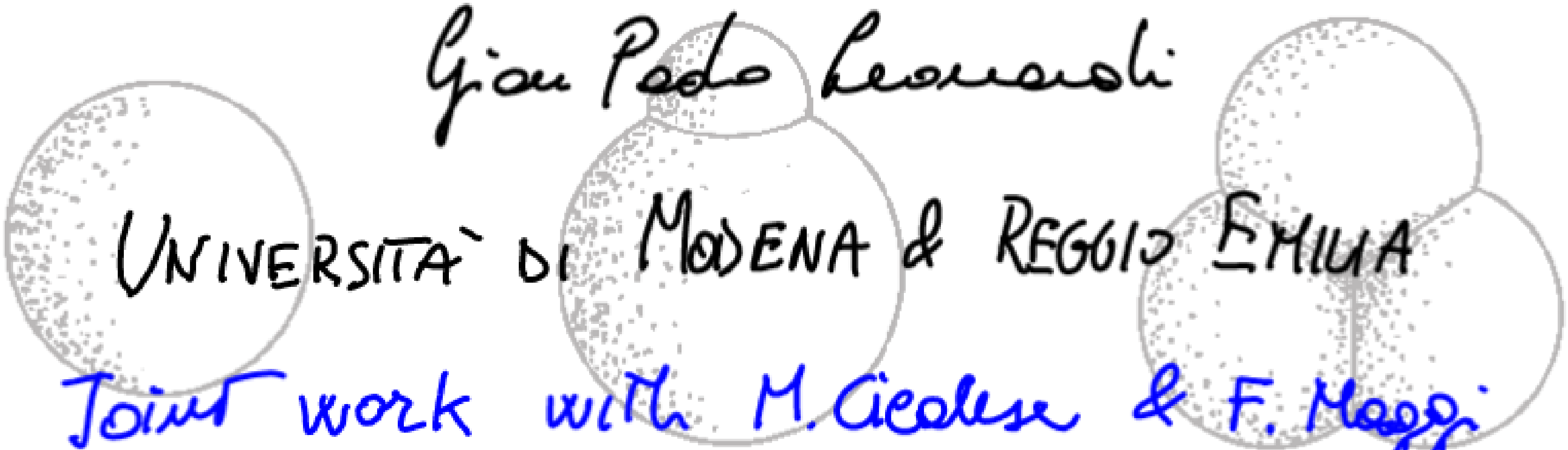
STABILITY INEQUALITIES FOR PERIMETER MINIMIZING CLUSTERS



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Joint work with M. Cicalese & F. Maggi

INTRODUCTION

- The Sharp Quantitative Isoperimetric Inequality

$$P(E) \geq P(B_E)\left[1 + C \cdot \min_{x \in \mathbb{R}^n} \frac{|E \Delta (x + B_E)|^2}{|E|^2}\right]$$

* Proved with different methods:
  - Symmetrization
  - Optimal Transport
  - Penalization + Regularity

* A long history starting from the beginning of the $\overline{\underline{\text{XX}}}$ Century ...

- An interesting application:

$$\min \ \mathcal{F}_\delta(E) = P(E) + \delta \cdot V(E)$$

$|E|$ fixed, $\delta > 0$ small, V potential.

Examples:

$*$ $V(E) = \int_E g(x)\,dx$ LOCAL, like gravity over a substrate

$*$ $V(E) = \iint_{E\times E} G(x,y)\,dx\,dy$ NON-LOCAL, like Ohta-Kawasaki

QUESTION: if $\delta \ll 1$, how close a minimizer E_δ of $\mathcal{F}_\delta$ is to a <u>ball</u>?

Figalli - Maggi (ARMA, 2011) for the local case

Cicalese - Spadaro (2011) for Ohta - Kawasaki

$\downarrow$

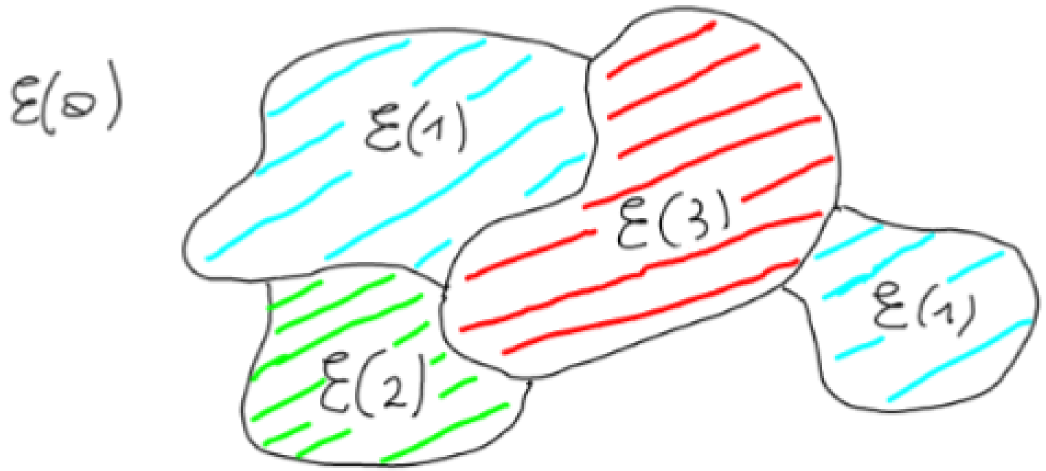
Estimates on the shape of a single droplet (phase)

N-clusters in $\mathbb{R}^n$

$\mathcal{E} = (\mathcal{E}(0), \mathcal{E}(1), \dots, \mathcal{E}(N))$ a partition of $\mathbb{R}^n$

$\mathrm{vol}(\mathcal{E}) = (|\mathcal{E}(1)|, \dots, |\mathcal{E}(N)|)$ volume vector

$$P(\mathcal{E}) = \sum_{i<j=0}^{N} \mathcal{H}^{n-1}\left(\partial^*\mathcal{E}(i) \cap \partial^*\mathcal{E}(j)\right)$$ perimeter



- $\mathcal{E}$ is **perimeter-minimizing** iff

$P(\mathcal{E}) \leq P(\mathcal{F}) \quad \forall \mathcal{F}$ with $\mathrm{vol}(\mathcal{F}) = \mathrm{vol}(\mathcal{E})$

(<u>Almgren</u>, Mem AMS 1976)

Existence & partial regularity of perimeter-minimizing N-clusters in $\mathbb{R}^n$

$\longrightarrow$ see new book by F. MAGGI

Sets of finite perimeter and geometric variational problems: an introduction to GMT, Cambridge 2012

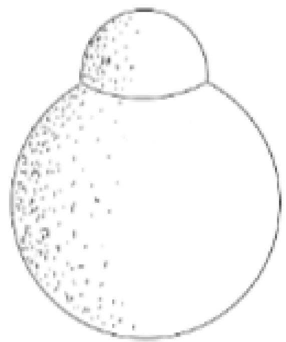
GOAL: extend the theory to perimeter-minimizing N-clusters in $\mathbb{R}^n$

WARNING

Only few of them are known and proved to be perimeter-minimizing at fixed volumes

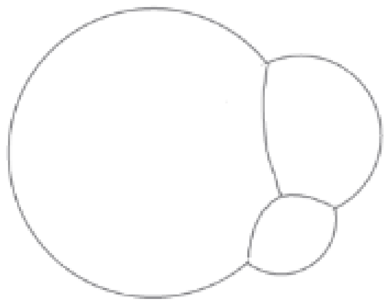
The known P-minimizing N-clusters, $N \geq 2$:

∗ $N = 2$



$\begin{cases} m=2 & \text{Foisy et al. 1993} \\ m=3 & \text{Hutchings - Morgan - Ritoré - Ros 2002} \\ m>3 & \text{Reichardt 2008} \end{cases}$

∗ $N = 3$



$m = 2$ Wichiramala 2004

NOTE: the proofs heavily rely on the REGULARITY THEORY

THE QUANTITATIVE STABILITY PROBLEM

$V = (v_1, \dots, v_N)$ volume vector

$\mathcal{E}_0$ N-cluster uniquely perimeter-minimizing among N-clusters with $\mathrm{vol}(\cdot) = V$.

For $\mathcal{E}$ with $\mathrm{vol}(\mathcal{E}) = V$, define

$$\delta(\mathcal{E}) = P(\mathcal{E}) - P(\mathcal{E}_0) \qquad \text{isop. deficit}$$

$$\alpha(\mathcal{E}) = \min_{\substack{x \in \mathbb{R}^n \\ f \in O(n)}} d(\mathcal{E}, x + f(\mathcal{E}_0)) \qquad \text{asymmetry}$$

where $d(\mathcal{E}, \mathcal{F}) = \sum_{i=1}^{N} |\mathcal{E}(i) \Delta \mathcal{F}(i)|$

GOAL: $\boxed{\exists\ k_0 = k_0(\mathcal{E}_0) > 0 : \quad \delta(\mathcal{E}) \geq k_0\, \alpha(\mathcal{E})^2}$

I. GENERAL RESULTS TOWARD STABILITY

II. STRONGER RESULTS in $\mathbb{R}^2$ and $\mathbb{R}^3$

I. GENERAL RESULTS TOWARD STABILITY for UNIQUELY P-min. N-clusters in $\mathbb{R}^n$

Optimality of QUADRATIC ESTIMATES

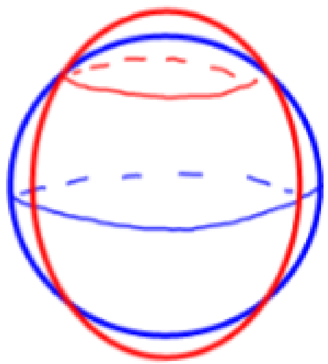
Reduction to L^1-approximating competitors via SOFT STABILITY

Reduction to approximating Λ-MINIMIZERS via SELECTION PRINCIPLE

OPTIMALITY of QUADRATIC ESTIMATES

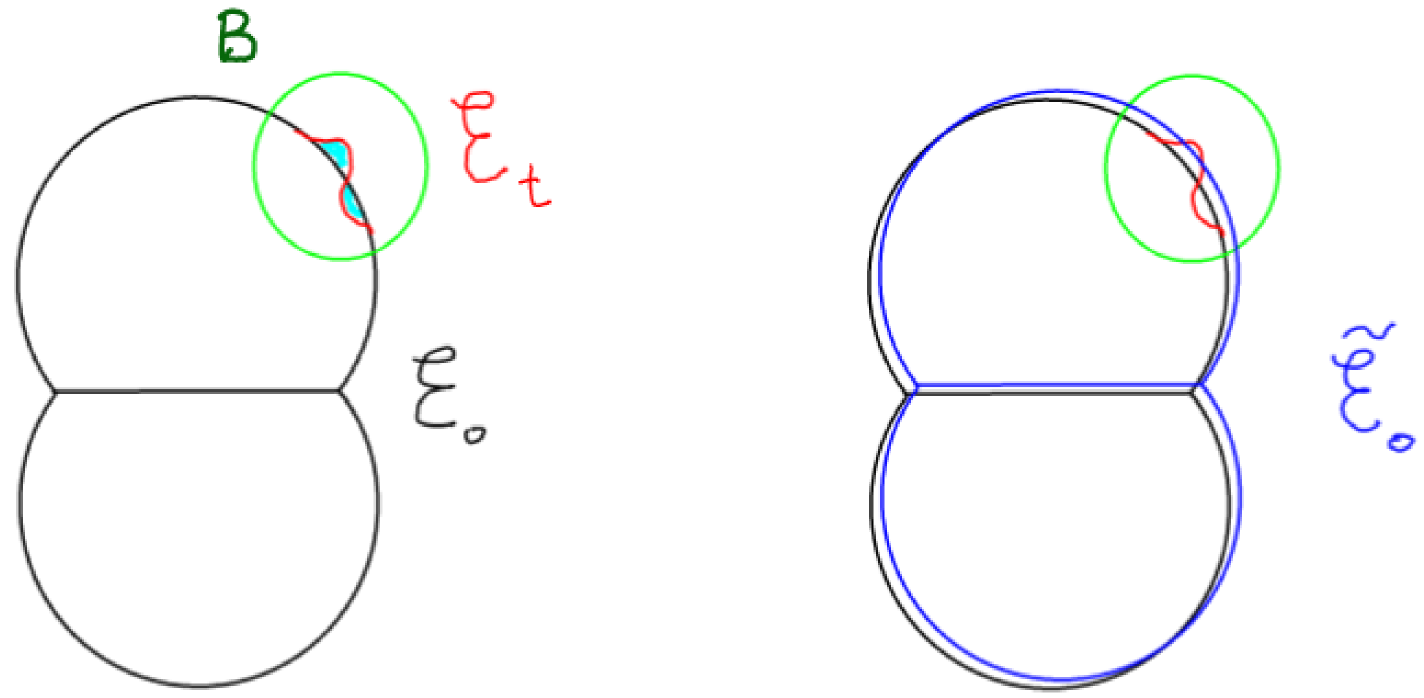
find $\mathcal{E}_t \to \mathcal{E}_0$ as $t \to 0$, s.t. $\delta(\mathcal{E}_t) \leq C \cdot \alpha(\mathcal{E}_t)^2$ for some uniform $C > 0$.

- EASY in the case $N = 1$, $n \geq 2$



one can check with **ellipsoids** (Hall, 1992)

- LESS EASY in the case $N \geq 2$ ...



$$\alpha(\mathcal{E}_t) = d(\mathcal{E}_t, \tilde{\mathcal{E}}_0) \geq d_{B^c}(\mathcal{E}_0, \tilde{\mathcal{E}}_0)$$

$$\begin{aligned}\sqrt{\delta(\mathcal{E}_t)} = \sqrt{P(\mathcal{E}_t) - P(\mathcal{E}_0)} &\leq c \cdot d(\mathcal{E}_t, \mathcal{E}_0) \\ &\leq c \cdot [\alpha(\mathcal{E}_t) + d(\mathcal{E}_0, \tilde{\mathcal{E}}_0)]\end{aligned}$$

GOAL: prove that $\boxed{d(\mathcal{E}_0, \tilde{\mathcal{E}}_0) \leq 2 \cdot \alpha(\mathcal{E}_t)}$

(at least for $\alpha(\mathcal{E}_t)$ small)

Observe that $d(\mathcal{E}_0, \tilde{\mathcal{E}}_0) = d_B(\mathcal{E}_0, \tilde{\mathcal{E}}_0) + d_{B^c}(\mathcal{E}_0, \tilde{\mathcal{E}}_0)$.

Since $d_{B^c}(\mathcal{E}_0, \tilde{\mathcal{E}}_0) \leq \alpha(\mathcal{E}_t)$, it is enough to prove

$$\boxed{d_B(\mathcal{E}_0, \tilde{\mathcal{E}}_0) \leq d_{B^c}(\mathcal{E}_0, \tilde{\mathcal{E}}_0)}$$

for B suitably chosen ...

Lemma (MAGIC BALL): $E \subseteq \mathbb{R}^n$ bounded set, $P(E) < +\infty$. Then for all $x \in \partial^* E$ one can find $\varepsilon > 0$ with the following property: for any isometry f with $\|f - \mathrm{Id}\| < \varepsilon$, it holds

$$\left|\left(E \triangle f(E)\right) \cap B(x,\varepsilon)\right| \leq \left|\left(E \triangle f(E)\right) \setminus B(x,\varepsilon)\right| .$$

Proof by contradiction, using in particular

* De Giorgi's Blow-up Theorem
* Besicovich Covering theorem
* Campbell-Hausdorff formula on the Lie Algebra of infinitesimal generators of isometries

SOFT STABILITY

$$\boxed{\forall \alpha_0 > 0 \ \exists \delta_0 > 0 : \quad \delta(\mathcal{E}) < \delta_0 \Rightarrow \alpha(\mathcal{E}) < \alpha_0}$$

Proof by contradiction, if $(\mathcal{E}_k)_k$ is such that $P(\mathcal{E}_k) \to P(\mathcal{E}_0)$ but $\alpha(\mathcal{E}_k) \geq \alpha_0 > 0 \ \forall k$, then with a careful application of Almgren's volume adjustment technique we build another sequence $\tilde{\mathcal{E}}_k$ such that $d(\tilde{\mathcal{E}}_k, \mathcal{E}_0) \to 0$ but $\alpha(\tilde{\mathcal{E}}_k) \geq \frac{\alpha_0}{2}$, a contradiction.

A consequence of SOFT STABILITY is

$$\delta(\varepsilon) \gtrsim C \cdot \alpha(\varepsilon)^2$$

$$\Updownarrow$$

$$\delta(\varepsilon) \gtrsim C_0 \cdot \alpha(\varepsilon)^2 + o\left(\alpha(\varepsilon)^2\right)$$

REDUCTION TO SMALL ASYMMETRIES

SELECTION PRINCIPLE

Cicalese - L. (ARMA 2012, JEMS 2013) for $N=1$
Cicalese - L. - Maggi for $N \geq 2$

$$\boxed{\delta(\mathcal{E}) \geq C_0 \cdot \alpha(\mathcal{E})^2 + o\left(\alpha(\mathcal{E})^2\right)}$$

OPTIMAL ASYMPTOTIC CONSTANT

- Fix $(W_k)_k$ recovery sequence for C_0
- Define $\quad \mathcal{F}_k(\mathcal{E}) = P(\mathcal{E}) + \left|\alpha(\mathcal{E}) - \alpha(W_k)\right|^{\frac{3}{2}}$
- Prove that a minimizer $\mathcal{E}_k$ of $\mathcal{F}_k$ under the volume constraints $\mathrm{vol}(\mathcal{E}_k) = \mathrm{vol}(\mathcal{E}_0)$ EXISTS.

Then $(\mathcal{E}_k)_k$ has the following properties:

① $\alpha(\mathcal{E}_k) \longrightarrow 0$ and $\dfrac{\delta(\mathcal{E}_k)}{\alpha(\mathcal{E}_k)^2} \longrightarrow C_0$.

② $P(\mathcal{E}_k) \leq P(\mathcal{E})$ $\forall \mathcal{E}$ with $\mathrm{vol}(\mathcal{E}) = \mathrm{vol}(\mathcal{E}_0)$ and $\alpha(\mathcal{E}) = \alpha(\mathcal{E}_k)$.

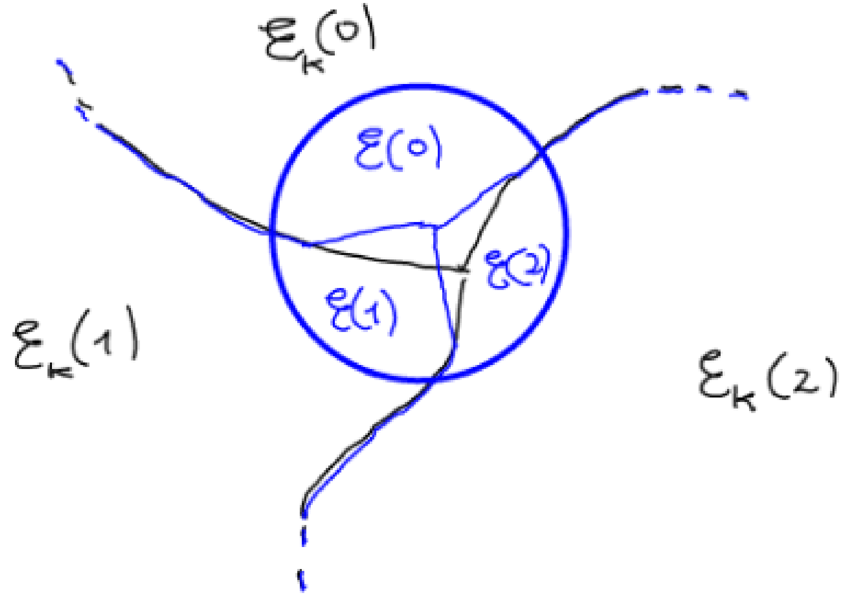
③ $\mathcal{E}_k$ is a (Λ, ρ)-minimizer of $P(\cdot)$ with Λ, ρ uniform in k

thus $\mathcal{E}_k$ is a qualified candidate for testing the positivity of C_0

$\mathcal{E}_k$ is a (Λ,ρ)-minimizer :

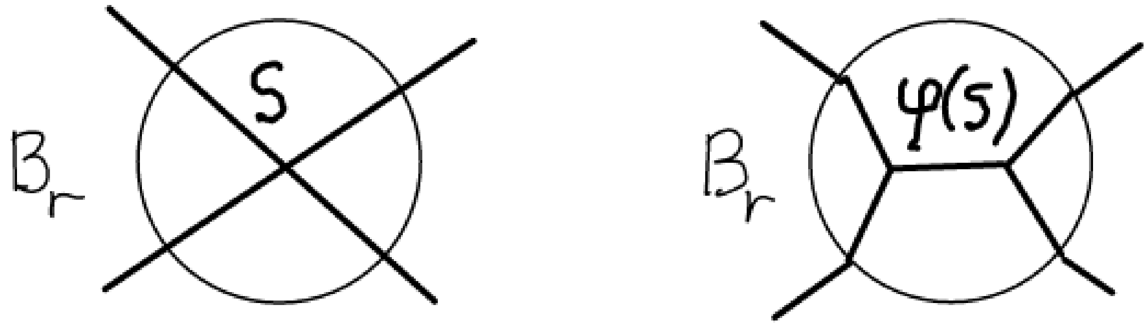
$\exists\ \Lambda, \rho > 0$ dimensional constants s.t.
for all $x \in \mathbb{R}^n$ and all N-cluster $\mathcal{E}$ with
$\mathcal{E}(i) \Delta \mathcal{E}_k(i) \subset\subset B(x,\rho)$, $i = 1, \dots, N$,

$$\boxed{P(\mathcal{E}_k) \leq P(\mathcal{E}) + \Lambda \cdot d(\mathcal{E}, \mathcal{E}_k)}$$



Other notions of almost-minimality

- (M, ε, δ)-minimality (Almgren)



$r < \delta$

$\varphi \in \mathrm{Lip}$

$$\mathcal{H}^m(S) \leq (1+\varepsilon(r))\,\mathcal{H}^m(\varphi(S))$$

$\Downarrow$

- Almost-minimality (David)

$$\mathcal{H}^m(S) \leq \mathcal{H}^m(\varphi(S)) + h(r)\, r^m$$

REGULARITY THEOREM I

1. if $\mathcal{E}$ is a (Λ, ρ)-minimizer, then
$\partial\mathcal{E} = \partial^*\mathcal{E} \cup \Sigma(\mathcal{E})$ with
   - $\partial^*\mathcal{E}$ union of $\mathcal{C}^{1,\beta}$ hypersurfaces
   - $\Sigma(\mathcal{E})$ closed set with $\mathcal{H}^{n-1}(\Sigma(\mathcal{E}))=0$

2. if $(\mathcal{E}_k)_k$ is a sequence of (Λ, ρ)-minimizers that L^1-converge to a limit cluster $\mathcal{E}_0$, then
   - $\partial\mathcal{E}_k \longrightarrow \partial\mathcal{E}_0$ Hausdorff
   - $\partial^*\mathcal{E}_k \longrightarrow \partial^*\mathcal{E}_0$ $\mathcal{C}^{1,\beta}_{loc}(\partial^*\mathcal{E}_0)$

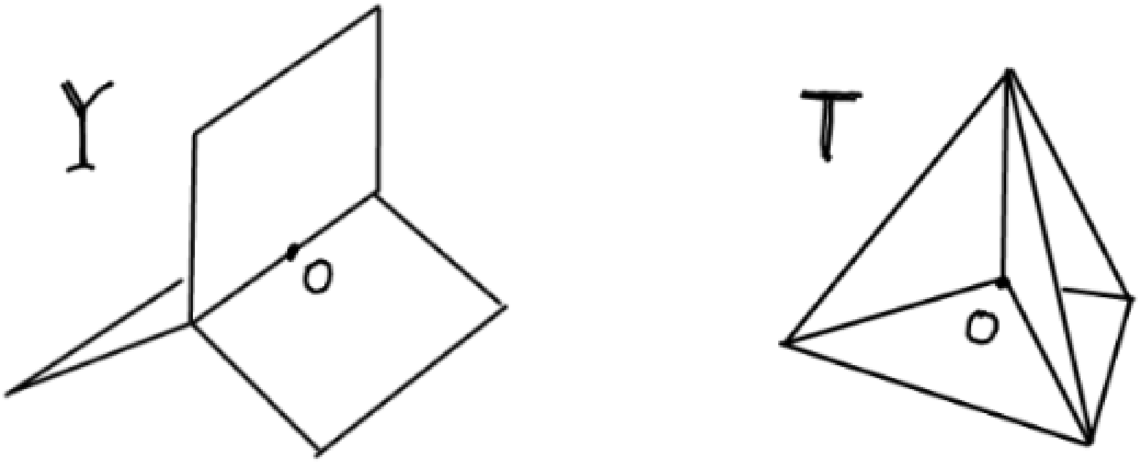
WARNING: not known in general whether

$$\Sigma(\mathcal{E}_k) \longrightarrow \Sigma(\mathcal{E}_0) \quad \text{Hausdorff}$$

NOTE: SIMONS' CONE $\mathcal{C}_0 = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| < |y|\}$ is a MINIMIZER $(\Lambda = 0)$ and is the limit of a 1-parameter family $\mathcal{C}_t$ of minimizers, with $\Sigma(\mathcal{C}_t) = \emptyset$.

REGULARITY THEOREM II

Taylor (1976)
David (2010)



THEOREM. $\mathcal{E}$ (Λ, ρ)-minimizing cluster in $\mathbb{R}^3$.
Assume $\vartheta(\partial\mathcal{E}, 0) \geq \vartheta(\mathcal{C}, 0)$ for some $\mathcal{C} \in \{Y, T\}$.
If for some $r > 0$ small enough we have

- $\partial\mathcal{E}$ close enough to $\mathcal{C}$ in B_r
- $P(\mathcal{E}; B_r)/r^2 - \vartheta(\partial\mathcal{E}, 0)$ small enough

Then $\partial\mathcal{E}$ is LOCALLY $C^{1,\beta}$-DIFFEOMORPHIC to $\mathcal{C}$.

II. STRONGER RESULTS in $\mathbb{R}^2$ and $\mathbb{R}^3$

Regularity Theorem II

Regularity Theorem I

STRATIFIED HAUSDORFF CONVERGENCE of $\partial\mathcal{E}_k$ to $\partial\mathcal{E}_0$

DIFFEOMORPHIC REPRESENTATION

$$\partial\mathcal{E}_k = f_k(\partial\mathcal{E}_0)$$

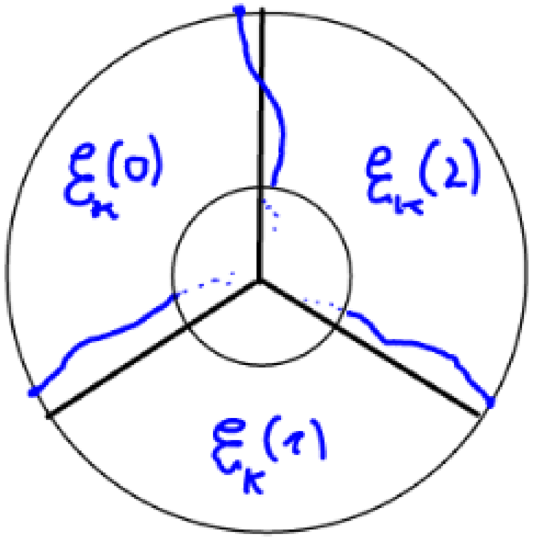
with f_k diffeo $C^{1,\beta}$

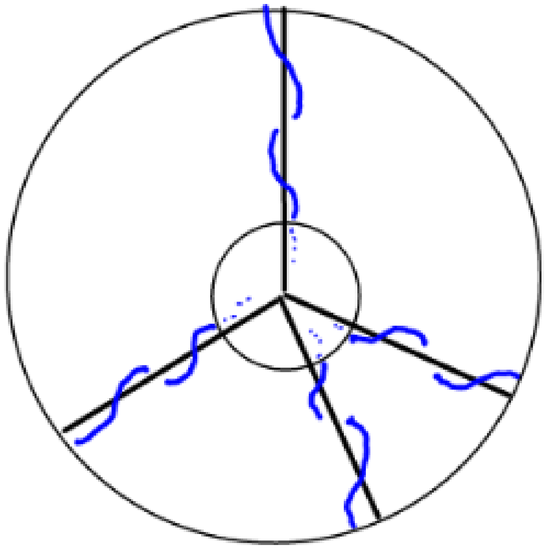
and $\|f_k - \mathrm{Id}\|_{C^1} \to 0$

SHARP QUANTITATIVE INEQUALITY FOR PLANAR 2-BUBBLES

STRATIFIED HAUSDORFF CONVERGENCE

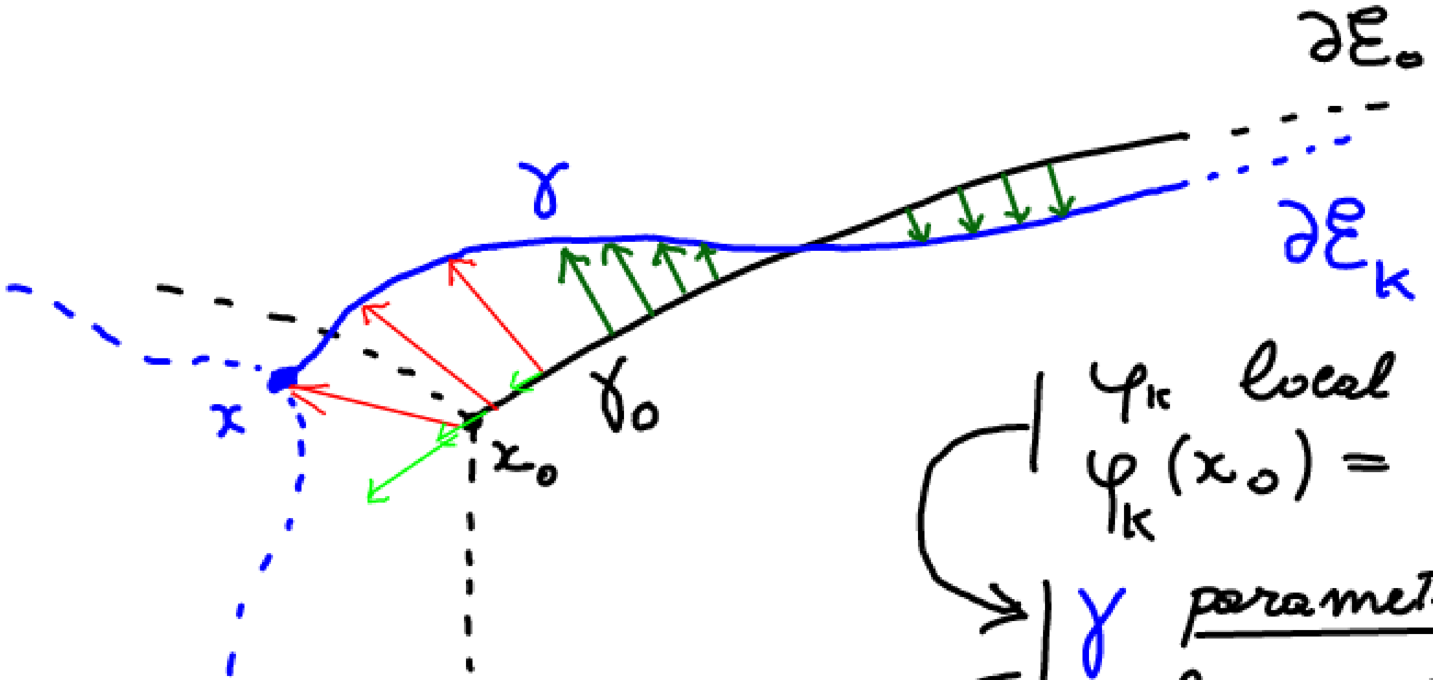
A combination of Regularity Theorems I and II, plus
a TOPOLOGICAL ARGUMENT
to exclude the "SIMONS' CONE PHENOMENON"





thus showing the existence of singularities in $\partial \mathcal{E}_k$ that are close to (and of the same type as) any singularity in $\partial \mathcal{E}_0$, if k large enough.

DIFFEOMORPHIC REPRESENTATION



φ_k local diffeo in $B_\varepsilon(x_0)$
$\varphi_k(x_0) = x$

γ parametrized on γ_0
loc. in x_0 by φ_k

γ as a ZERO LOCUS
near x

γ re-parametrized
on γ_0 via CUT-OFF of TANGENTIAL COMPONENT
plus IMPLICIT FUNCTION THEOREM

SHARP INEQUALITY FOR PLANAR DOUBLE BUBBLES

THEOREM (CICALESE - L. - MAGGI)
Assume $\mathcal{E}_0$ is a STANDARD 2-BUBBLE in the plane.
then there exists $k_0 = k_0(\mathcal{E}_0) > 0$ such that

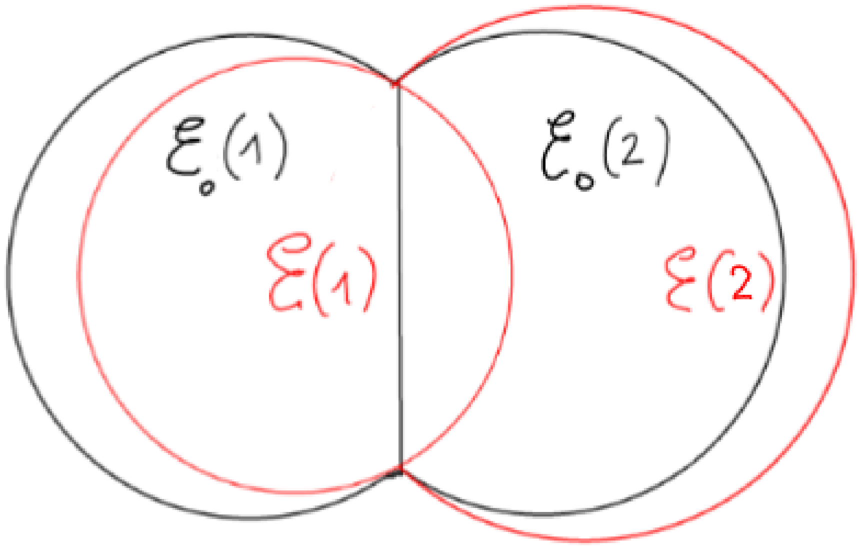
$$\delta(\mathcal{E}) \geqslant k_0 \, \alpha(\mathcal{E})^2$$

for all 2-clusters $\mathcal{E}$ with $\mathrm{vol}(\mathcal{E}) = \mathrm{vol}(\mathcal{E}_0)$.

The proof combines the SELECTION PRINCIPLE with a NON-TRIVIAL FUGLEDE-TYPE ARGUMENT that corresponds to evaluating the STRICT POSITIVITY OF THE II VARIATION of the double bubble, with respect to volume-preserving diffeomorphisms.

TECHNICAL DIFFICULTY: control a system of intertwined POINCARÉ-TYPE ESTIMATES on the 3 boundary arcs.

A geometric idea of the difficulty



$$P(\mathcal{E}(1)) < P(\mathcal{E}_0(1))$$

$$P(\mathcal{E}(2)) > P(\mathcal{E}_0(2))$$

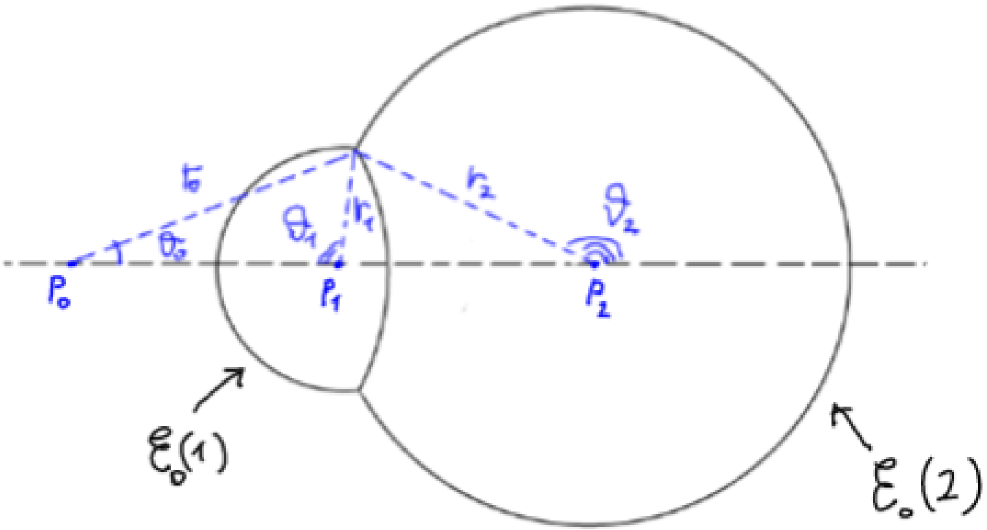
$\longrightarrow$ a SUBTLE COMPETITION even when the singularities do not move

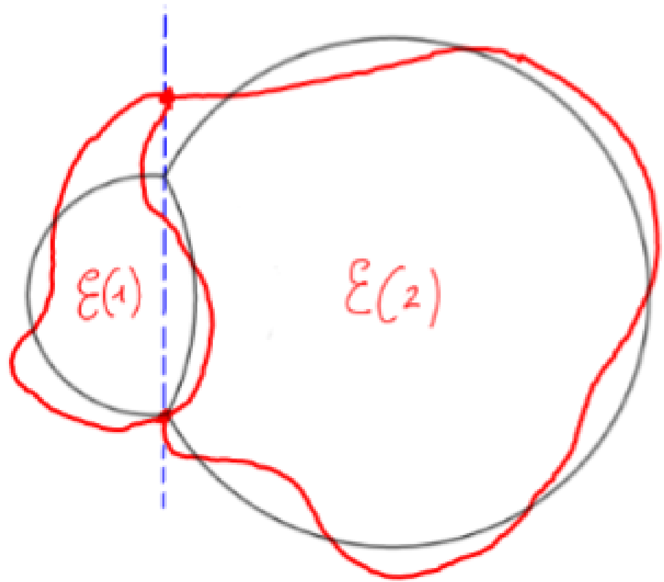
Some details on FUGLEDE for the 2-bubble

ε, σ small $\rightarrow$ $\mathcal{E}$ is a (ε,σ)-perturbation of $\mathcal{E}_0$
iff $\mathrm{vol}(\mathcal{E}) = \mathrm{vol}(\mathcal{E}_0)$ and

$$\partial\mathcal{E} = (1+\sigma)\cdot\bigcup_{k=0}^{2}\left\{(1+u_k(t))\cdot(p_k + r_k e^{it}) : t\in[-\vartheta_k, \vartheta_k]\right\}$$

with $u_k(\pm\vartheta_k) = 0$ and $\|u_k\|_{C^1} < \varepsilon$.





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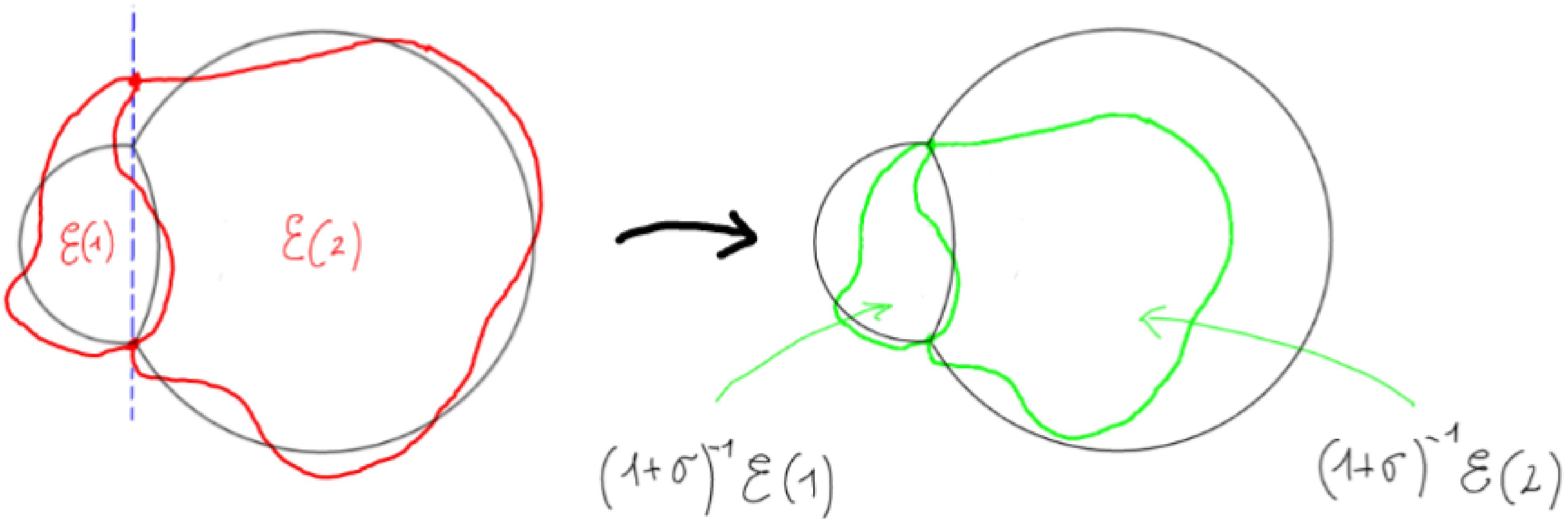

Proposition 1: if $\varepsilon, \sigma \ll 1$ and $\mathcal{E}$ is (ε,σ)-perturbation of $\mathcal{E}_0$, then

$$\delta(\mathcal{E}) \approx \sum_{k=0}^{2} r_k \int_{-\vartheta_k}^{\vartheta_k} \left(\dot{u}_k^2 - u_k^2\right) + P(\mathcal{E}_0)\cdot \sigma^2$$

and

$$\sigma^2 \approx \frac{\left[r_1^2 \int_{-\vartheta_1}^{\vartheta_1} \left(u_1 + \frac{u_1^2}{2}\right) + r_2^2 \int_{-\vartheta_2}^{\vartheta_2} \left(u_2 + \frac{u_2^2}{2}\right) \right]^2}{4\,(V_1 + V_2)^2}$$

PROBLEM: $\int_{-\vartheta_k}^{\vartheta_k} \left(\dot{u}_k^2 - u_k^2\right)$ can be NEGATIVE !

$$\delta(\varepsilon) \approx \sum_{k=0}^{2} r_k \int_{-\vartheta_k}^{\vartheta_k} (\dot{u}_k^2 - u_k^2) \quad + P(\varepsilon_0) \cdot \sigma^2$$

... a system of

intertwined Poincaré-Type estimates

plus contribution of

singularity displacement σ

To be analyzed ...

Two key Poincaré-Type estimates

Lemma 1 if $\vartheta \in (0,\pi)$ then

$$\inf\left\{\int_{-\vartheta}^{\vartheta} \dot{u}^2 - u^2 : \; u \in W_0^{1,2} \text{ and } \int_{-\vartheta}^{\vartheta} u = s\right\} = \frac{-s^2}{2(\vartheta - \tan\vartheta)}$$

> 0 if $\vartheta < \frac{\pi}{2}$ but

< 0 if $\vartheta > \frac{\pi}{2}$

Lemma 2 if $\vartheta \in (0,\pi)$ then for all $u \in W_0^{1,2}$

$$\left(\int_{-\vartheta}^{\vartheta} u\right)^2 \leq \frac{\vartheta(\pi-\vartheta)}{\pi}\int_{-\vartheta}^{\vartheta} u^2 \implies \int_{-\vartheta}^{\vartheta}\left(\dot{u}^2 - u^2\right) \geq \frac{\pi^2-\vartheta^2}{2(\pi^2+\vartheta^2)}\|u\|_{W^{1,2}}^2$$

CONCLUSION: for a PLANAR 2-BUBBLE $\mathcal{E}_0$

Fuglede-type estimate: $$\delta(\mathcal{E}) \geqslant c_F \cdot \sum_{i=0}^{2} \int_{-\theta_k}^{\theta_k} u_k^2$$

$$\Downarrow$$

Sharp quantitative inequality: $$\boxed{\begin{array}{l} \delta(\mathcal{E}) \geqslant \kappa_0 \, \alpha(\mathcal{E})^2 \\ \forall \mathcal{E} \text{ with } \mathrm{vol}(\mathcal{E}) = \mathrm{vol}(\mathcal{E}_0) \end{array}}$$