“Una caracterización de superficies isoparamétricas en curvatura constante vía superficies mínimas”

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Introduction

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- We prove that necessarily the initial surface is isoparametric.
- It is also shown, that the curves are necessarily geodesics.
- On other hand, using the classification of isoparametric surfaces it is possible to prove that they have the above property along every geodesic.
  So, we have a characterization.

Normal ruled surface along a curve

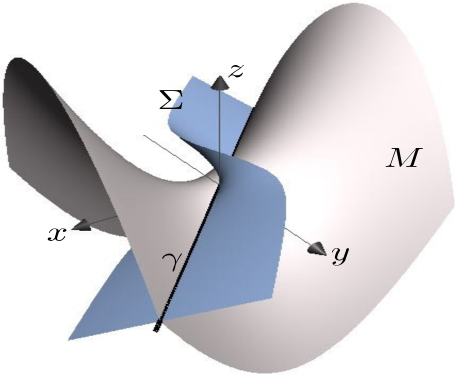
Definition

Let $M \subset \overline{M}^3$ be an oriented surface, isometrically immersed in a complete, oriented, three dimensional riemannian manifold and let $\gamma \subset M$ be a curve. Let ξ be the unitary vector field orthogonal to M compatible with the orientation. We define the surface

$$\Sigma := \Sigma_\gamma = \{\exp_{\gamma(s)}(t\xi(s)) \in \overline{M}, t \in (-\epsilon, \epsilon)\},$$

where $\exp_{\gamma(s)} : T_{\gamma(s)}\overline{M} \longrightarrow \overline{M}$ denotes the exponential map of $\overline{M}$ at $\gamma(s)$. We call it “the ruled normal surface (to M) along γ”. Observe that Σ is an embedded surface near γ. See figure 5.

Figure: Ruled surface Σ along a curve γ in M



A motivation by Bonnet

Proposition

(Bonnet)
Let M be an immersed surface in $\mathbb{R}^3$*. A curve* $\gamma \subset M$ *is line of curvature if and only if* Σ_γ *is flat.*

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(Lamoneda-RH)
Let $M \subset \overline{M}^3$, $\gamma \subset M$ and $\Sigma = \Sigma_\gamma$. The curve γ is a line of curvature of M if and only if, along γ, the curvature of Σ is equal to the sectional curvature of $\overline{M}$ in the tangent planes of Σ.

We want to improve and generalize the next result

Theorem

(R. López-RH)

Let M be a connected surface in Euclidean 3-space $\mathbb{R}^3$. If there exist four geodesics through each point of M with the property that the ruled normal surface constructed along these geodesics is a surface with constant mean curvature, then M is a plane, a sphere or a right circular cylinder.

Notation

- M is a surface isometrically immersed in the ambient 3-manifold $\overline{M}$;

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- $S^M_\xi(X) = -\overline{\nabla}_X \xi$ denotes the shape operator of the submanifold M with respect to the normal vector field ξ.

A result inspired by Bonnet

Figure: Pierre Ossian Bonnet

Lemma

(Lamoneda-RH)
Let $M \subset \overline{M}^3$, $\gamma \subset M$ and Σ as before. Then γ is a geodesic of M if and only if Σ has zero mean curvature along γ.

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Let M be an isoparametric surface

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- in $\mathbb{S}^3$. Then M is either a totally geodesic 2-sphere, or a totally umbilical 2-sphere or Hopf tori over circles.

Observation

The umbilic surfaces in space forms are either

- totally geodesic surfaces in any of the three space forms, or
- spheres in euclidean $\mathbb{R}^3$, or
- spheres in $\mathbb{S}^3$, or
- spheres, horospheres and equidistant surfaces to totally geodesic hyperbolic planes in $\mathbb{H}^3$.

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Properties of tubes around geodesics in space forms

In any ambient space form:

- An equidistant surface of an isoparametric surface is again isoparametric.

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- An equidistant surface to a tube, around a geodesic, is again a tube around the same geodesic.
- If γ is a geodesic of a tube M then the corresponding curve γ_ϵ in a equidistant surface M_ϵ of the tube is a geodesic of M_ϵ too.

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- M is an isoparametric surface.
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- A geodesic of M is a helix of the ambient and makes a constant angle with any line of curvature of M. In this case we are considering geodesics of M that are not lines of curvature.

Definition

A *ruled* minimal surface Σ in a three dimensional Riemannian manifold is a minimal surface which has a foliation by geodesics of the ambient.

Ruled minimal surfaces in space forms

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Parametrization by Lawson, Do Carmo-Dajczer The classification of ruled minimal surfaces is well known in the case when the ambient is a space form:

- In Euclidean space, Σ is (an open part of) a plane or a helicoid.

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- (H. B. Lawson) In $\mathbb{S}^3$, Σ is (an open part of) a surface parametrized by

$$\Psi(x, y) = (\cos(ax)\cos(y), \sin(ax)\cos(y), \cos(x)\sin(y), \sin(x)\sin(y))$$

for some $a > 0$.

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- (Do Carmo, Dajczer) In $\mathbb{H}^3$, Σ is (an open part of) a surface parametrized by $\Psi(x,y) =$ $(\cosh(ax)\cosh(y), \sinh(ax)\cosh(y), \cos(x)\sinh(y), \sin(x)\sinh(y))$.

Figure: Obrigado Manfredo Do Carmo!

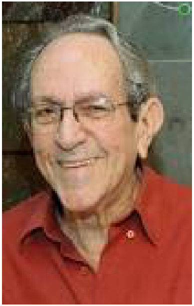
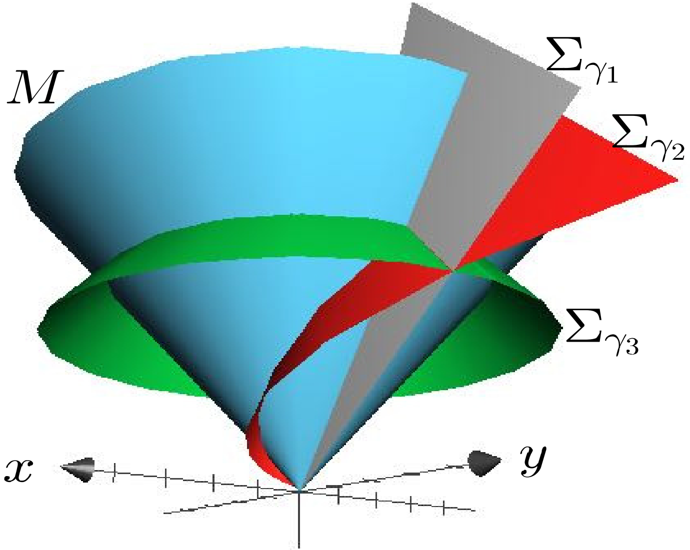


Figure: A tube M in $\mathbb{H}^3$ with three orthogonal minimal surfaces along geodesics $\gamma_1, \gamma_2, \gamma_3$



Isoparametric surface in $\mathbb{H}^3$

Example

A tube around a vertical geodesic

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- *The geodesics* γ_1, γ_3 *are lines of curvature of the* M*.*

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- *The geodesics* γ_1, γ_3 *are lines of curvature of the* M*.*
- *The geodesic* γ_2 *is a helix in* $\mathbb{H}^3$*.*

Example

...

- *Three geodesics* $\gamma_1, \gamma_2, \gamma_3$.
- *Three ruled surfaces* $\Sigma_{\gamma_1}, \Sigma_{\gamma_2}, \Sigma_{\gamma_3}$.
- $\Sigma_{\gamma_1} = \{(4\cos(\ln(4))s(t+1), 4\sin(\ln(4))s(t+1), 4s(1-t)), s \in (0, 1.6), t \in (-0.3, 0.3)\}$,
- $\Sigma_{\gamma_2} = \{(e^s\cos(s)(t+1), e^s\sin(s)(t+1), e^s(1-t)) | s \in (-10, 1.8), t \in (-0.3, 0.3)\}$,
- $\Sigma_{\gamma_3} = \{(4\sin(s)(t+1), 4\cos(s)(t+1), 4(1-t)), s \in (0, 10), t \in (-0.3, 0.3)\}$.
- *The surface* Σ_1 *is part of a vertical plane.*
- *The surface* Σ_2 *is only a minimal surface.*
- Σ_3 *is part of a half sphere orthogonal to the plane* xy.

Let us assume now that $\overline{M}^3 = \mathcal{Q}$ is a space form, i.e. $\mathcal{Q} = \mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$.
We are going to use the standard embeddings of $\mathbb{S}^3$ in $\mathbb{R}^4$ and of $\mathbb{H}^3$ in $\mathbb{R}^{3,1}$:
$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$ and
$\mathbb{H}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 - w^2 = -1, w > 0\} \subset \mathbb{R}^{3,1}$.
We also denote by D the Levi-Civita connection on $\mathbb{R}^4$ (or $\mathbb{R}^{3,1}$) and we let $\langle , \rangle$ denote either Euclidean or Lorentzian inner product in $\mathbb{R}^4$ (or $\mathbb{R}^{3,1}$).

Extrinsic parametrization of the surface Σ

As before, assume we have a surface $M \subset \mathcal{Q}$, γ a curve in M and Σ the ruled normal surface to M along γ. Parametrize Σ using the embedding of $\mathcal{Q}$ in $\mathbb{R}^4$ (resp. $\mathbb{R}^{3,1}$) as follows

$$\phi(s,t) = f(t)\gamma(s) + g(t)\xi(s),$$

where

$$\begin{array}{rcc} & S^3 & \mathbb{H}^3 \\ f(t) = & \cos t & \cosh t \\ g(t) = & \sin t & \sinh t \end{array}.$$

In the case of $\mathcal{Q} = \mathbb{R}^3$, we set $\phi(s,t) = \gamma(s) + t\xi(s)$.

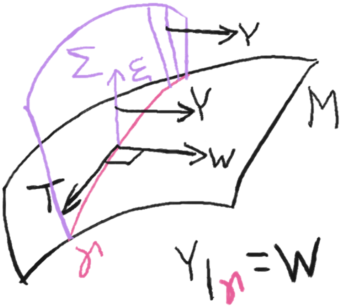
Recall that, along γ, we have a unit vector field W orthogonal to Σ. Extend W to a vector field Y (tangent to $\mathcal{Q}$) orthogonal to Σ everywhere.

Lemma

Let $M \subset \mathcal{Q}$ be an oriented surface isometrically immersed in the space form $\mathcal{Q}$ and let ξ be an unitary orthogonal vector field to M. Let $\gamma \subset M$ and Σ be as in Definition 1. Extend W to a unit vector field Y normal to Σ at every point. Let ϕ be the parametrization of Σ inside R^3, $\mathbb{R}^4$ or $\mathbb{R}^{3,1}$ defined above. Then Σ is a minimal surface if and only if $\langle Y, \phi_{ss} \rangle = 0$.

The mimimality condition

Figure: Y is normal along Σ



Definition

A curve in a three dimensional Riemannian manifold is called a *helix* if its curvature and torsion are non zero constant functions.

Proposition

If Σ is a minimal surface in $\mathcal{Q}$ (i.e. $H_{\Sigma\subset\mathcal{Q}} = 0$), then γ is either

- *a helix in $\mathcal{Q}$ (i.e. κ and τ are non-zero constants); or,*
- *a planar (i.e. $\tau = 0$) line of curvature of M, .*

Corollary

If Σ is a minimal surface in $\mathcal{Q}$ and the torsion of γ vanishes then Σ is totally geodesic.

Lemma

Let $M \subset \overline{M}^3$ be an oriented surface isometrically immersed in a three dimensional riemannian manifold and let ξ be a unitary orthogonal vector field to M. Let $\gamma \subset M$ be a geodesic in M with unit tangent vector T. If γ is a helix in $\overline{M}^3$ then along γ we have that $(\nabla_T S_\xi^M)(T) = 0$.

Definition

A surface in a three dimensional manifold is called parallel if its shape operator is parallel: $\nabla_X S^M_\xi = 0$, for every direction $X \in T_pM$.

Corollary

Let $M \subset \mathcal{Q}$ be an oriented surface isometrically immersed in a three dimensional space form and let ξ be a unitary orthogonal vector field to M. Let us assume that at some point $p \in M$ two geodesics go through, which, moreover, are helices in $\mathcal{Q}$. Then, at p, the shape operator is parallel.

Proposition

A surface in a three dimensional space form is parallel if and only if either

- *it is isoparametric, the lines of curvature are geodesics and, therefore, it is flat; or,*
- *it is umbilic.*

Characterization of isoparametric surfaces in space forms

Theorem

(Lamoneda-RH)

Let M be a complete surface immersed in a three dimensional space form $\mathcal{Q}$. Let us assume that through every point $p \in M$, there are three different curves whose ruled normal surface is minimal. Then M is an isoparametric surface.

For two geodesics is false

Example

- *If one only assumes that through each point of M one has 2 geodesics whose normal surface is minimal, then the above result is false.*
- *The easiest example, perhaps, is to take as M the product of any curve α in the xy-plane cross the z-axis. Through each point of M you have two geodesics (the vertical axis and a parallel to α) for which the normal surfaces are planes.*

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