BERNSTEIN PROBLEMS IN HIGHER CODIMENSION

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ABSTRACT. The recent decades admit intensive research devoted to the study of **minimal submanifolds in higher codimension**, in particular, **special Lagrangians**. Lawson and Osserman [7] studied non-existence, non-uniqueness and irregularity of solutions of the minimal surface system. Unlike Bernstein's Theorem in $\mathbb{R}^3$, for general higher codimension $n \geq 2$, there exist plenty of entire 2-dimensional minimal non-planar graphs in $\mathbb{R}^{n+2}$. For Bernstein type results in higher codimension, we refer to [1, 3, 5, 6, 9, 10, 12, 13, 14, 15, 16].

In this talk, we survey various Bernstein type theorems in higher codimension. In particular, we meet a geometric proof of the characterization of entire special Lagrangian graphs in $\mathbb{R}^4$. An important property of special Lagrangian submanifolds is the interesting fact that they are volume minimizing in their homology classes.

Theorem 0.1 (**Bernstein type problem for entire special Lagrangian graphs** [1, 16]). *When a minimal surface in Euclidean space $\mathbb{R}^4$ becomes an entire gradient graph (x, y, f_x, f_y) for some function $f : \mathbb{R}^2 \to \mathbb{R}$, the potential function f should be harmonic or quadratic. Or equivalently, given any constant $\theta \in \mathbb{R}$, the entire solutions of special Lagrangian equation*

$$\cos\theta\,(f_{xx} + f_{yy}) + \sin\theta\left(1 - f_{xx}f_{yy} + {f_{xy}}^2\right) = 0$$

are harmonic or quadratic.

We show that it can be reduced to the following classical result.

Theorem 0.2 (Jörgens' Theorem [4]). *The only entire solutions of the unimodular Hessian equation $f_{xx}f_{yy} - {f_{xy}}^2 = 1$ are quadratic polynomial functions.*

We provide a minimal-surface proof of Jörgens' Theorem. The Harvey-Lawson Theorem [2] shows that the entire gradient graph Σ given by (x, y, f_x, f_y) becomes a minimal surface in $\mathbb{R}^4$. We then employ the extended Osserman's Lemma [8] to show that its generalized Gauss map [11] $\mathcal{G} : \Sigma \to \mathcal{Q}_2 \subset \mathbb{CP}^3$ is constant. Here, the variety $\mathcal{Q}_2 = \{\, [\zeta] = [\zeta_1 : \cdots : \zeta_4] \in \mathbb{CP}^3 : {\zeta_1}^2 + \cdots + {\zeta_4}^2 = 0 \,\}$ is a model for the Grassmannian manifold $\mathcal{G}_{2,2}$ of oriented planes in $\mathbb{R}^4$.

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