A study of real hypersurfaces in terms of *-Ricci tensor

George Kaimakamis

(joint work with K. Panagiotidou)

Faculty of Mathematics and Engineering Sciences
Hellenic Army Academy

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Definition

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$M_n(c), c \neq 0 \longrightarrow$ Non-Flat Complex Space Form

Definition

Almost contact structure *or* (φ, ξ, η) *-* ***structure*** *is a tensor field* φ *of type (1,1), a vector field* ξ *and a 1-form* η*, which satisfy the following relations*

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1,$$

for any vector field $X \in \mathfrak{X}(M)$*.*

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Compatible Metric *of an almost contact manifold is a Riemannian metric such that*

$$\eta(X) = g(X, \xi), \ \ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), X, Y \in \mathfrak{X}(M).$$

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Gauss and Weingarten equations

$$\overline{\nabla}_Y X = \nabla_Y X + g(AY, X)N,$$

$$\overline{\nabla}_X N = -AX,$$

where ∇ is the Levi-Civita connection of M, A is the shape operator of M and g the induced Riemannian metric on M.

Definition of the (φ, ξ, η, g) - structure on a real hypersurface

Structure vector field ξ : $\xi = -JN$

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$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \varphi = 0, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \;\; g(X, \varphi Y) = -g(\varphi X, Y),$$

$$\nabla_X \xi = \varphi AX, \qquad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

for any tangent vector field X, Y on M.

Gauss equation:

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X$$

$$-g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z] + g(AY,Z)AX - g(AX,Z)AY$$

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Codazzi equation:

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X,Y)\xi],$$

where R denotes the Riemannian curvature tensor on M.

The tangent space T_PM : $T_PM = \text{span}\{\xi\} \oplus \mathbb{D}$,
$\mathbb{D} = \ker \eta = \{X \ \epsilon \ T_PM : \eta(X) = 0\}$ (**holomorphic distribution**).

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Definition

*A real hypersurface is a **Hopf hypersurface**, if the structure vector field ξ is principal, i.e. $A\xi = \alpha\xi$.*

If M is a Hopf hypersurface then α is constant

Complex Projective Space $\mathbb{C}P^n, n \geq 2$

Takagi (1973) [15], Cecil, Ryan (1982) [2]

- $(A1)$ geodesic spheres of radius r, where $0 < r < \frac{\pi}{2}$,
- $(A2)$ tubes of radius r over totally geodesic complex projective space $\mathbb{C}P^k$, where $0 < r < \frac{\pi}{2}$ and $1 \leq k \leq n-2$,
- (B) tubes of radius r over complex quadrics and $\mathbb{R}P^n$, where $0 < r < \frac{\pi}{4}$,
- (C) tubes of radius r over the Serge embedding of $\mathbb{C}P^1 \times \mathbb{C}P^{\frac{(n-1)}{2}}$, where $0 < r < \frac{\pi}{4}$ and $n \geq 2\kappa + 3$, $\kappa \ \epsilon \ \mathbb{N}^*$,
- (D) tubes of radius r over the *Plucker* embedding of the complex Grassmannian manifold $G_{2,5}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,
- (E) tubes of radius r over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.

Complex Hyperbolic Space $\mathbb{C}H^n, n \geq 2$

Montiel (1985) [11], Berndt (1989) [1]

- (A0) horospheres
- (A1,0) geodesic spheres of radius $r > 0$,
- (A1,1) tubes of radius $r > 0$ over totally geodesic complex hyperbolic hyperplanes $\mathbb{C}H^{n-1}$,
- (A2) tubes of radius $r > 0$ over totally geodesic submanifold $\mathbb{C}H^k$, $1 \leq k \leq n-2$,
- (B) tubes of radius $r > 0$ over totally real hyperbolic space $\mathbb{R}H^n$.

In case of $\mathbb{C}P^n$,$n \geq 2$,$\longrightarrow$ Maeda [10] and in case of $\mathbb{C}H^n$,$n \geq 2$,$\longrightarrow$ Montiel [11].

Theorem

Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then
i) If W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then $(\lambda - \frac{\alpha}{2})A\varphi W = (\frac{\lambda\alpha}{2} + \frac{c}{4})\varphi W$.
ii) If the vector field W satisfies $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ then $\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}$.

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Theorem

Let M be a real hypersurface in $M_n(c)$, $n \geq 2$, then $\varphi A = A\varphi$ if and only if M is an open subset of real hypersurfaces of type (A).

*-Ricci Tensor

Definition [Tachibana (1959) [14]]

The ****-Ricci tensor*** *of a Kaehler manifold N is given by*

$$g(S^{*}X, Y) = \frac{1}{2}(trace\{J \circ R(X, JY)\}),$$

for any X, Y tangent vector fields to N.

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Definition [Hamada (2002) [5]]

The ****-Ricci tensor*** *of a real hypersurface* M *in a complex space form is given by*

$$g(S^*X, Y) = \frac{1}{2}(trace\{\varphi \circ R(X, \varphi Y)\}),$$

for any X, Y *tangent vector fields to* M.

Definition [Hamada (2002) [5]]

A real hypersurface is called ****-Einstein*** *when the *-Ricci tensor satisfies the following*

$$g(S^*X,Y) = \frac{\rho^*}{2(n-1)g(X,Y)}, \textit{for } X \,\epsilon\, \mathbb{D} \textit{ and } \rho^* \textit{ is constant.}$$

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Ivey and Ryan (2011) [6]:

$$S^*X = -[\frac{cn}{2}\varphi^2 X + (\varphi A)^2 X], \ \text{for } X \in TM.$$

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The *-Einstein Hopf hypersurfaces in $M_n(c)$, where $n \geq 2$ are precisely

- the Hopf hypersurfaces whose Hopf principal curvature α vanishes and,
- the open connected subsets of homogeneous Hopf hypersurfaces of types (A_0), (A_1) and (B).

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Parallel *-Ricci tensor

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A tensor field P of type (1,s) of a real hypersurface is ***parallel****, if* $\nabla_X P = 0$*, for any* $X \epsilon TM$*.*

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Theorem [- , Panagiotidou, Taiw. J. Math. (2014)]

There do not exist real hypersurfaces in $\mathbb{C}P^2$ whose *-Ricci tensor is parallel. In $\mathbb{C}H^2$ only the geodesic hypersphere has parallel *-Ricci tensor with $\coth(r) = 2$.

Semi-parallel *-Ricci tensor

Definition

A tensor field P of type (1,s) is ***semi-parallel****, if $R \cdot P = 0$, where R is the Riemannian curvature and acts as derivation on P.*

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Theorem [- , Panagiotidou, (submitted)

There do not exist real hypersurfaces in $\mathbb{C}P^2$ with semi-parallel *-Ricci tensor. In $\mathbb{C}H^2$ only the geodesic hypersphere has parallel *-Ricci tensor with $\coth(r) = 2$.

Pseudo-parallel *-Ricci tensor

Definition

A tensor field P of type (1,s) is ***pseudo-parallel****, if there exists a function L such that $R \cdot P = L\{(X \wedge Y) \cdot P\}$,where $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$.*

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$$\begin{aligned} R(X,Y)S^*Z - S^*[R(X,Y)Z] &= L\{g(Y,S^*Z)X - g(X,S^*Z)Y \\ &\quad -S^*[g(Y,Z)X - g(X,Z)Y]\}. \end{aligned}$$

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$$R(X,Y)S^*Z - S^*[R(X,Y)Z] = L\{g(Y,S^*Z)X - g(X,S^*Z)Y - S^*[g(Y,Z)X - g(X,Z)Y]\}.$$

Theorem [- , Panagiotidou, (submitted)]

Every real hypersurface M in $M_2(c)$ with pseudo-parallel *-Ricci tensor is a Hopf hypersurface. Furthemore, M is locally congruent to either a real hypersurface of type (A) or to a Hopf hypersurface satisfying relation $A\xi = 0$, with L constant.

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Theorem [- , Panagiotidou, (submitted)]

Every real hypersurface M in $M_2(c)$ with ξ-parallel *-Ricci tensor is a Hopf hypersurface. Moreover, M is locally congruent to i) a real hypersurface of type (A) or ii) to a real hypersurface of type (B) or iii) to a Hopf hypersurface whose principal curvatures corresponding to the holomorphic distribution are non-constant and the derivative of them in the direction of ξ is equal to zero

Types of Parallelness of *-Ricci tensor

Condition of S^*	$\mathbb{C}P^2$	$\mathbb{C}H^2$
Parallel	**No**	✔
Semi - parallel	**No**	✔
Pseudo - parallel	✔	✔
ξ **- parallel**	✔	✔

Previous Work

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Definition

A Riemannian metric g of a Riemannin manifold is called ***Ricci soliton*** *if there exist a smooth vector field V such that the following relation is satisfied*

$$\frac{1}{2}\mathcal{L}_V g + Ric - \lambda g = 0,$$

where λ is a constant. Furthermore, V is called ***potential vector field****.*

Theorem

There do not exist Einstein real hypersurfaces in non-flat complex space forms.

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Definition [Cho, Kimura (2009) [3]]

*An η-**Ricci soliton** on a real hypersurface in non-flat complex space form is a pair (η,g) which satisfies the following relation*

$$\frac{1}{2}\mathcal{L}_\xi g + Ric - \lambda g - \mu\eta \otimes \eta = 0,$$

where λ, μ are constant and $Ric(X, Y) = g(SX, Y)$ and η is the 1-form on M.

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Definition [Cho, Kimura (2009) [3]]

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where λ*,* μ *are constant and* $Ric(X, Y) = g(SX, Y)$ *and* η *is the 1-form on* M*.*

Theorem [Cho, Kimura (2009) [3]]

Let M be a real hypersurface in a non-flat complex space form $M_n(c)$. If M admits an η-Ricci soliton, them M is a Hopf hypersurface and is locally congruent to a real hypersurface of type (A).

Theorem [Cho, Kimura (2009) [3]]

Let M be a real hypersurface in a non-flat complex space form $M_n(c)$. If M admits an η-Ricci soliton, them M is a Hopf hypersurface and is locally congruent to a real hypersurface of type (A).

Corollary

A real hypersurface in a non-flat complex space form does not admit a Ricci soliton with potential vector field ξ.

Theorem [Cho, Kimura (2009) [3]]

Let M be a real hypersurface in a non-flat complex space form $M_n(c)$. If M admits an η-Ricci soliton, them M is a Hopf hypersurface and is locally congruent to a real hypersurface of type (A).

Corollary

A real hypersurface in a non-flat complex space form does not admit a Ricci soliton with potential vector field ξ.

Theorem [Cho, Kimura (2011) [4]]

A compact Hopf hypersurface in a non-flat complex space form does not admit a Ricci soliton

*-Ricci soliton

Definition

A Riemannian metric g on a real hypersurface M is called ****-Ricci solition*** *if*

$$\frac{1}{2}\mathcal{L}_V g + Ric^* - \lambda g = 0,$$

where $Ric^* = g(S^*X, Y)$ where S^* is the *-Ricci tensor on M and λ is constant.

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where $Ric^* = g(S^*X, Y)$ *where* S^* *is the* **-Ricci tensor on* M *and* λ *is constant.*

Theorem [- , Panagiotidou, J. Phys. and Geom. (2014)]

There do not exist real hypersurfaces in $\mathbb{C}P^n$, $n \geq 2$ admitting a *-Ricci soliton with potential vector field being the structure vector field ξ.

Theorem [- , Panagiotidou, J. Phys. and Geom. (2014)]

Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 2$ admitting a *-Ricci soliton with potential vector field being the structure vector field ξ. Then M is loccaly congruent to a geodesic hypersphere with $2n = coth^2(r)$.

1^{st} Step

Proposition

*Every real hypersurface in $M_n(c)$, $n \geq 2$, admitting a *-Ricci soliton with potential vector field ξ is a Hopf hypersurface.*

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Using the formula $S^* = -[\frac{cn}{2}\varphi^2 + (\varphi A)^2]$ and $\mathcal{L}_\xi g = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)$ we obtain

$$g(\varphi AX, Y) + g(\varphi AY, X) + ncg(X,Y) - nc\eta(X)\eta(Y)$$
$$+2g(\varphi AX, A\varphi Y) - 2\lambda g(X,Y) = 0.$$

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For certain combination of $\xi, U, \varphi U$ we lead to the conclusion that $\mathcal{N}$ is empty.

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Case I: $\alpha^2 + c \neq 0$.
In this case only the geodesic hypersphere in $\mathcal{C}H^n, n \geq 2$, admits a *-Ricci soliton with potential vector field ξ.

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Case I: $\alpha^2 + c \neq 0$.
In this case only the geodesic hypersphere in $\mathcal{C}H^n, n \geq 2$, admits a *-Ricci soliton with potential vector field ξ.
Case II: $\alpha^2 + c = 0$

- $\lambda \neq \frac{\alpha}{2}$
- $\lambda = \frac{\alpha}{2} \quad \Rightarrow \quad Horosphere$

Types of Parallelness of *-Ricci tensor

Condition of S^*	$\mathbb{C}P^2$	$\mathbb{C}H^2$	$M_n(c), n \geq 3$
Parallel	No	✔	?
Semi - parallel	No	✔	?
Pseudo - parallel	✔	✔	?
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Ideas for further work

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- Is there a Ricci flow which corresponds to *-Ricci soliton?

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- Is there a Ricci flow which corresponds to *-Ricci soliton?
- Are there real hypersurfaces in $M_n(c)$ admitting a *-Ricci soliton whose potential vector field V belongs to the holomorphic distribution?

References I

- Berndt, J.: *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*. J. Reine Angew. Math. **395**, 132-141 (1989).
- Cecil, T.E., Ryan, P.J: *Focal sets and real hypersurfaces in complex projective space*. Trans. Amer. Math. Soc. **269**, 481-499 (1982).
- Cho, J.T., Kimura, M.:*Ricci solitons and real hypersurfaces in a complex space form*. Tohoku Math. J. **61**, 205-212 (2009).
- Cho, J.T., Kimura, M.:*Ricci solitons of compact real hypersurfaces in Kahler manifolds*. Math. Nachr. **284**, 1385-1393 (2011).

References II

- Hamada, T.: *Real hypersurfaces of complex space forms in terms of Ricci *-tensor*. Tokyo J. Math. **25**, 473-483 (2002).
- Ivey, T., Ryan, P. J.: *The *-Ricci tensor for hypersurfaces in* $\mathbb{C}P^n$ *and* $\mathbb{C}H^n$. Tokyo J. Math. **34**, 445-471 (2011).
- Kaimakamis, G., Panagiotidou,K.: **-Ricci solitons of real hypersurfaces in non-flat complex space forms*. J. Geom. and Phus. **86**, 408-413 (2014).
- Kaimakamis, G., Panagiotidou,K.: *Parallel *-Ricci tensor of real hypersurfaces in* $\mathbb{C}P^2$ *and* $\mathbb{C}H^2$. Taiwan. J. Math. **18**, 1991-1998 (2014).

References III

- Kaimakamis, G., Panagiotidou,K.: Conditions of parallelism of *-Ricci tensor of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. Submitted.
- Maeda, Y.: *On real hypersurfaces of a complex projective space.* J. Math. Soc. Japan **28**,529-540, (1976).
- Montiel, S.: *Real hypersurfaces of a complex hyperbolic space.* J. Math. Soc. Japan **35**, 515-535, (1985).
- Montiel, S. and Romero, A.: *On some real hypersurfaces of a complex hyperbolic space.* Geom. Dedicata **20**, 245-261, (1986).

References IV

- Okumura, M.: *On some real hypersurfaces of a complex projective space*. Trans. Amer. Math. Soc. **212**, 355-364, (1975).
- Tachibana, S.: *On almost-analytic vectors in almost Kählerian manifolds*. Tohoku Math J. **11**, 247-265 (1959).
- Takagi, R: *Real hypersurfacesin a complex prjective space with constant principal curvatures*. J. Math. Soc, Japan **27**, 43-53 (1975).

THANK YOU