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STABLE SUBMANIFOLDS WITH CONSTANT MEAN CURVATURE

A variational approach to the isoperimetric problem in higher codimension

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HYPERSURFACES with CMC in $\mathbb{R}^{m+1}$

$F : M^m \to \mathbb{R}^{m+1}$ immersed compact hypersurface

ν unit normal, $\partial M = \emptyset$ or not

$$\mathbf{A}(\mathbf{F}) = \int_{\mathbf{M}} \mathbf{dM} = |\mathbf{M}|$$
$$\mathbf{V}(\mathbf{F}) = \frac{1}{m+1} \int_{\mathbf{M}} \langle \mathbf{F}, \nu \rangle \mathbf{dM} = |\mathbf{CF}|$$

$F_t(x) = F(t,x)$, $F_0 = F$, fixing ∂M, $\frac{d}{dt}_{|t=0} F_t = f\nu$

$$\frac{\mathbf{d}}{\mathbf{dt}}_{|\mathbf{t=0}} \mathbf{A}(\mathbf{F_t}) = -\mathbf{m} \int_{\mathbf{M}} \mathbf{H\, f\, dM},$$
$$\frac{\mathbf{d}}{\mathbf{dt}}_{|\mathbf{t=0}} \mathbf{V}(\mathbf{F_t}) = \int_{\mathbf{M}} \mathbf{f\, dM}$$

$$\mathcal{F} = \{ f \in H_0^1(M) : \int_M f\, dM = 0 \} = \{1\}^{\perp_{L^2}}$$

If $V(F_t) = V(F)$, then $f \in \mathcal{F}$,

F is critical pt. of $A(F)$ iff $H = constant$.

Barbosa-doCarmo (1984)

$(1),(2),(3)$ are equivalent:

(1) F has Constant Mean Curvature $H = h_0 = \frac{1}{|D|}\int_D HdM$

(2) F is a critical point of $A(F)$, $\forall$ var. $F_t(x)$ with $V(F_t) = V(F)$

(3) F is a critical point of $J(F) = A(F) + mh_0 V(F)$ $\forall$ var. F_t (*).
((*)$\forall$ var. F_t with $W = f\nu$, $f \in \mathcal{F}$.)

Have proved (2) $\Rightarrow$ (1) but missing:

Lemma

If $f \in \mathcal{F}$, i.e. $f_{\partial M} = 0$, and $\int_M fdM = 0$,
there exists a variation F_t that fixes ∂M, s.t. $\frac{dF}{dt}_{|t=0} = f\,\nu$
and F_t preserves the volume: $V(F_t) = V(F)$

(2)$\Rightarrow$ (1): $\frac{d}{dt}_{|t=0} A(F_t) = -m\int_M H\, fdM = 0$. so $H \perp \mathcal{F} = \{constants\}^{\perp}$.
Thus $H = constant$.

(3) $\Rightarrow$ (2) (obvious), (1) $\Rightarrow$ (3) (easy).

Isoperimetric problems

“Seek closed hypersufaces
of minimum area
with fixed enclosed volume.”

”=” Stability

$\{$ crit. pts. of $A(F)$ for a fixed volume $V(F)$ $\}$ $=$

$\{$ crit. pts. of $J(F) = A(F) + mh_0 V(F)$ $\}$

Alexandrov (1958): $M^m \subset \mathbb{R}^{m+1}$, closed, **embbeded** CMC $\Rightarrow$ $M = \mathbb{S}^m$

Wente's tori (1986) immersed with CMC. ($g = 1$)

F a crit.pt. of $J = A + mh_0 V$,
F is **stable** if $\forall F_t$ with $\frac{dF}{dt}_{|t=0} = f\nu$, $f \in \mathcal{F}(*)$

$$
\begin{aligned}
I(f,f) &= \frac{d^2}{dt^2}_{|t=0} J(F_t) \quad (= \tfrac{d^2}{dt^2}_{|t=0} A(F_t) \quad \text{only if } V(F_t) = V(F)\ \forall t) \\
&= \int_M (-f\Delta f - \|B\|^2 f^2) dM \\
&= \int_M (\|\nabla f\|^2 - \|B\|^2 f^2) dM \geq 0
\end{aligned}
$$

Barbosa-do Carmo (1984)
$F : M^m \to \mathbb{R}^{m+1}$ closed immersed hypersurface CMC, is stable iff $F(M)$ a sphere.

Proof. Spheres are stable because $\lambda_1(\mathbb{S}^m) = m = \|B\|^2$.
Converse: $h = \langle F, \nu \rangle$, then $f = h_0 h + 1 \in \mathcal{F}$ and stability using f implies $\int_M (\|B\|^2 - m\|H\|^2) \leq 0$. Hence $\|B\|^2 = m\|H\|^2$, i.e. $B = H g$ tot. umbilical.

- "enclosed" may not be a cone (we may choose other)

Why(*)? : The case of spheres: $D \supset$ hemisphere (not "J-stable") D stable $\Longrightarrow$ $D' \subset D$ stable.

- **IN a SPACE FORM** $F : M^m \to \bar{M}^{m+1}$ hypersurface

$(\bar{M}^{m+1}, \bar{g})$ **any Riemannian manifold**
How to define "**enclosed** $(m+1)$**-volume**" $V(F)$**?**

Barbosa-doCarmo-Eschenburg (1988): For each variation, fix. ∂M
$\bar{F}(t,x) = F_t(x) : [0,\epsilon) \times M \to \bar{M}$ define the *path volume* at time t

$$\mathbf{V(\bar{F}(t))} = \int_{[0,t]\times M} \mathbf{\bar{F}^* d\bar{M}}$$

$$\frac{\mathbf{d}}{\mathbf{dt}}_{|\mathbf{t=0}} \mathbf{A(F_t)} = -\mathbf{m} \int_M \mathbf{H\, f\, d\bar{M}}$$

$$\frac{\mathbf{d}}{\mathbf{dt}}_{|\mathbf{t=0}} \mathbf{V(\bar{F}(t))} = \int_M \mathbf{f\, d\bar{M}}$$

$$\frac{dF}{dt}_{|t=0} = f\nu$$

Definition
F_t is volume preserving if $V(\bar{F}(t)) = V(\bar{F}(0)) = 0 \;\; \forall t$

Theorem (**Barbosa-doCarmo-Eschenburg (1988)**)

$(1), (2), (3)$ *are equivalent:*

(1) *F has Constant Mean Curvature* $H = h_0 = \frac{1}{|D|} \int_D H dM$

(2) *F is a critical point of* $A(F)$, $\forall$ *var.* $F_t(x)$ *with* $V(\bar{F}(t)) = 0$

(3) *F is a critical point of* $J(F) = A(F) + m h_0 V(F)$ $\forall$ *var.*

In this case, for $f \in \mathcal{F}$

$$J''(0)(f) = \int_M (-f \Delta f - (\bar{R} + \|B\|^2) f^2) dM$$

where $\bar{R} = Ricci^{\bar{M}}(\nu, \nu)$

Theorem (**Barbosa-doCarmo-Eschenburg (1988)**)

$\bar{M}$ space form (simply conn.) of const. sec. curv. **c**,
M closed, immersed hypersurface with CMC. Then:
M is stable iff M is a geodesic sphere $S_R = r^{-1}(R)$

Proof. Geodesic spheres satisfy

$$\begin{cases} \text{CMC} \quad \|H\| \qquad \text{CSC} = \mathbf{c} + \|H\|^2 \\ \text{totally umbilical } (B = H\,g) \quad \|B\|^2 = m\|H\|^2 \\ \lambda_1 = m(\mathbf{c} + \|H\|^2) = \bar{R} + \|B\|^2 \end{cases}$$

Then $\int_M (-f\Delta f - (\bar{R} + \|B\|^2)f^2) \geq 0$.

Reciprocally (case $\mathbf{c} = 0$)

Take $h = \langle F, \nu \rangle$ (support function). Then $f = \|H\|h + 1 \in \mathcal{F}$. Stability applied to f implies $\int_M m\|H\|^2 - \|B\|^2 dM \geq 0$, i.e. $\|B\|^2 = m\|H\|^2$, M is umbilical, thus a geodesic sphere.

In a model space $\bar{M} = \mathbb{S}^{m+1} \subset \mathbb{R}^{m+2}$ or $\bar{M} = \mathbb{H}^{m+1} \subset \mathbb{R}_1^{m+2}$, similar proof using support functions and the inner products of $\mathbb{R}^{m+2}$ and of $\mathbb{R}_1^{m+2}$.

• HIGHER CODIMENSION

[Morgan, Perimeter-minimizing curves and surfaces in $\mathbb{R}^n$ enclosing presc. multi-volume, Asian J. Math. (2000)]
[S., Stability of submanifolds with parallel mean curvature in calibrated manifolds, Bul.Braz.Mat.Soc. (2010)]
[Morgan+S., The isoperimetric problem in higher codimension (2012)]

$F : M^m \to N^{m+n}$ compact submanifold. Enclosed Volume:

• $N = \mathbb{R}^{m+n}$, M closed, $\exists\ \bar{M}^{m+1}$ (gener. submanifold): $M = \partial \bar{M}$ and $|\bar{M}| = inf_R |R|$ *for* $\partial R = M$.

$$\mathbf{V(M)} := |\bar{\mathbf{M}}^{\mathbf{m+1}}|$$

• ω m-form on N [R.Gulliver, F.Duzaar, M.Fuchs]

$$\mathbf{V}_\omega(\mathbf{F}) := \int_{\mathbf{M}} \mathbf{F}^*\omega$$

• $N = \mathbb{R}^{m+n}$, Ω $(m+1)$-form (constant), e_i F-o.n [F.Morgan]

$$\mathbf{V}_\Omega(\mathbf{F}) := \frac{1}{m+1} \int_{\mathbf{M}} \boldsymbol{\Omega}(\mathbf{F}, \mathbf{e_1}, ...\mathbf{e_m})\mathbf{dM} = \frac{1}{m+1} \int_{\mathbf{M}} \langle \mathbf{F}, \mathbf{G}_{\Omega,\mathbf{F}} \rangle \mathbf{dM}$$

• Ω $(m+1)$-form on N, var. $F_t(x) = F(t,x) := \bar{F}$, *path volume* [S]

$$\bar{\mathbf{V}}_\Omega(\mathbf{F_t}) := \int_{[\mathbf{0},\mathbf{t}]\times\mathbf{M}} \bar{\mathbf{F}}^*\boldsymbol{\Omega}$$

If $\Omega = d\omega$,

$$\begin{aligned} \bar{V}_\Omega(F_t) &= \int_{[0,t]\times M} \bar{F}^*\Omega \\ &= \int_M F_t^*\omega - \int_M F_0^*\omega \\ &= V_\omega(F_t) - V_\omega(F_0) \end{aligned}$$

Minimizing Area under Volume constraint

$M_t = M$ with metric $g_t = F_t^* g$.

$$\mathbf{A(F_t)} = \int_{\mathbf{M}} \mathbf{dM_t} = |\mathbf{M_t^m}|$$

With volume constraint : $V(F_t) = c$, $\bar{V}_\Omega(F_t) = 0$. $W = \frac{dF_t}{dt}_{|t=0}$.

$$\frac{d}{dt}_{t=0} A(F_t) = -m \int_M g(H, W) dM_0$$

Question: *Characterize infinitesimal vect. variations* W *of vol. pres. variations* i.e. W s.t. $\exists\ F_t\ V(F_t) = c$ and $W = \frac{dF_t}{dt}_{|t=0}$

The infinitesimal variations $W = \frac{dF_t}{dt}_{|t=0}$

$\bar{V}_\Omega(F_t) = \int_{[0,t]\times M} \bar{F}^*\Omega \quad V_\omega(F) = \int_M F^*\omega \quad V_\Omega(F) = \frac{1}{m+1}\int_M \Omega(F, e_1, \ldots, e_m)$

$$\frac{\mathbf{d}}{\mathbf{dt}}_{\mathbf{t=0}}\bar{\mathbf{V}}_\Omega(\mathbf{F_t}) = \int_{\mathbf{M}} \boldsymbol{\Omega}(\mathbf{W}, \mathbf{e_1}, ..., \mathbf{e_m})\mathbf{dM_0}$$
$$\frac{\mathbf{d}}{\mathbf{dt}}_{\mathbf{t=0}}\mathbf{V}_\omega(\mathbf{F_t}) = \int_{\mathbf{M}} \mathbf{d}\omega(\mathbf{W}, \mathbf{e_1}, ..., \mathbf{e_m})\mathbf{dM_0}$$

If $N = \mathbb{R}^{m+n}$

$$(m+1)\frac{d}{dt}_{t=0} V_\Omega(F_t) = (m+1)\int_M \Omega(W, e_1, ..., e_m)dM_0 - \int_M d\Omega(F, e_1, \ldots, e_m)dM_0 + \int_M \bar{\nabla}_F\Omega(W, e_1, \ldots, e_m)dM_0$$

If $\boldsymbol{\Omega} = \mathbf{d}\omega$ need $\int_M \bar{\nabla}_F\Omega(W, e_1, \ldots, e_m)dM_0 = 0$ (e.g. Ω const.) to have

$$\frac{\mathbf{d}}{\mathbf{dt}}_{|\mathbf{t=0}}\bar{\mathbf{V}}_\Omega(\mathbf{F_t}) = \frac{\mathbf{d}}{\mathbf{dt}}_{|\mathbf{t=0}}\mathbf{V}_\omega(\mathbf{F_t}) = \frac{\mathbf{d}}{\mathbf{dt}}_{|\mathbf{t=0}}\mathbf{V}_\Omega(\mathbf{F_t}) = \int_{\mathbf{M}} \boldsymbol{\Omega}(\mathbf{W}, \mathbf{e_1}, \ldots, \mathbf{e_m}) = \langle \mathbf{G}, \mathbf{W}\rangle_{\mathbf{L^2}}$$

$$\mathbf{G} = \boldsymbol{\Omega}^\sharp(\mathbf{e_1}, \ldots, \mathbf{e_m}) = \mathbf{G_{\Omega,F}} \in \mathbf{NM} \subset \mathbf{F^{-1}TN}$$
$$g(G, W) := \Omega(W, e_1, \ldots, e_m) \quad \forall W \in F^{-1}TN$$

$V(F_t) = c$ implies (with $F = F_0$)

$$\mathbf{W} \in \mathbf{H^1_T}(\mathbf{F^{-1}TN}) = \left\{\mathbf{W} \in \mathbf{H^1_0}(\mathbf{F^{-1}TN}) : \int_{\mathbf{M}} \mathbf{g}(\mathbf{W}, \mathbf{G})\mathbf{dM} = \mathbf{0}\right\} = \mathbf{G}^\perp \text{ in } \mathbf{L^2}$$

Reciprocally: given $W \perp G$**,** $\exists F_t$**:** $V(F_t) = V(F)$**,** $W = \frac{d}{dt}_{t=0} F_t$**?**

$\mathbf{N} = \mathbb{R}^{\mathbf{m+n}}$, if $\langle F, G\rangle_{L^2} \neq 0$ (non-degeneracy cond.), for $\langle W, G\rangle_{L^2} = 0$

$$F_t(x) = \phi(t)(F(x) + tW_x) \qquad \text{Homotheties}$$

$$\phi'(t) = \phi(t) \frac{\int_M \Omega(W, dF(e_1) + tdW(e_1), \ldots dF(e_m) + tdW(e_m))dM}{\int_M \Omega(F + tW, dF(e_1) + tdW(e_1), \ldots dF(e_m) + tdW(e_m))dM}$$

with $\phi(0) = 1$. Then $F_0 = F$, $\frac{d}{dt}_{|t=0} F_t = W$, $\phi'(0) = 0$ and $\frac{d}{dt} V(F_t) = 0$ $\forall t$, i.e. $\forall t$

$$\begin{cases} \bar{V}_\Omega(F_t) = 0 & \text{for any } \Omega \\ V_\omega(F_t) = V_\omega(F) & \text{for } \Omega = d\omega \\ V_\Omega(F_t) = V_\Omega(F) & \text{if } \Omega \text{ constant} \end{cases}$$

• Morgan: F_t has prescribed *multi-volume* if $V_\Omega(F_t) = V_\Omega(F)$ $\forall \Omega$ constant. In this case $W \perp \Omega^\sharp(e_1, \ldots, e_m) = G_{\Omega,F}$ $\quad \forall \Omega$ *constant*. Do not know if exist multi-volume preserving variation $\forall$ such W.

• The case $V(M) = |\bar{M}^{m+1}|$. $A(M) = |M^m|$ is cont. function for prescribed volume. Need to assume, $\exists G$: $\forall W$ infinit. var. of F, $V(F_t)$ smooth and $\frac{d}{dt}_{t=0} V(F_t) = -\int_M \langle G, W\rangle$.

FIRST VARIATION OF AREA FOR PRESCRIBED VOLUME

Lagrange Multipliers

$N \hookrightarrow \mathbb{R}^N$ embbed closed set. $C^r(M, \mathbb{R}^N)$ Banach (norm unif.c. deriv. $<= r$)
$C^r(M, N) \subset C^r(M, \mathbb{R}^N)$, $0 \leq r < \infty$, closed set, C^∞ (Finsler) manifold modeled on a separable Banach manifold.

$\mathbf{N} = \mathbb{R}^{\mathbf{m+n}}$, $H^1(M, \mathbb{R}^{m+n})$, $L^2(M; \mathbb{R}^{m+n})$ separable Hilbert spaces
$U = Imm(M, N) \cap H^1(M, N)$ or $Embb(M, N) \cap H^1(M, N)$ open sets

$$\mathbf{A},\ \mathbf{V} : \mathbf{H^1}(\mathbf{M}, \mathbb{R}^{\mathbf{m+n}}) \to \mathbb{R} \quad \textbf{area, volume functionals}$$

$A(F) = \int_M \sqrt{g_{ij}} dx^{1..m}$ continuous. Also differentiable on U.
$V(F) = \int_M g(F, G) dM = \int_M \Omega(F, \partial_i) dx^{1..m}$ diff on H^1 (assumed if $V(F) = |\bar{M}|$)

$$\begin{aligned} dA(F)(W) &= \frac{d}{dt}_{t=0} A(F_t) = -m \int_M g(H, W) dM \\ dV(F)(W) &= \frac{d}{dt}_{t=0} V(F_t) = \int_M g(G, W) dM \end{aligned}$$

Assume F crit. pt of area A, for constraint $V(F) = c$.
$dA(F)(W) = 0 \ \forall W$ infinitesimal volume preserving.

If $\langle \mathbf{F}, \mathbf{G} \rangle_{\mathbf{L^2}} \neq \mathbf{0}$, then $\mathbf{G}^\perp = \{$**W comes from vol. preserv.** $\mathbf{F_t}\}$
$dV(F) : L^2(M; \mathbb{R}^{m+n}) \to \mathbb{R}$ is continuous, onto, with Kernel $G^\perp$.
$dA(F) : L^2 \to \mathbb{R}$ bounded, $dA(F)(G^\perp) = 0$, then $\mathbf{dA(F) = -\lambda dV(F)}$

$F : M \to \mathbb{R}^{m+n}$ **is a critical pt of** A **under constraint** $V(F) = c$
iff $\mathbf{H} = \frac{\lambda}{\mathbf{m}} \mathbf{G} = \frac{\lambda}{\mathbf{m}} \mathbf{\Omega}^\sharp(\mathbf{e_1}, \ldots, \mathbf{e_m})$ **(prescribed MC)**
iff F **critical pt. of** $J = A + \lambda V$

In this case, if:

- $\mathbf{\Omega}$ a **calibration**, and $|\lambda| \leq \frac{\mathbf{m}}{|\mathbf{M}|} \int_{\mathbf{M}} \|\mathbf{H}\| \mathbf{dM}$, then $\|\mathbf{H}\| = \frac{|\lambda|}{\mathbf{m}}$ **constant**.
- M **closed**, **embedded**, $\Omega = dx^1 \wedge \ldots \wedge dx^{m+1}$ **simple**, then $\mathbf{M} = \mathbb{S}^{\mathbf{m}}$.

Proof. $\Omega = d\omega$, $\omega = x_1 dx^2 \wedge \ldots \wedge dx^{m+1}$,
$F(p) = (x(p), y(p)) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-1}$.
$F_t(p) = (x(p), \frac{y(p)}{1+t})$ is V_ω-preserv.
$A(F_t) = A(F) - \phi(t)$, with $\phi(t) > 0$ if $t > 0$ and $|dy| \neq 0$,
More, $\frac{d}{dt}_{t=0} A(F_t) = 0$ iff $dy \equiv 0$. y must be constant (0 say).
$F : M^m \to \mathbb{R}^{m+1}$ (embedded) with CMC. By Alexandrov, $M = \mathbb{S}^m$

If N **any Riemannian manifold** Ω $(m+1)$-form ($\Omega = d\omega$ resp.),

$$\frac{d}{dt}_{|t=0} A(F_t) = -m\int_M g(W,H)dM$$
$$\frac{d}{dt}_{|t=0} \bar{V}_\Omega(F_t) = \int_M \Omega(W, e_1, \ldots, e_m)dM \quad (= \frac{d}{dt}_{|t=0} V_\omega(F_t))$$

$\mathcal{IVP} := \{\mathbf{W} : \textbf{infinitesimal vol. pres. var.}\} \subset \{\int_{\mathbf{M}} \mathbf{g}(\mathbf{W},\mathbf{G})\mathbf{dM} = \mathbf{0}\} = \mathbf{G}^\perp$

$C^2(M;N)$ $(L^2(M;N))$ not a v.s., Banach man. closed set of $C^2(M,\mathbb{R}^N)$.
No use of Lagrange multip. No use of homotheties in construct. vol. pres. var.
If $\nu \in F^{-1}TN$ s.t. $a_\nu := g(\nu, G) > 0$ (non-degenracy)

$$\mathcal{F}_\Omega := \{\mathbf{W} = \mathbf{f}\,\frac{\nu}{\mathbf{a}_\nu} : \mathbf{f} \in \mathcal{F}\} \subset \mathcal{IVP} \subset \mathbf{G}^\perp(\textbf{ week sense})$$

$$F(t,x) = \rho(\xi(t,x),x), \ \rho(\xi,x) = exp_{F(x)}(\xi\nu_x)$$
$$\frac{d}{dt}\xi(t,x) = \frac{f(x)}{\Omega(\frac{d\rho}{d\xi}(\xi(t,x),x), d\rho_{\xi(t,x)}(x)(e_1),\ldots, d\rho_{\xi(t,x)}(x)(e_m))}, \qquad \xi(0,x)=0$$

satisfies $\frac{d}{dt}_{|t=0} F(t,x) = W$ and $\bar{V}_\Omega(F_t) = 0 \ \forall t.$

$\mathbf{\frac{d}{dt}_{t=0} A(F_t) = 0} \ \forall\, W \in G^\perp$ iff $\mathbf{H} = \lambda\mathbf{G} = \lambda\mathbf{\Omega^\sharp(e_1,\ldots,e_m)}$ **(prescribed)**
In this case, $\frac{d}{dt}_{t=0} A(F_t) = 0$ for $\bar{V}_\Omega(F_t) = 0$ (critical point for Vol. pres. var.). Do not know if this is sufficient for Presc. MC.

Example: $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ has PMC and no Ω-prescribed MC, $\forall \Omega = d\omega$ constant 3-form. Because $\Omega(H, e_1, e_2)$ never constant. Hence, $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ no crit. pt of A for any V-preserving var., with Ω const.

- **Prescribed CMC**: Ω calibration, M calibrated extended tangent space $TM \oplus H$: then $\nu := \frac{H}{\|H\|} = \Omega^\sharp(e_1, \ldots, e_m) = G$. $\quad \Omega(G, e_1, \ldots, e_m) = 1$

(1) $F : M \to N$ is a critical pt of A under constraint $\bar{V}_\Omega(F) = 0$

iff (2) $\|H\|$ constant

iff (3) F crt. pt. of $J = A + m\, h_0\, V$, where $h_0 = \frac{1}{|M|}\int_M \|H\| dM$.

Proof.

(1) $\Rightarrow$ (2) If $f \in \mathcal{F}$, then $W = f\nu \in \mathcal{IVP}$, so $-m\int_M fg(H, \nu) = 0$, i.e. $\|H\| = g(H, \nu) \perp \mathcal{F}$. Thus $\|H\| = constant$.

(2) $\Rightarrow$ (1) $\forall W$, $g(\nu, W) = \Omega(W, e_1, \ldots, e_m)$, and $\|H\| = g(H, \nu)$. If $\bar{V}_\Omega(F_t) = 0$, $W = \frac{d}{dt}_{t=0} F_t \in G^\perp$,

$$\frac{d}{dt}_{|t=0} A(F_t) = -m \int_M g(H, W) = -m\|H\| \int_M \Omega(W, e_1, \ldots, e_m) = 0$$

(3) $\Rightarrow$ (1) trivial. (2) $\Rightarrow$ (3) since by conditions

$$\frac{d}{dt}_{|t=0} J(F_t) = -m \int_M g(H, W) + m\|H\| \int_M g(W, \nu) dM = 0$$

• SECOND VARIATION for $J = A + m\, h_0 V$

$F : M^m \to N^{m+n}$ **PMC**, Ω-calibrated ext.tg. (conditions)
$TM \oplus \nu, \quad \nu = G = H/\|H\| \quad \Omega(\nu, e_1 ... e_m) = 1.\ NM = \nu \oplus E.$
$\mathcal{F}_\Omega := \{W = f(\nu + N) : f \in \mathcal{F}, N \in E\} \subset \mathcal{IVP}$. Then

$$\mathbf{H^1_T(NM)} := span\{\mathcal{F}_\Omega\} = \{\int_M \Omega(W, e_1 ... e_m) dM = 0\} = \mathbf{G}^\perp$$

$$J''(0) = \int_D g(\mathcal{J}_\Omega(W), W) dM = \mathbf{I_\Omega(W, W)} \qquad (\text{for } W \in \mathcal{F}_\Omega, H^1_T(NM))$$

$$= A''(0) \qquad (\text{for } \bar{V}_\Omega(F_t) = 0\ \forall t)$$

$$= \int_M (\|\nabla^\perp W\|^2 - \bar{g}(\bar{R}(W), W) - \bar{g}(\bar{B}(W), W) + m\|H\| \bar{g}(C_\Omega(W), W)) dM$$

$$\mathcal{J}(\mathbf{W}) = -\mathbf{\Delta}^\perp \mathbf{W} - \bar{\mathbf{R}}(\mathbf{W}) - \bar{\mathbf{B}}(\mathbf{W}) \quad \text{Jacobi operator}$$

$$\mathcal{J}_\Omega(\mathbf{W}) = \mathcal{J}(\mathbf{W}) + \mathbf{m}\|\mathbf{H}\| \mathbf{C}_\Omega(\mathbf{W}) \quad \mathbf{\Omega}\text{-Jacobi operator}$$

$$\bar{R}(W) = \sum_i (R^N(e_i, W) e_i)^\perp \qquad \bar{B}(W) = \sum_{ij} \bar{g}(W, B(e_i, e_j)) B(e_i, e_j)$$

$C_\Omega : C^\infty_0(NM) \to C^\infty_0(NM)$ L^2-s.a. first-order differential operator

$$\begin{aligned} 2g(C_\Omega(W), W') \quad = \quad & \textstyle\sum_i 2\Omega(W', e_1, \ldots, \nabla^\perp_{e_i} W, \ldots, e_m) \\ & -\textstyle\sum_i \bar{\nabla}_{e_i} \Omega(W, e_1, \ldots, W', ..., e_m) \\ & -\textstyle\sum_i \sum_{j \neq i} \Omega(W, e_1, \ldots, B(e_i, e_j), ..., W', ..., e_m) \\ & +\bar{\nabla}_W \Omega(W', e_1 ... e_m) + \bar{\nabla}_{W'} \Omega(W, e_1 ... e_m) \end{aligned}$$

$g(C_\Omega(W), W') - g(C_\Omega(W'), W) = div(X_{WW'})$

Ω-Stability

F with CMC and Ω-calibrated ext. tg. space. $TM \oplus G$
Assume $\nu = G$ parallel (then F has PMC) $NM = \mathbb{R}\nu \oplus E$.

F is **Ω-stable** if $I_\Omega(W,W) \geq 0 \;\; \forall W \in H^1_T(NM)$

• M closed, $C_\Omega = 0$, NM allows another unit parallel normal $\nu^E \perp \nu$
if $\int_M \sum_i \bar{R}(e_i, \nu^E, e_i, \nu^E) dM > 0$, then M is Ω-unstable.

• $\pi : N \to \Sigma$ Riemannian submersion, $(m+1)$-dim fibres $\pi^{-1}(p)$

$$\Omega_\pi(X_1, \ldots, X_{m+1}) = dV_\pi(X_1^v, \ldots, X_{m+1}^v)$$
$$d\Omega_\pi(e'_\alpha, e'_1, \ldots, e'_{m+1}) = -(m+1)g(H^v, e'_\alpha)$$

$d\Omega_\pi = 0$ iff minimal fibres + TN^h integrable (tot. geod.).

If $\sum_i \bar{R}(W^E, e_i, W^E, e_i) \leq 0$, $F(M) \subset M'$, with M' a totally geodesic $(m+1)$-dim. fibre of π, then M is Ω_π-stable. *Proof.* $C_\Omega = 0$.

Ex. $N = \bar{M}^{m+1} \times P^n$, $\Omega((X_1, Y_1), \ldots, (X_{m+1}, Y_{m+1})) = dV_{\bar{M}}(X_1 \ldots, X_{m+1})$,
calibrates slices $\bar{M} \times q$. $M^m \subset N$ has calib.ext.tg space iff $M \subset$ slice.

• $\pi : \mathbb{R}^{m+n} \to \Sigma$ Riemannian fibration with a totally geodesic fibre $\mathbb{R}^{m+1}$. Then $\mathbb{S}^m \subset \mathbb{R}^{m+1} \subset \mathbb{R}^{m+n}$ is Ω_π-stable in $\mathbb{R}^{m+n}$.

• unique fibrations of spheres w/ totally geodesic fibres: **Hopf** fibrations

$\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4(\frac{1}{2}) \quad \mathbb{S}^3 \hookrightarrow \mathbb{S}^{4k+3} \to \mathbb{HP}^k \quad \mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{S}^8(\frac{1}{2})$

$\mathbb{HP}^k$ quaternionic project. s. sec. curv. $1 \leq K \leq 4$, $\mathbb{S}^4(\frac{1}{2})$, $\mathbb{S}^8(\frac{1}{2})$ spheres curvat. 4.

M^m geodesic sphere of one of such fibres is Ω_π-unstable.

M stabe in the fibre M', not stable in $\bar{M}$

• **M^m geodesic sphere of $\mathbb{H}^{m+1}$ fibre of $\pi : \mathbb{H}^{m+n} \to N$, $n \geq 2$ is Ω_π-stable in $\mathbb{H}^{m+n}$.**

Existence and Regualrity

If N compact, or $N = \mathbb{R}^{m+n}$ and $\Omega = d\omega$ constant.
Minimizing sequneces of loc. integ. currents, Compactness Theorem, $\exists$ M^m isoperimetric submanifold for prescribed vol. $V(M)$ (all volumes), M is compact (monotonicity formula) and $C^{1,\alpha}$ on open dense set ($H = G$ weakly bounded, Allard regulararity thm).

- **Round spheres are uniquely isoperimetric for** $V(M) = |\bar{M}^{m+1}|$

$D^{m+1} \subset \mathbb{R}^{m+1} \subset \mathbb{R}^{m+n}$ unit disk, $\mathbb{S}^m = \partial D$. M^m submanifold of $\mathbb{R}^{m+n}$.

(Almgren) $A(M^m) = A(\mathbb{S}^m)$ **then** $|\bar{M}| \leq |D|$**, with equality iff** $\bar{M} = D$**.**

(1986) Round spheres are uniquely isoperimetric: i.e
$|\bar{M}| = |D|$**, then** $A(M) \geq A(\mathbb{S}^m)$**.**

Proof. Assume $A(M) < A(\mathbb{S}^m)$. $t > 1$: $A(tM) = A(\mathbb{S}^m)$. Then
$|t\bar{M}| \leq |D|$. So, $|\bar{M}| < |t\bar{M}| \leq |D|$, impossible.

- There are non circle curves that minimize area for prescribed multi-volume

- V_ω**-isoperimetric problem on** N

ω m**-calibration, calibrates** M^m**.**
Then M **is isoperimetric for prescribed volume** V_ω**.**

Proof. Let M' s.t. $V_\omega(M') = V_\omega(M)$. then

$$A(M') \geq \int_{M'} \omega = V_\omega(M') = V_\omega(M) = \int_M \omega = A(M)$$

M minimizes area for prescribed volume V_ω.

• Spheres are Minimizing for fixed V_ω, and Stable

$\Omega = d\omega$ calibration, calibrates a disk $D^{m+1} \subset \mathbb{R}^{m+n}$.
Then $\mathbb{S}^m = \partial D$ minimizes area among all closed M^m with same V_ω-volume.

Proof.
(Almgren) $A(M^m) = A(\mathbb{S}^m)$ **then** $|\bar{M}| \leq |D|$**, with equality iff** $\bar{M} = D$**.**

Assume $\exists\ F : M^m \to \mathbb{R}^{m+n}$ closed:

$$V_\omega(M) = V_\omega(\mathbb{S}^m) \quad \text{and} \quad A(M) < A(\mathbb{S}^m).$$

$t > 1:\ A(tM) = A(\mathbb{S}^m)$. By Almgren $|t\bar{M}| \leq |D|$. $|\bar{M}| < |t\bar{M}| \leq |D|$.
By assumption $\int_M F^*\omega = \int_{\mathbb{S}^m} \omega$. Thus,

$$|\bar{M}| \geq \int_{\bar{M}} \Omega = \int_M F^*\omega = \int_{\mathbb{S}^m} \omega = \int_D \Omega = |D|$$

Impossible. Thus $A(M) = A(\mathbb{S}^m)$, and so $M = \mathbb{S}^m$

$\mathbb{S}^m$ is $\bar{V}_\Omega$-stable

Proof. $H^1_T(NM) = G^\perp = \{\text{Vol.pres.variat.}\}$ (by homotheties argument)

STABLE SPHERES = CAUCHY RIEMANN INEQUALITIES

Ω (not closed nec.) $(m+1)$-calibration on $\mathbb{R}^{m+n}$.
$\mathbb{S}^m \subset \mathbb{R}^{m+1}$, $\mathbb{R}^{m+1}$ Ω-calibrated. $\mathbb{R}^{m+n} = \mathbb{R}^{m+1} \times \mathbb{R}^{n-1}$.
W_α, const. o.n.b $F = \mathbb{R}^{n-1}$

$$\alpha < \beta \qquad \text{1-form} \qquad \xi_{\alpha,\beta}(\mathbf{X}) := \boldsymbol{\Omega}(\mathbf{W}_\alpha, \mathbf{W}_\beta, *\mathbf{X}) \qquad \mathbf{X} \in \mathbf{T}\mathbb{S}^{\mathbf{m}}$$

$\mathbb{S}^m$ is $\bar{V}_\Omega$-stable iff $\forall W = f\nu + \sum_\alpha f_\alpha W_\alpha \in H^1_T(N\mathbb{S}^m)$, $f \in \mathcal{F}$, f_α any

$$I_\Omega(W^F, W^F) = \sum_\alpha \int_{\mathbb{S}^m} \|\nabla f_\alpha\|^2 dM + \sum_{\alpha<\beta} 2m \int_{\mathbb{S}^m} f_\alpha \, \xi_{\alpha\beta}(\nabla f_\beta) dM \geq 0$$

is Ω-stable iff $\xi_{\alpha,\beta}$, are co-exact 1-forms, $\xi_{\alpha,\beta} = \delta\omega_{\alpha,\beta}$ and $\forall f_\alpha$ we have the **long integral Cauchy-Riemann inequality** holds:

$$\sum_{\alpha<\beta} -2m \int_{\mathbb{S}^m} \omega_{\alpha,\beta}(\nabla f_\alpha, \nabla f_\beta) dM \leq \sum_\alpha \int_{\mathbb{S}^m} \|\nabla f_\alpha\|^2 dM$$

(This holds when $\nabla\Omega = 0$; or $C_\Omega = 0$ e.g $\Omega = \Omega_\pi$.)

Fixing $\alpha < \beta$ and take $f_\gamma = 0$ for $\gamma \neq \alpha, \beta$, and since $(m = \lambda_1)$ we have the **short integral C-R inequality**

$$\lambda_1 \left| \int_{\mathbb{S}^m} \omega_{\alpha,\beta}(\nabla f, \nabla h) dM \right| \leq \sqrt{\int_{\mathbb{S}^m} \|\nabla f\|^2 dM} \sqrt{\int_{\mathbb{S}^m} \|\nabla h\|^2 dM}$$

Example: associative calibration on $\mathbb{R}^7$

$$\boldsymbol{\Omega = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}}$$

Ω-stability of $\phi : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3 \subset \mathbb{R}^7$ means that $\forall (f_4, f_5, f_6, f_7) : \mathbb{S}^2 \to \mathbb{R}^4$

$$\begin{aligned} & 4\int_{\mathbb{S}^2} \phi_1(\langle J\nabla f_4, \nabla f_5\rangle + \langle J\nabla f_6, \nabla f_7\rangle)dM \\ & +4\int_{\mathbb{S}^2} \phi_2(\langle J\nabla f_4, \nabla f_6\rangle - \langle J\nabla f_5, \nabla f_7\rangle)dM \\ & -4\int_{\mathbb{S}^2} \phi_3(\langle J\nabla f_4, \nabla f_7\rangle + \langle J\nabla f_5, \nabla f_6\rangle)dM \\ \leq \quad & \int_{\mathbb{S}^2} (\|\nabla f_4\|^2 + \|\nabla f_5\|^2 + \|\nabla f_6\|^2 + \|\nabla f_7\|^2)dM \end{aligned}$$

where J is the complex structure of $\mathbb{S}^2$. The Short one

$$2\left|\int_{\mathbb{S}^2} \phi_k \langle J\nabla f, \nabla h\rangle dM\right| \leq \sqrt{\int_{\mathbb{S}^2} \|\nabla f\|^2 dM}\sqrt{\int_{\mathbb{S}^2} \|\nabla h\|^2 dM}$$

equality holds iff $f = \pm\phi_i$, $h = \pm\phi_j$ (k, i, j) + permut. $(1, 2, 3)$.
$\left(\int \langle J\nabla f, \nabla h\rangle = \sqrt{\int \|\nabla f\|^2}\sqrt{\int \|\nabla h\|^2}\right.$ iff $J\nabla f = \nabla h$, i.e. $f + ih$ holom.$\left.\right)$

A direct proof using spherical harmonics:
$\lambda_k = k(k+m-1)$, $k = 0, 1, 2...$ eigenvalues of $\mathbb{S}^m$ (closed eigenv.),
E_{λ_k} eigenspaces, Rayleigh:

$$\lambda_k = \inf_{f \in E_{\lambda_k^+}} \frac{\int \|\nabla f\|^2}{\int f^2} \qquad E_{\lambda_k}^+ := E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_{k-1}}$$

proving that:

- $\phi \in E_{\lambda_1}$, $h \in E_{\lambda_k} \implies \langle J\nabla\phi, \nabla h\rangle \in E_{\lambda_k}$
- $(\phi, f, h) \to \int \phi \langle J\nabla f, \nabla h\rangle$ skew on the 3 functions,
- $\phi \in E_{\lambda_1}$, with $\|\nabla\phi(p)\|^2 = 1 - \phi^2(p) \leq 1 \ \ \forall p$. If $f, h \in E_{\lambda_k}^+$, then

$$\begin{aligned} 2\int \phi\langle J\nabla f, \nabla h\rangle &= -\int f\langle J\nabla\phi, \nabla h\rangle - \int h\langle J\nabla f, \nabla\phi\rangle \\ &\leq |f|_{L^2}\|\nabla h\|_{L^2} + |h|_{L^2}\|\nabla f\|_{L^2} \\ &\leq \frac{2}{\lambda_k}\|\nabla f\|_{L^2}\|\nabla h\|_{L^2} \end{aligned}$$

Example: The Kähler 4-calibration of $\mathbb{C}^3 = \mathbb{R}^6$

$$\tilde{\omega} = dx^{12} + dx^{34} + dx^{56}$$
$$\mathbf{\Omega} = \tfrac{1}{2}\tilde{\omega} \wedge \tilde{\omega} = \mathbf{dx^{1234}} + \mathbf{dx^{1256}} + \mathbf{dx^{3456}}$$

Ω-stability of $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) : \mathbb{S}^3 \hookrightarrow \mathbb{R}^4 = \mathbb{C}^2 \subset \mathbb{C}^3 = \mathbb{R}^6$ means

$$\int_{\mathbb{S}^3} \tilde{\omega}(\nabla \mathbf{f}, \nabla \mathbf{h}) \mathbf{dM} \leq \frac{\mathbf{2}}{\mathbf{3}} \sqrt{\int_{\mathbb{S}^3} \|\nabla \mathbf{f}\|^2 \mathbf{dM}} \sqrt{\int_{\mathbb{S}^3} \|\nabla \mathbf{h}\|^2 \mathbf{dM}}$$

with equality iff $f, h \in E_{\lambda_1}$ with $f = \mu_1\phi_1 + \mu_2\phi_2 + \mu_3\phi_3 + \mu_4\phi_4$ and $h = \mu_2\phi_1 - \mu_1\phi_2 + \mu_4\phi_3 - \mu_3\phi_3$.

A Direct proof (not using Almgren):

Proof:

Ω-Weitzenböck formulas:

Ω parallel on $\mathbb{R}^{m+n}$, $\xi_{\alpha\beta}(X) = \Omega(\epsilon_\alpha, \epsilon_\beta, *X)$, $X \in T\mathbb{S}^m$, ϵ_α in $\mathbb{R}^{n-1}$.

If $f \in E_{\lambda_k}$, then $\xi_{\alpha\beta}(\nabla f) \in E_{\lambda_k}$.

For, $f, h \in E_{\lambda_3^+}$, $(\xi = \xi_{\alpha\beta} = \delta\omega$, and $\omega = \frac{1}{2}\phi^*\tilde{\omega})$

$$\int_{\mathbb{S}^3} -\frac{3}{2}\tilde{\omega}(\nabla f, \nabla h) = \int_{\mathbb{S}^3} -3f\xi(\nabla h) \leq \frac{3}{\sqrt{\lambda_3}} \|\nabla f\|_{L^2} \|\nabla h\|_{L^2} \leq \|\nabla f\|_{L^2} \|\nabla h\|_{L^2}$$

Inspection for $f, h \in E_{\lambda_1}, E_{\lambda_2}$ (polinomials of degree 1 and 2)

UNIQUENESS

Conjecture [M-S] **In $\mathbb{R}^{m+n}$, if M^m, closed, prescribed MC $H = \lambda G$, is V-stable, ($V(M) = |\bar{M}|$, or $V(M) = \bar{V}_\Omega, V_\Omega$, $\Omega = constant$, not multi-volume), then $M = \mathbb{S}^m$.**

Possible approach: Tot. umbilical and closed $\Rightarrow$ sphere.
Pseudo-umbilical + PMC may not be a sphere.

THEOREM. *$N = \mathbb{R}^{m+n}$, $C_\Omega = 0$, $F : M^m \to \mathbb{R}^{m+n}$ closed, PMC, $NM = \mathbb{R}\nu \oplus E$, calibrated extended tg. $TM \oplus \mathbb{R}\nu$.*
height functions

$$h = \langle F, \nu \rangle \quad \text{and} \quad S = \sum_{ij} \langle F, (B(e_i, e_j))^E \rangle B^\nu(e_i, e_j)$$

*If F is $\bar{V}_\Omega$-**stable** and $\int_M \mathbf{S}(2 + \mathbf{h}\|\mathbf{H}\|)\mathbf{dM} \leq \mathbf{0}$, then*

F is pseudo-umbilical and a minimal calibrated extension $\bar{M}$ of M exists with $Ricci^{\bar{M}}(\nu, \nu) = -\|B^{\bar{M}}(\nu, \nu)\|^2 = 0$ and $S = 0$.
Furthermore, if NM is a trivial bundle, then $\bar{M} = \mathbb{R}^{m+1}$, and $M = \mathbb{S}^m$

Morse Index Theorem

$D \subset M$ domain, $W \in H^1_{0,T}(NM_{/D})$ is a Ω-Jacobi field if $I_\Omega(W, W') = 0$ $\forall W' \in H^1_{0,T}(NM_{/D})$,
that is $\mathcal{J}_\Omega(W) = c\nu$, c constant, or equivalently if

$$\mathcal{J}'_\Omega(W) = \mathcal{J}_\Omega(W) - |D|^{-1} \int_D \bar{g}(\mathcal{J}_\Omega(W), \nu) dM = 0$$

If the dimension d of the space of Ω-Jacobi fields on D is nonzero then ∂D is called a conjugate boundary, and d its order. The operator $\mathcal{J}'_\Omega(W)$ has uniqueness in the Cauchy problem and a Morse index theorem can be stated for submanifolds with parallel mean curvature and calibrated extended space using Ω-Jacobi fields, namely:

Let $F : D \subset M^m \to N^{m+n}$ be a Ω-stable submanifold with parallel mean curvature and calibrated extended tangent space.
Then taking a contraction D_t of D (sufficiently small), there exists only a finite number of conjugate boundaries ∂D_{t_i} and the index of D (the maximal dimension of a subspace of $H^1_{0,T}(NM_{/D})$ where I_Ω is negative definite) is the sum of the orders of these conjugate boundaries.

A Forced Mean Curvature Flow

$\Omega \quad (m+1)$-Calibration on $\mathbb{R}^{m+n}$, $\quad \Omega^\sharp : (\mathbb{R}^{m+n})^m \to (\mathbb{R}^{m+n})$
satisfies $\quad \Omega^\sharp(u_1, \ldots, u_m) \perp \mathit{span}\{u_1, \ldots, u_m\}$
Set

$$G(F) = \Omega^\sharp(e_1, \ldots, e_m)$$
$$V_\Omega(F) = \frac{1}{m+1} \int_M \Omega(F, e_1, \ldots, e_m) dM$$
$$J(F) = A(F) + mV_\Omega(F)$$

where e_i is F^*g-d.o.n. Evolution eq. F_t for $F : M^m \to \bar{M}^{m+n}$,

$$\begin{cases} \frac{dF}{dt} = H_t - G(t) \\ F_0 = F \end{cases}$$

Then
$$\frac{d}{dt} J(t) = -\int_M \|H_t - G(t)\|^2 dM_t$$

so $J(t)$ nonincreasing with t.
If solution exists F_∞ and $H_\infty - G(\infty) = 0$, then:

- F_∞ has calibrated extended tg
- $\|H_\infty\| = 1$ is constant
- $\Omega^\sharp(e_1^\infty, \ldots, e_m^\infty) = H_\infty$ (prescribing the mean curvature)