Minimal surfaces in finite volume hyperbolic 3-manifolds N^3 and in $M^2 \times \mathbb{S}^1$, M^2 a complete hyperbolic surface of finite area

Harold Rosenberg

Joint work with P. Collin and L. Hauswirth

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We will discuss properly immersed minimal surfaces in such N and $M \times \mathbb{S}^1$

Ends of N

Ends E of N can be parametrized by

$$\{(x, y, t) \in \mathbb{R}^3; y \geq y_0 > 0\}/\Gamma$$

Metric in N : $g = \frac{dx^2+dy^2+dt^2}{y^2}$

Γ=group generated by two linearly independent translations of the $(x, ., t)$ plane.

$$E = \cup_{y \leq y_0} \mathbb{T}(y)$$

$\mathbb{T}(y)$= CMC-1 tori quotient of the horosphere by Γ.

This end E of N is called a hyperbolic cusp end.

Ends of $M \times \mathbb{S}^1$

Ends E^* of $M \times \mathbb{S}^1$ can be parametrized by

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Metric in $M \times \mathbb{S}^1$: $g = \frac{dx^2+dy^2}{y^2} + dt^2$

Γ=group generated by two linearly independent translations of the $(x, ., t)$ plane of the form

$$(x, y, t) \to (x, y, t+h) \text{ and } (x, y, t) \to (x+\lambda, y, t)$$

$$E^* = \cup_{y \leq y_0} \mathbb{T}(y)$$

$\mathbb{T}(y)$= CMC-1/2 tori quotient of the horosphere by Γ.

Remarks

If L is a line at infinity $\{y = 0\}$, and $P = L \times [y_0, \infty)$ then P is a totally geodesic (cusp) end in E and P is a minimal surface in E^*.

P is totally geodesic in E^* only when L is horizontal or vertical.

P is vertical at infinity (as $y \to \infty$) in E^*.

FINITE TOTAL CURVATURE THEOREM (Collin,Hauswirth, -)

A finite topology properly immersed orientable minimal surface Σ in N, or in $M \times \mathbb{S}^1$, has finite total curvature equal to $2\pi\chi(\Sigma)$.

Moreover, the ends of Σ are asymptotic to standard ends : In N there is one standard end (a cusp), and in $M \times \mathbb{S}^1$ there are three standard ends

1. A horizontal end= (a cusp end of M)$\times\{point\}$,
2. A vertical end=(a geodesic ray of M)$\times\mathbb{S}^1$,
3. A helicoidal end with axis at infinity.

RIGIDITY THEOREM (Mazet, -)

Let X be a complete Riemannian 3-manifold with sectional curvature $K_X \leq -1$. Let T be a constant mean curvature one torus embedded in X. Then T separates X and its mean convex side is isometric to a hyperbolic cusp E.

Mazet and I proved more, inspired by an (unpublished) result of Calabi.

If $\mathbb{S}^2$ is given a metric of curvature between 0 and 1 and γ is a simple closed geodesic of $\mathbb{S}^2$, then the length of γ is at least 2π.

Calabi proved that when the length is 2π (on a complete surface M with curvature between 0 and 1) then M is isometric to the unit sphere or to $\mathbb{S}^1 \times \mathbb{R}$.

RIGIDITY THEOREM (Mazet, -)

We proved that if M is a complete 3-manifold of sectional curvature between 0 and 1 and Σ is a minimal embedded 2-sphere in M, then the area of Σ is at least 4π.

If the area equals 4π then M is isometric to the unit 3-sphere or to a quotient of $\mathbb{S}^2 \times \mathbb{R}$ by the group of isometries

$$(p, t) \to (\alpha(p), t + t_0)$$

α an isometry of $\mathbb{S}^2$, $t_0 \neq 0$.

Some Introductary Remarks

The existence of closed orientable incompressible surfaces in a 3-manifold is a great help in understanding the manifold.

Haken used incompressible surfaces in closed irreducible 3-manifolds (now called Haken manifolds) to determine knot and link invariants.

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It was thought that all irreducible 3-manifolds were Haken manifolds, (with the exception of some Seifert fibrations), until Thurston discovered many hyperbolic link complements, which (after performing Dehn surgery) contained no incompressible surfaces.

Some Introductory Remarks

This encouraged considerable research on totally geodesic surfaces in hyperbolic manifolds. Here are some (of many) theorems on totally geodesic submainifolds.

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Alan Reid : There are infinitely many hyperbolic knot complements that contain no closed totally geodesic surfaces and do contain non compact such surfaces.

Gordon, Litherhand : The complement of the figure eight contains many closed totally geodesic surfaces. There is no embedded example.

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The reason there is no embedded such surface, is that a closed embedded essential surface (incompressible and not boundary parallel) in N, implies any cyclic branched cover is Haken. For the figure eight knot the double branch cover is a Lens Space, which has finite fundamental group.

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Perspective : Understand minimal surface theory in hyperbolic 3-manifolds.

Applications of the Finite Total Curvature Theorem

The theorem, together with the Gauss equation, gives topological obstructions for the existence of proper minimal immersions of prescribed topology in N or in $M \times \mathbb{S}^1$:

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The theorem, together with the Gauss equation, gives topological obstructions for the existence of proper minimal immersions of prescribed topology in N or in $M \times \mathbb{S}^1$:

•A plane $\mathbb{R}^2$ can not be so realized in N or in $M \times \mathbb{S}^1$,

•An annulus $\mathbb{S}^1 \times \mathbb{R}$ can not be realized in N. If Σ is a properly immersed minimal annulus in $M \times \mathbb{S}^1$, then $\Sigma = \gamma \times \mathbb{S}^1$, γ a complete geodesic of M.

Applications of the Finite Total Cuvature Theorem

More generally, suppose Σ is an orientable surface of genus g with n punctures. Then if Σ can be properly immersed in N or $M \times \mathbb{S}^1$ as a minimal surface, we have

$$\int_\Sigma K_\Sigma = 2\pi(2 - 2g - n) = \int_\Sigma K_{ext} + \int_\Sigma K_{sect}$$

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Since $K_{ext} \leq 0$ and

$$-1 \leq K_{sect} \leq 0 \text{ in } M \times \mathbb{S}^1$$
$$K_{sect} = -1 \text{ in } N$$

we have

$$2 - 2g - n \leq 0$$

and equality $\iff K_{ext} = K_{sect} = 0$.

Applications of the Finite Total Cuvature Theorem

In N, we obtain an area estimate :

$$|\Sigma| = \int_\Sigma K_{ext} + 2\pi(2g+n-2) \leq 2\pi(2g+n-2)$$

and equality means Σ is totally geodesic.

Question : When do such totally geodesic surfaces exist ?

When N is the complement of the figure eight knot, with a complete hyperbolic metric of finite volume, then there is no totally geodesic closed embedded surface.

Existence results

Theorem : There is a properly embedded minimal surface Σ in N of finite topology. Σ is incompressible and stable.

Questions : Can an embedded incompressible surface in N be isotoped to an area minimizing surface ? In particular, can one find a once punctured embedded minimal torus in the complement of the figure eight knot ?

Is there a compact embedded minimal surface in any N ?

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Remark Thurston proved there are compact hyperbolic 3-manifolds that contain no incompressible surface.

Bounded curvature and Stable surfaces

Theorem : (C.-H.-R. + Laurent Mazet) Let Σ be a properly embedded minimal surface in N of bounded curvature. Then Σ has finite topology.

Corollary A properly embedded stable minimal surface in N has finite topology (hence finite total curvature)

Question : Does a properly embedded stable surface in $M \times \mathbb{S}^1$, of bounded curvature, have finite topology ?

In $M \times \mathbb{S}^1$ we have many examples of properly embedded minimal surfaces, including many stable ones as above

Examples in $M \times \mathbb{S}^1$

We describe two examples in $M \times \mathbb{S}^1$ and then examples in N.

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1. The simplest complete hyperbolic surface is a 3-punctured sphere. It has a unique complete hyperbolic metric of area π, and any other such surface has area greater than π.

Let M be the complete hyperbolic 3-punctured sphere. We will realize M in the unit disc model of $\mathbb{H}^2$.

Σ will be an embedded minimal surface in $M \times \mathbb{S}^1$ with three ends ; two helicoidal and the other horizontal.

Examples in $M \times \mathbb{S}^1$

The domains and notation we now introduce will be used in all the examples we describe.
Let Γ be the ideal triangle in the disk model of $\mathbb{H}^2$ with vertices $A = (0, 1), B = (0, -1), C = (-1, 0)$ and sides a, b, c as indicated in figure1.

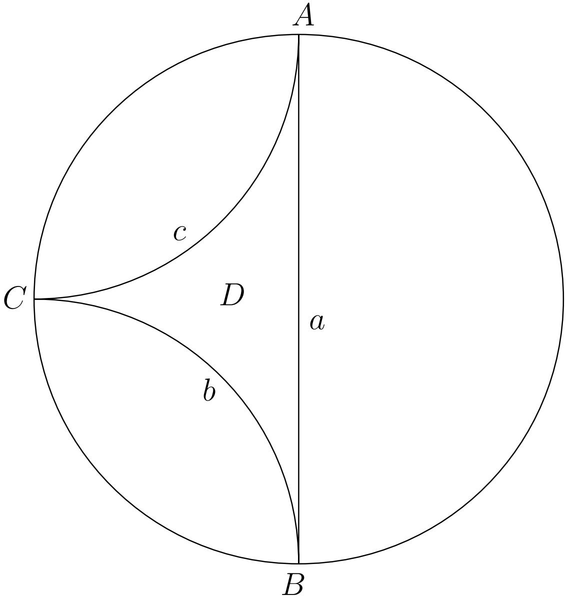

FIGURE: Ideal triangle (ABC) in $\mathbb{H}^2$

Examples in $M \times \mathbb{S}^1$

Let Σ, be the minimal graph over the domain D bounded by Γ, taking the values 0 on b and c and $h > 0$ on a.

Extend Σ, to an entire minimal graph $\tilde{\Sigma}$ over $\mathbb{H}^2$ by rotation by π in all the sides of Γ, and the sides of the triangles thus obtained.

In figure 2, we indicate some of the reflected triangles and the values of the graph $\tilde{\Sigma}$ on their sides.

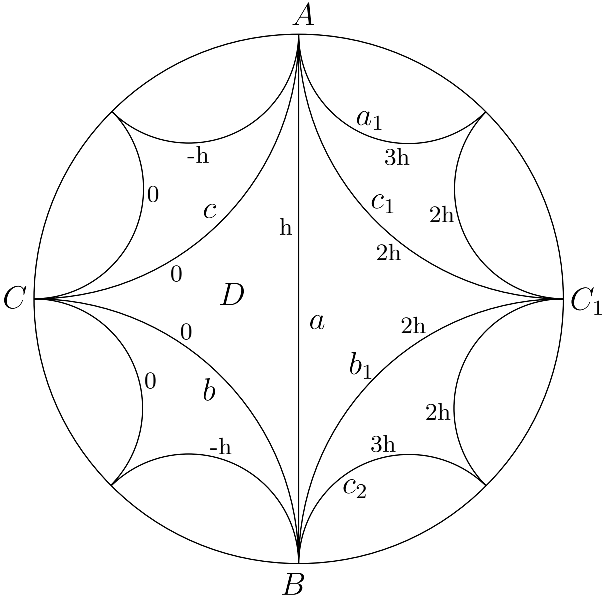

FIGURE: Value of the graph $\tilde{\Sigma}$ on geodesics

Examples in $M \times \mathbb{S}^1$

Let D be the domain bounded by Γ. Let ψ_A be the parabolic isometry with fixed point A which takes the geodesic c to c_1 and a to a_1 ; $\psi_A = R_{c_1} R_a$, where R_γ denotes reflection in the geodesic γ.

Let ψ_B be the parabolic isometry of $\mathbb{H}^2$ leaving B fixed, taking b to b_1 and a to c_2 ; $\psi_B = R_{b_1} R_a$.

Notice that the group of isometries of $\mathbb{H}^2 \times \mathbb{R}$, generated by $T(2h) \circ \psi_A$ and $T(2h) \circ \psi_B$, leaves $\tilde{\Sigma}$ invariant.

Examples in $M \times \mathbb{S}^1$

The quotient of this graph by the group yields a minimal surface Σ embedded in $M \times (\mathbb{R}/T(2h))$.

Here M is the 3-punctured sphere obtained by identifying the sides of $D \cup R_a(D)$ by ψ_A, ψ_B (c with c_1, b with b_1). M is hyperbolic and has finite area.

Σ is a 3-punctured sphere with 3-ends ; two helicoidal and the other horizontal.

Σ has total curvature -2π and is stable : (Σ is transverse to the killing field $\partial/\partial t$).

Example 2 in $M \times \mathbb{S}^1$

A helicoid with one helicoidal end in $M \times \mathbb{S}^1$, M a once punctured torus.

The surface Σ will be topologically the connected sum of a 2-torus and projective plane, punctured in one point.

M is the once punctured torus obtained from the ideal quadrilateral Q_1 in $\mathbb{H}^2$, with 4 vertices $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$ by identifying opposite sides.

Let S be the third quadrant of Q_1 :
$S = \{(x, y) \in Q_1; x \leq 0, y \leq 0\}$. For $h > 0$, let Σ_1 be the minimal graph over S with boundary values indicated in figure 3.

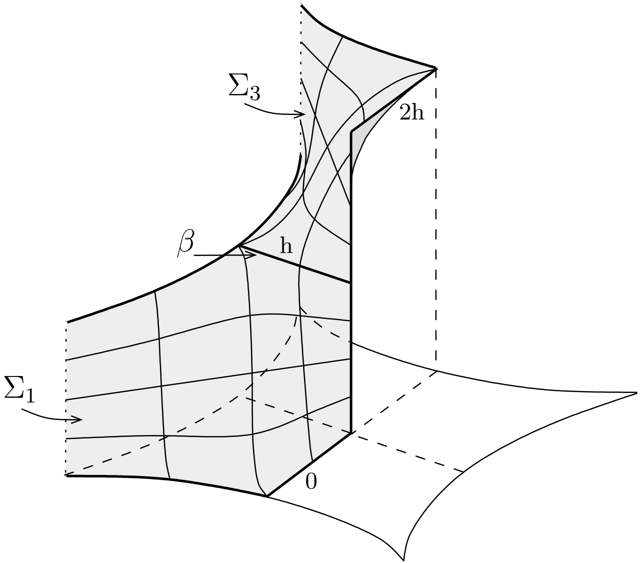

FIGURE: Σ_1 be a minimal graph over S

Example 2 in $M \times \mathbb{S}^1$

Let Σ_3 be the reflection of Σ_1 through β (cf figure 3) ; Σ_3 is between heights h and $2h$ and is a graph over the second quadrant of Q_1.

Then rotate $\Sigma_1 \cup \Sigma_3$ by π through the vertical axis between $(0,0)$ and $(0,2h)$, to obtain $\Sigma_2 \cup \Sigma_4$; Σ_4 is a graph over the fourth quadrant of Q_1. Σ is the union of the four pieces Σ_1, through Σ_4, identified along the boundaries as follows.

First we consider identifying opposite sides of Q_1 be the hyperbolic translations sending the opposite side to the other.

Example 2 in $M \times \mathbb{S}^1$

Then we can quotient by $T(2h)$ or by $T(4h)$. The first quotient gives an non orientable surface in $M \times \mathbb{S}^1$ with one helicoid type end.

The second gives an orientable surface of total curvature -8π with two helicoidal type ends (it is a double cover of the first example). Topologically the first example is the connected sum of a once punctured torus and a projective plane. The second surface is 2 punctured orientable surface of genus two.

Example 2 in $M \times \mathbb{S}^1$

The reader can see the helicoidal structure of Σ by going along a horizontal geodesic on Σ at $h = 0$, from one puncture to the other. Then spiral up Σ along a helice going to the horizontal geodesic at height h. Continue along this geodesic to the other (it's the same) puncture and spiral up the helices on Σ to height $2h$. If we do this right, we are back where we started.

Examples in N

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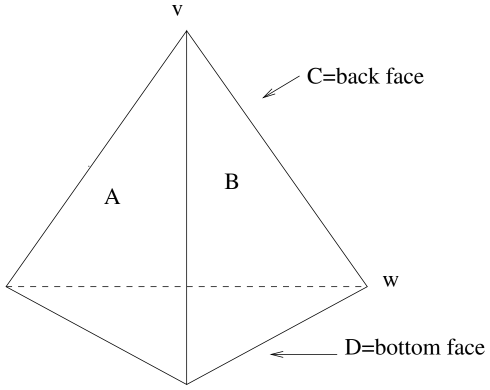
One can think in the Klein model (or in the Poincaré model) of $\mathbb{H}^3$ as the open unit ball. Take T to be a regular Euclidean tetrahedron with 4 vertices on the unit sphere= $\partial_\infty(\mathbb{H}^3)$, and dihedral angle $\alpha = \pi/3$.


v
C=back face
B
A
w
D=bottom face

Examples in N

1. Identify the face A with B by rotation about v by $2\pi/3$.
2. Identify C with D by rotation about w by $2\pi/3$.
3. The quotient has one vertex (at infinity), two faces, and one 3-cell.

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Along each point of the edge in the quotient, there are 6 faces making an angle $\pi/3$. So a neighborhood of the point is a ball in $\mathbb{H}^3$, and the quotient N is a hyperbolic manifold with one end. The link of the vertex at infinity is a Klein bottle so N is non orientable.

Examples in N

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C. Adams proved it has the smallest possible volume (1.01) among complete hyperbolic 3-manifolds of finite volume.

Examples in N

The orientable two sheeted covering of N is the complement of the figure eight knot in $\mathbb{S}^3$. This manifold $\tilde{N}$ was shown to have a hyperbolic structure by Riley and Jorgensen. It is discussed in detail in the book of Thurston.

Examples in N

Theorem (Jorgensen) Given $V > 0$, there are a finite number of complete, non compact hyperbolic 3-manifolds N_i with volume $\leq V$.

Theorem (Thurston) Any compact hyperbolic 3-manifold with volume $\leq V$, is obtained from one of the N_i above by hyperbolic Dehn surgery.

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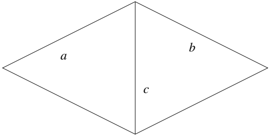
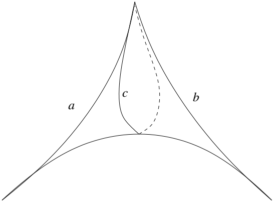
Σ is obtained by gluing together the two faces, A and B, C and D, giving a losange. The edges of this losange are identified as in the figure which gives a 3-punctured sphere :





Minimal surfaces in N

Here is a smooth immersion of a Klein bottle punctured in 2 points as a totally geodesic surface.

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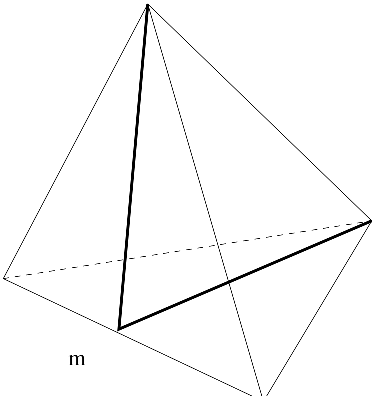
For each edge, take the plane of $\mathbb{H}^3$ passing through the bisectrice of the edge and the edge of T opposite to the first edge



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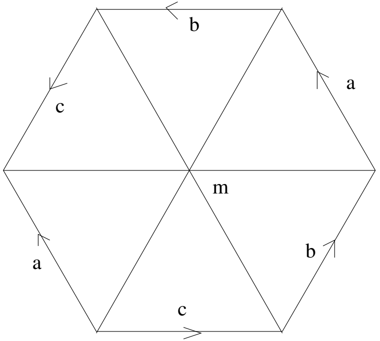
The six triangles make a hexagon at m with the boundary identifications indicated in figure



Minimal surfaces in N

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This is a Klein bottle with a puncture at $[bc]$ and another at $[ab]$.

Theorem There is no closed embedded totally geodesic surface in N.

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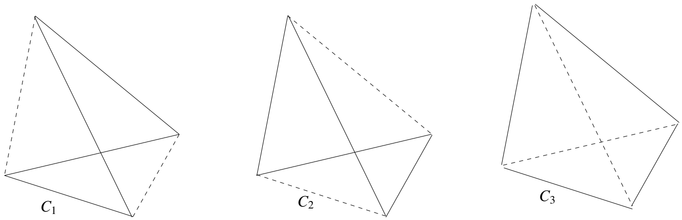
Consider the 3 geodesic polygons of the 1-skeleton of T without the vertices :



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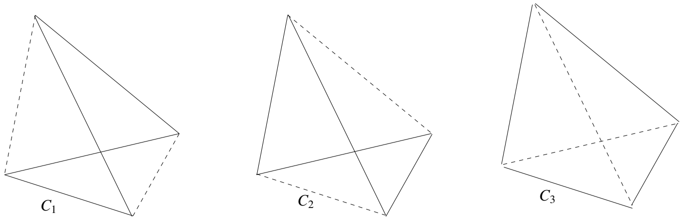
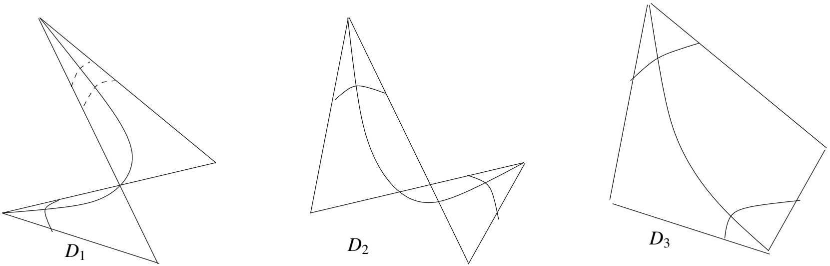


Notice that rotation by $2\pi/3$ about v sends C_1, to C_3, and rotation by $2\pi/3$ about w sends C_1 to C_2 (and $Rot_v(2\pi/3)(C_2) = C_1$).

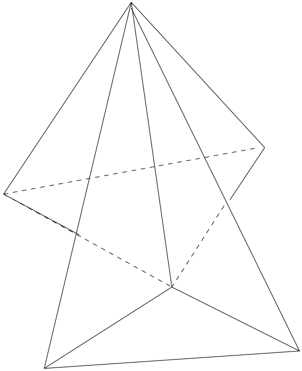
Minimal surfaces in N

Each C_k bounds a unique minimal disk D_k in $\mathbb{H}^3$ (contained in T) :



Minimal surfaces in N

Rotation by π about an edge of T smoothly extends the D_i passing through this edge to the unique $D_j \neq D_i$ sharing this same edge



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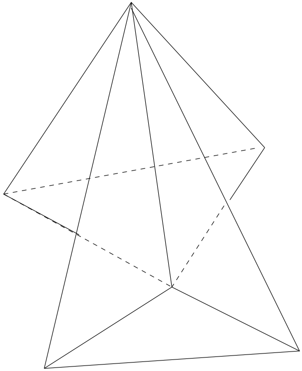

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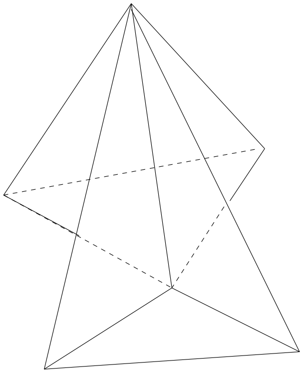

Here D_1 (in the back tetrahedron) extends to D_2. Thus the 3 D_i's give an immersed minimal surface Σ in N.

Existence of properly embedded minimal surfaces in N^3

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Suppose M^3 is a compact orientable irreducible Riemannian 3-manifold, $\partial M^3 \neq \emptyset$, and M^3 not topologically a 3-ball.

Then $H^1(\partial M, \mathbb{R}) \to H^1(M)$ gives a non zero closed one form α on M. A surface S dual to α, can be chosen to be embedded, 2 sided, and incompressible in M and in ∂M.

Existence of properly embedded minimal surfaces in N^3

If ∂M is mean convex then a theorem of Meeks-Yau gives a least area embedded minimal surface Σ (or a double cover of) in the isotopy class of S. When $\partial S \subset \partial M$, the isotopy can be chosen to leave ∂S fixed.

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Thus sequences of surfaces of decreasing area may go off to infinity in the ends of N and solving Plateau-type problems may be difficult.

Existence of properly embedded minimal surfaces in N^3

An end of N is homeomorphic to $\mathbb{T} \times \mathbb{R}^+$, $\mathbb{T}$ a 2-torus, and the metric is $e^{-2s}g_0 + ds^2$, g_0 a flat metric on $\mathbb{T}$.

Existence of properly embedded minimal surfaces in N^3

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The tori $\mathbb{T}(s) = \mathbb{T} \times (s)$ are the quotient of horospheres of $\mathbb{H}^3$ (with the same point at infinity) by 2 independent parabolic isometries (fixing the point at infinity). Going a distance $s > 0$ from $\mathbb{T}(0)$, by the geodesic s-flow ; shrinks lengths on $\mathbb{T}(0)$ by e^{-s}.

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A small circle C on $\mathbb{T}(0)$ of length ℓ (for the metric $g(0)$) bounds the infinite cylinder $C \times [0, \infty)$, whose area equals ℓ. The disk on $\mathbb{T}(0)$ bounded by C has area $\pi\ell^2$; which is less than ℓ for $\ell << 1$.

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The projection of $\tilde{\Gamma}$ to N is a solution to the Plateau problem for Γ

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Is there an infinite genus Σ in N, embedded and minimal ?

Idea of the proof

N is homeomorphic to the complement of a non trivial link in $\mathbb{S}^3$. The link bounds a compact orientable embedded surface S in $\mathbb{S}^3$, S incompressible.

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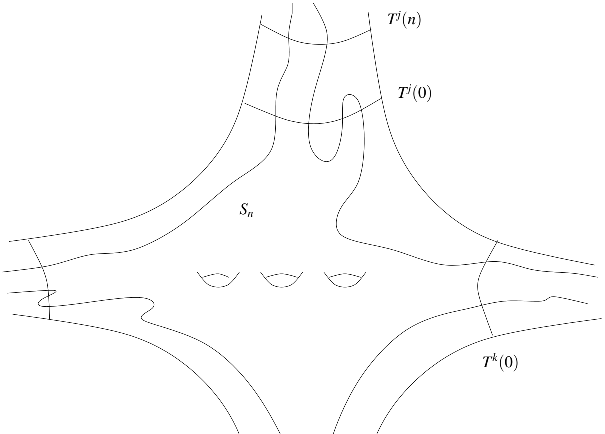
Let $E_1, ..., E_m$ be the cusp-ends of N, each E_j foliated by the constant mean curvature one tori $\{\mathbb{T}^j(s),\ 0 \leq s < \infty\}$.

For each $n > 1$, let $M(n)$ be the thick part of N bounded by $\cup_{j=1}^m T^j(n)$.

Existence of properly embedded minimal surfaces in N^3

Isotop S to a surface $\tilde{S}_n$ such that $\tilde{S}_n$ meets each $T^j(n)$ transversally, in one Jordan curve, and the complement in $\tilde{S}_n$ of $\tilde{S}_n \cap M(n)$ consists of m annuli : $\mathbb{S}^1 \times [0, \infty)$



Existence of properly embedded minimal surfaces in N^3

Change the hyperbolic metric of $N(n)$ in a neighborhood of $\partial M(n)$ so that ∂M_n becomes mean convex, and the metric of $M(n)$ is unchanged at points a distance greater than $1/2$ from $\partial M(n)$.

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Using the fact that the $T^j(s)$ are shrinking exponentially as $s \to \infty$, and curvature bounds for stable surfaces (such curvature bounds do not require positive injectivity radius of the ambient space (Rosenberg,Souam, Toubiana), we can prove :

Existence of properly embedded minimal surfaces in N^3

Proposition : As $n \to \infty$, a subsequence of the Σ_n, in the region bounded by $T^j(n-2)$ and $T^j(n-1)$ converges to a union of flat annuli $\gamma_\ell \times [n-2, n-1]$, where the γ_ℓ are disjoint compact geodesics of T^j.

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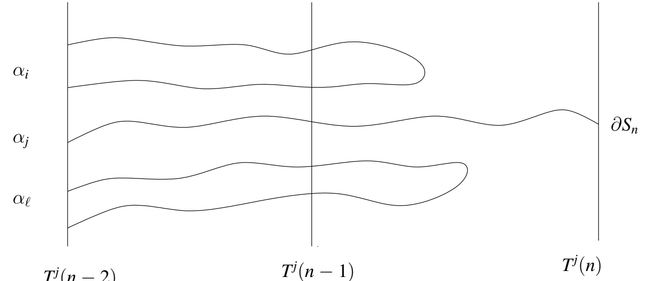
Thus for n large, we have a section of $\tilde{\Sigma}_n$ as in figure :



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The sequence of minimal surfaces Σ_n have uniformly bounded curvature and area so a subsequence converges to a properly embedded minimal surface Σ.

Σ may not have the same topology as S. The compact annuli bounded by $\alpha_i \cup \alpha_j$ may escape to infinity as $n \to \infty$ to become non compact. Σ is stable, hence has bounded curvaure. Thus, by the bounded curvature theorem, Σ has finite topoogy.

Idea of the proof of the Rigidity Theorem

Let $n(p)$ be the mean curvature vector of $T \subset N$, at $p \in T$. Define

$$\phi : T \times \mathbb{R}^+ \to N$$
$$\phi(p, t) = exp_p(tn(p))$$

and $\epsilon_0 = \sup\{\epsilon > 0; \phi$ is an immersion on $T \times [0, \epsilon)\}$.

Pull back the metric of N to $T \times [0, t_0)$

$$d\sigma^2 = dt^2 + d\sigma_t^2.$$

Idea of the proof of the Rigidity Theorem

Let $T(t) = T \times \{t\}$=the equidistance to $T(0)$ and $H(p,t)$=the mean curvature of $T(t)$ at (p,t).

Let λ satisfy $H+\lambda$ and $H-\lambda$ are the principal curvatures of $T(t)$ at (p,t).

Then the Gauss equation and Gauss Bonnet

$$0 = \int_{T(t)} \tilde{K}(t) = \int_{T(t)} H^2 - \lambda^2 + K_t,\ K_t = K_{sect}$$

Since $K_t \leq 1, \Longrightarrow$

$$\int_{T(t)} \lambda^2 = \int_{T(t)} H^2 + K_t \leq \int_{T(t)} H^2 - |T(t)|$$

Idea of the proof of the Rigidity Theorem

Let $F(t) = \int_{T(t)} H^2 - |T(t)|$, so $F(0) = 0$ and $F(t) \geq 0, t \geq 0$.

Calculate

$$F'(t) = \int_{T(t)} (2H \frac{\partial H}{\partial t} - 2H^3) + \int_{T(t)} 2H = \int_{T(t)} H(Ric(\frac{\partial}{\partial t} + 2) + 2\lambda^2)$$

$H(p, 0) = 1$ so there exists $t \in (0, \epsilon_0)$, and $C > 0$, such that

$0 < H \leq C$ on $T \times [0, \epsilon]$.

Idea of the proof of the Rigidity Theorem

$Ricci(\partial_t) + 2 \leq 0$, so

$$F'(t) \leq \int_{T(t)} 2H\lambda^2 \leq 2CF(t).$$

Hence $F(t) \leq F(0)e^{2Ct}$ for $t \in [0, \epsilon]$, $\Rightarrow F(t) = 0$ for $t \in [0, t]$.

Then $\lambda = 0$ and the $T(t)$ are umbilical, and $Ricci(\partial_t) = -2$ since $H > 0$.

So H satisfies $\frac{\partial H}{\partial t} = -2 + 2H^2$, $\Rightarrow H = 1$ on $T \times [0, \epsilon]$.

Idea of the proof of the Rigidity Theorem

Let $\epsilon \to \epsilon_0$; $\Longrightarrow Ric(\partial_t) = -2$ and $H = 1$ on $T \times [0, t_0]$.

Since $0 = \int_{T(t)} (H^2 + K_{sect}(t)), \Longrightarrow K_{sect} = -1$.

$\Longrightarrow d\sigma_0$ is flat and $d\sigma_t^2 = e^{-2t} d\sigma_0^2$,

$\Longrightarrow \phi : T \times \mathbb{R}^+ \to \text{hyperbolic cusp}$, is an immersion and a local isometry.

One proves ϕ is injective.

Gracias