networks

We now make various assumptions on these data and set notation:

i) If $\alpha \neq \beta$, then $\text{dist}\,(\gamma_\alpha, \gamma_\beta) := \eta_{\alpha\beta} \in (0, \eta_0)$, where η_0 is the maximal separation between vertical planes which support a horizontal catenoid.

ii) The segment $\tau_{\alpha\beta}$ realizes the distance $\eta_{\alpha\beta}$ between γ_α and γ_β, and hence is perpendicular to both these geodesic lines.

iii) Set $p_\alpha(\beta) = \tau_{\alpha\beta} \cap \gamma_\alpha$ and $p_\beta(\alpha) = \tau_{\alpha\beta} \cap \gamma_\beta$, and then define

$$D_\alpha = \min_{(\alpha\beta),(\alpha,\beta')\in A'} \{\text{dist}(p_\alpha(\beta), p_\alpha(\beta'))\}, \quad \text{and} \quad D = \min_\alpha D_\alpha.$$

This number D is called the minimal neck separation of the configuration $\mathcal{F}$.

iv) We also write $\eta := \sup \eta_{\alpha\beta}$, and call it the maximal neck parameter.

Geodesic networks

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We now complete the perturbation analysis to show how to pass from the nearly minimal surfaces $\Sigma_{\mathcal{F}}$ to **actual minimal surfaces in** $\mathbb{H}^2 \times \mathbb{R}$ when $\mathcal{F}$ is a geodesic network with minimal neck separation D **sufficiently large**, and with a **uniform lower bound** $\eta_{\alpha\beta} \geq c > 0$ on the neck parameters.

Perturbation of $\Sigma_{\mathcal{F}}$ to a minimal surface

Fixing $\mathcal{F}$, let $\Sigma = \Sigma_{\mathcal{F}}$ and let ν be the unit normal on Σ with respect to a fixed orientation. For any $u \in \mathcal{C}^{2,\mu}(\Sigma)$, consider the normal graph over Σ with graph function u,

$$\Sigma(u) = \{\exp_q(u(q)\nu(q)),\ q \in \Sigma\}.$$

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Assuming that all $\eta_{\alpha\beta} \geq c > 0$, then there exists $C = C(c) > 0$ such that if $||u||_{2,\mu} < C$, then $\Sigma(u)$ **is embedded.**

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Question

When $\Sigma(u)$ is minimal ?

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iii) if ϵ is sufficiently small and $||u||_{2,\mu} < \epsilon$, then

$$||\mathcal{N}(u)||_{0,\mu} \leq C\epsilon \quad \text{and} \quad ||Q(u)||_{0,\mu} \leq C\epsilon^2.$$

Strategy

$$\mathcal{N}(u) = 0 \Leftrightarrow L_\Sigma u = -H_\Sigma - Q(u).$$

The strategy is now a standard one: we shall define certain weighted Hölder spaces X and Y, and first prove that $L_\Sigma : X \to Y$ is Fredholm. A more careful analysis shows that, at least when the minimal neck separation D is sufficiently large, this map is invertible and moreover its inverse $G_\Sigma : Y \to X$ has norm which is uniformly bounded by a constant depending only on the lower bounds D for the minimal neck separation and c for the maximal neck parameter. Given these facts, we then rewrite $\mathcal{N}(u) = 0$ as

$$u = -G_\Sigma(H_\Sigma + Q(u)),$$

and solve this equation by a standard contraction mapping argument.

Spectrum of the Jacobi operator of K_η

Recall that for any minimal surface Σ, its **Jacobi operator** (for the minimal surface equation) is the elliptic operator

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We now study the L^2 spectrum of the Jacobi operator L_η. By the general considerations

$$\mathrm{spec}(-L_\eta) = \{\lambda_j(\eta)\}_{j=1}^N \cup [1, \infty).$$

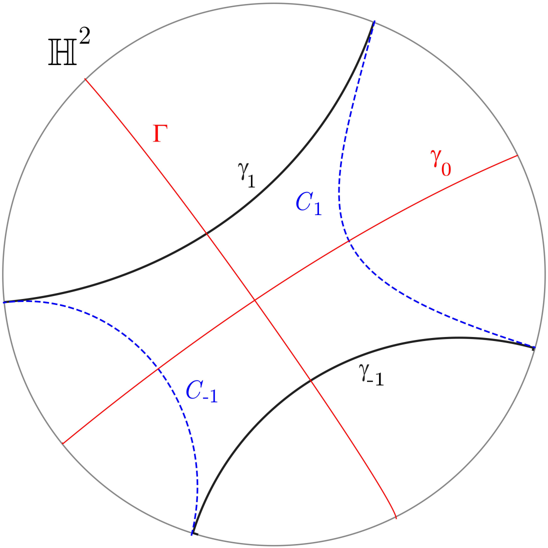
This ray consists of **absolutely continuous spectrum**, while the **discrete spectrum** all lies in $(-\infty, 1)$; note that, even counted according to multiplicity, the number of eigenvalues may depend on η.

Spectrum of the Jacobi operator of K_η

Theorem

For each $\eta \in (0, \eta_0)$, the only one of the eigenvalues of $-L_\eta$ which is negative is $\lambda_0(\eta)$, and only $\lambda_1(\eta) = 0$. All the remaining eigenvalues are strictly positive. The ground-state eigenfunction $\phi_0 = \phi_0(\eta)$ is even with respect to all three reflections, R_t, R_s and R_o; the eigenfunction ϕ_1, which is the unique L^2 Jacobi field, is generated by vertical translations and is odd with respect to R_t but even with respect to R_s and R_o. In particular, if we restrict $-L_\eta$ to functions which are even with respect to R_t, then L_η is nondegenerate.

Proof:



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We can decompose the spectrum of $-L_\eta$ into the parts which are either even or odd with respect to each of the isometric reflections R_t, R_s and R_o. Indeed, for each such reflection, there is an even/odd decomposition

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The reduction of $-L_\eta$ to the odd part of any one of these decompositions corresponds to this operator acting on functions on the appropriate half $K_\eta^{j,+}$ of K_η with Dirichlet boundary conditions.

Spectrum of the Jacobi operator of K_η

Claim

The restriction of $-L_\eta$ to $L^2(K_\eta)_{j-\mathrm{odd}}$ with $j = s, o$ is strictly positive, and is nonnegative if $j = t$, with one-dimensional **nullspace** spanned by the Jacobi field Φ_t generated by vertical translations.

To prove this, note first that since $\Phi_t \in L^2(K_\eta)_{t-\mathrm{odd}}$ and Φ_t is strictly positive on $K_\eta^{t,+}$, it must be the ground state eigenfunction for this reduction and is thus necessarily simple, with all the other eigenvalues strictly positive.
On the other hand, we have proved above that Φ_s and Φ_o are strictly positive solutions of this operator on the appropriate halves of K_η, vanishing on the boundary, but of course do not lie in L^2.

Spectrum of the Jacobi operator of K_η

Lemma (Murata, Sullivan)

Consider the operator $-L = -\Delta + V$ on a Riemannian manifold M, where V is smooth and bounded. Assume either that M is complete, or else, if it has boundary, then we consider $-L$ with Dirichlet boundary conditions at ∂M. Suppose that there exists an L^2 solution u_0 of $Lu_0 = 0$ such that $u_0 > 0$, at least away from ∂M. If v is any other solution of $Lv = 0$ with $v > 0$ in M and $v = 0$ on ∂M, then $v = cu_0$ for some constant c.

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This Lemma implies that it is impossible for $-L_\eta$ to have lowest eigenvalue equal to 0 on either of the subspaces $L^2(K_\eta)_{j-\mathrm{odd}}$, $j = s, o$, since if this were the case, then we could use the corresponding eigenfunction as u_0 in this Lemma and let $v = \Phi_j$ to get a contradiction since $\Phi_j \notin L^2$.

We shall justify later that when η is very close to its maximal value η_0, the lowest eigenvalue of $-L_\eta$ on $L^2(K_\eta)_{j-\mathrm{odd}}$ is strictly positive.

Spectrum of the Jacobi operator of K_η

We shall justify later that when η is very close to its maximal value η_0, the lowest eigenvalue of $-L_\eta$ on $L^2(K_\eta)_{j-\mathrm{odd}}$ is strictly positive. Using the continuity of the ground state eigenvalue as η decreases combined with the argument above, we see that this lowest eigenvalue can never be negative on any one of these odd subspaces, and the only odd L^2 Jacobi field is Φ_t. This proves the claim.

Spectrum of the Jacobi operator of K_η

We have finally reduced to studying the spectrum of $-L_\eta$ on $L^2(K_\eta)_{\mathrm{ev}}$, i.e. the subspace which is even with respect to all three reflections (we call this "totally even"). Because of the existence of an L^2 solution of $L_\eta u = 0$ which changes signs, namely $u = \Phi_t$, we know that the bottom of the spectrum of $-L_\eta$ is strictly negative, and we have proved above that the corresponding eigenfunction must live in the totally even subspace. (This is also obvious because of the simplicity of this eigenspace and the fact that the corresponding eigenfunction is everywhere positive.) Thus $\lambda_0(\eta) < 0$ as claimed.

Spectrum of the Jacobi operator of K_η

Now suppose that the next eigenvalue $\lambda_1(\eta)$ lies in the interval $(\lambda_0(\eta), 0]$, and if $\lambda_1(\eta) = 0$, assume that there exists a corresponding eigenfunction which is totally even. Since this is the *second* eigenvalue, we know that the corresponding eigenfunction $\phi_1(\eta)$ has exactly two nodal domains. However, it is straightforward to see using the symmetries of K_η that if ϕ is any function on K_η which is totally even and changes sign, then it cannot have exactly two nodal domains. Indeed, if that were the case, then the nodal line $\{\phi = 0\}$ would have to either be a connected simple closed curve or else two arcs, and these would then necessarily be the fixed point set of one of the three reflections. This is clearly incompatible with ϕ being totally even.

Spectrum of the Jacobi operator of K_η

We are almost finished. It remains finally to prove that the lowest eigenvalue of $-L_\eta$ on any one of the odd subspaces is nonnegative when η is sufficiently large.
As a first step, we first prove that $\lambda_0(\eta) \nearrow 0$ as $\eta \nearrow \eta_0$. Recall that in this limit, K_η converges (once we translate vertically by an appropriate distance) to the limiting Scherk surface K_{η_0}. Moreover, K_{η_0} is strictly stable because the Jacobi field Φ_t generated by vertical translation is strictly positive on it.

Now suppose that $\lambda_0(\eta) \leq -c < 0$. When η is sufficiently close to η_0, we can construct a cutoff $\widetilde{\phi}_0(\eta)$ of the corresponding eigenfunction $\phi_0(\eta)$ which is supported in the region $t > 0$ (we are still assuming that K_η is centered around $t = 0$); this function lies in L^2 and regarding it as a function on K_{η_0}, it is straightforward to show that

$$\frac{\int_{K_{\eta_0}} (-L_{\eta_0}\widetilde{\phi_0})\widetilde{\phi_0}}{\int_{K_{\eta_0}} |\widetilde{\phi_0}|^2} \leq -c/2 < 0.$$

This contradicts the strict stability of K_{η_0}, and hence proves that $\lambda_0(\eta) \nearrow 0$.

Spectrum of the Jacobi operator of K_η

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Now suppose that there is some sequence $\eta^\ell \nearrow \eta_0$ and a corresponding sequence of eigenvalues $\lambda^\ell \in (\lambda_0(\eta^\ell), 0)$ and eigenfunctions $\phi^\ell \in L^2(K_\eta)_{j-\mathrm{odd}}$, $j = s, o$. We know that $\lambda^\ell \nearrow 0$. Suppose that the maximum of $|\phi^\ell|$ is attained at some point $p^\ell \in K_{\eta^\ell}$. Normalize by setting $\widehat{\phi}^\ell = \phi^\ell / \sup |\phi^\ell|$ and take the limit as $\ell \to \infty$. Depending on the limiting location of p^ℓ, we obtain a bounded solution of the limiting equation on the pointed Gromov-Hausdorff limit of the sequence (K_{η^ℓ}, p^ℓ). There are, up to isometries, only two possible such limits: either the limiting Scherk surface K_{η_0} or else a vertical plane $P = \gamma \times \mathbb{R}$.

In the latter case, the limiting function $\widehat{\phi}$ satisfies $L_P \widehat{\phi} = 0$. However, $L_P = \Delta_P - 1$ and there are no bounded solutions of this equation, so this case cannot occur. Therefore, we have obtained a function $\widehat{\phi}$ on K_{η_0} which is a solution of the Jacobi equation there and which is bounded. We now invoke

Theorem (Manzano, Pérez, Rodríguez)

Let (M^n, ds^2) be a Riemannian parabolic manifold. Consider an operator $L = \Delta + V$, where $V \in C^\infty(M)$. Let $u, v \in C^\infty(M)$ such that u is bounded, $v > 0$ and $uvLu \geq u^2 Lv$ on M. Then, u/v is constant.

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This proves that $\widehat{\phi}$ must equal the unique *positive* L^2 Jacobi field on K_{η_0}, but this is impossible because of the oddness of ϕ^ℓ with respect to either R_s or R_o.
This completes the proof of the theorem.