Scalar conformal differential invariants

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Presentation

I will expose a collection of scalar conformal differential invariants, which, to my knowledge, has been no discussed up to now.

We will take the following steps:

- Define the concept of conformal invariant that we are going to consider.
- Prove the existence of a distinguishing metric g' in the class $[g]$ of a generic conformal structure.
- Show that metric invariants of g' of r-order are conformal invariants of $[g]$ of $(r+2)$-order
- For some well-known spacetimes, I have calculated the scalar curvature of g', which turns out a fourth order scalar conformal invariant of $[g]$ and we could call the *conformal scalar curvature* (CSC). We will compare it with usual metric scalars.

Introduction

It is not true that in a conformal class of (pseudo-)Riemannian metrics there is no preferred metric.

In March 1921 Einstein wrote (Berliner Bericht, 1921, 261–264):

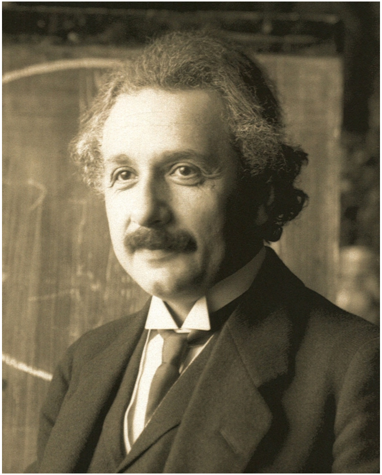
> *If we put $g'_{\mu\nu} = Jg_{\mu\nu}$ [J given next] then $d\sigma^2 = g'_{\mu\nu}dx_\mu dx_\nu$ is an invariant that depends only upon the ratios of the $g_{\mu\nu}$ [the conformal class]. All Riemann tensors formed as fundamental invariants from $d\sigma$ in the customary manner are—when seen as functions of the $g_{\mu\nu}$—Weyl tensors of weight 0 [conformally invariant]. ... Therefore, to every law of nature $T(g) = 0$ of the general theory of relativity, there corresponds a law $T(g') = 0$, which contains only the ratios of the $g_{\mu\nu}$.*

Einstein proposed to add $J = J_0$ (constant) to the field equations of GR:

> *Our only intention was to point out a logical possibility that is worthy of publication; it may be useful for physics or not.*

No one followed this idea, nor Einstein himself!

Albert Einstein



Albert Einstein during a lecture in Vienna in 1921.

https://commons.wikimedia.org/wiki/File:Einstein1921_by_F_Schmutzer_2.jpg

Bundle of G-structures

- Let $G \subset Gl_n$ be closed. The *bundle* M_G *of* G*-structures* on M: associated to the linear frame bundle $LM \equiv F^1 M$, with fiber ${}^{Gl_n}/_{G}$. Local sections of M_G correspond to local G-structures on M.

- *Bundle* M_{O_n} *of metrics* on M : $O_n =$ orthogonal group of signature (p, q). There is a diffeomorphism between ${}^{Gl_n}/_{O_n}$ and the symmetric invertible matrices of signature (p, q). A metric g of M corresponds to a section of $\sigma_g \colon M \to M_{O_n}$ given in coordinates by the matrix (g_{ij}).

- *Bundle* M_{C_n} *of conformal structures* : $C_n = \mathbb{R}^+ \cdot O_n$, conformal orthogonal group of signature (p, q). There is a diffeomorphism between ${}^{Gl_n}/_{C_n}$ and the symmetric invertible matrices of signature (p, q) and absolute value of its determinant one. A conformal structure $[g]$ on M corresponds to a section $\sigma_{[g]} \colon M \to M_{C_n}$, given in coordinates by $(c_{ij}) = |\det(g_{ij})|^{-1/n}(g_{ij})$.

- *The bundle* M^r_G *of* r*-jets of local sections of* M_G is associated to $F^{r+1}M$, with fiber $(\mathbb{R}^n_G)^r_0 \equiv J^r_0(\mathbb{R}^n, {}^{Gl_n}/_{G})$ of r-jets at 0 of maps of $\mathbb{R}^n$ to ${}^{Gl_n}/_{G}$. The r-frame bundle $F^r M$ (of r-jets at 0 of inverses of charts of M) is a principal bundle with the group G^r_n (of r-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ fixing 0).

Invariants of G-structures

A paradigm : The scalar curvature $R_g \colon M \to \mathbb{R}$ **is an invariant** because:

$$R_{\varphi^* g} = R_g \circ \varphi, \text{ for any local diffeomorphism } \varphi \text{ of } M. \tag{1}$$

But $R_g(m)$ is made from the 2-jet of g at $m \in M$, then it defines the function:

$$R \colon M^2_{O_n} \to \mathbb{R}, \quad R(j^2_m \sigma_g) := R_g(m); \tag{2}$$

and the property (1) becomes:

$$R \circ \widehat{\varphi}^2 = R, \quad \forall\, \varphi, \tag{3}$$

with $\widehat{\varphi}^2$ being the case $r = 2$ of the action of φ on M^r_G given by:

$$\widehat{\varphi}^r \colon M^r_G \to M^r_G, \quad j^r_p \sigma \mapsto j^r_{\varphi(p)}(\bar{\varphi} \circ \sigma \circ \varphi^{-1}), \tag{4}$$

with $\bar{\varphi}$ the lifting of φ to M_G. **Following (2) and (3) as model, we define:**

Definition

An *scalar invariant of r-order of G-structures on M* is a function $f \colon M^r_G \to \mathbb{R}$ verifying $f \circ \widehat{\varphi}^r = f$, for any local diffeomorphism φ of M.

Distinctive metric of a generic conformal structure

A tensor T on M, obtained from the r-jet of g by an specific formula, is said *conformally invariant* if, $\forall \bar{g} \in [g]$, T is equal to the tensor $\bar{T}$ obtained from the r-jet of $\bar{g}$ by the same formula.

- The conformal curvature or Weyl tensor ${C^i}_{jkl}$ is conformally invariant.
- The *square of the Weyl tensor*, $H = C_{ijkl}C^{ijkl}$, is not conformally invariant because it depends of the metric g used to raising and lowering indices.

A conformal structure $[g]$ will be called *generic* if $H(m) \neq 0, \forall\, m \in M$. The following theorem assert that a generic conformal structure owns a distinctive metric.

Theorem

Let $[g]$ be a generic conformal structure on M. The metric $g' := Jg \in [g]$, where $J = |H|^{1/2}$, do not depends of the metric g in the class of $[g]$.

Note that the theorem is of local character and can be formulated in a neighborhood of a generic point.

Distinctive metric of a generic conformal structure

Demostration

$$H = C_{ijkl}C^{ijkl} = g_{ia}C^{a}_{\ jkl}C^{i}_{\ bcd}g^{bj}g^{ck}g^{dl}.$$

If we change g by $\bar{g} = \alpha g$ then $\bar{g}_{ij} = \alpha g_{ij}$ and $\bar{g}^{ij} = \alpha^{-1}g^{ij}$.
Using that Weyl tensor is conformally invariant: $\bar{C}^{i}_{\ jkl} = C^{i}_{\ jkl}$, we obtain:

$$\bar{H} = \bar{C}_{ijkl}\bar{C}^{ijkl} = \bar{g}_{ia}C^{a}_{\ jkl}C^{i}_{\ bcd}\bar{g}^{bj}\bar{g}^{ck}\bar{g}^{dl} = \alpha^{-2}H.$$

Taking absolute values and square roots, we obtain $\bar{J} = \alpha^{-1}J$.
Finally, we get the result:

$$g' := Jg = \alpha^{-1}J\alpha g = \bar{J}\bar{g}.$$

Furthermore, the scalar J' corresponding to g' is constant one, hence we can say that **the distinctive metric of a generic conformal class is the only metric that normalizes to ± 1 the square of the Weyl tensor**.

Scalar conformal invariants

Theorem

To each scalar metric invariant of r-order on M corresponds an scalar conformal invariant of $(r+2)$-order on M.

(Sketch of the proof:) Because J is made from the partial derivatives up to second order of the metric g of M, the well-defined map between conformal structures and metrics, given by $[g] \mapsto g' = Jg$, depends of the 2-jet of the components c_{ij} that characterize $[g]$. Then, it can be show that the map between conformal structures and metrics induces, for successive order r, the maps

$$\Gamma\colon M^{r+2}_{C_n} \longrightarrow M^r_{O_n}, \qquad j^{r+2}_m \sigma_{[g]} \longmapsto j^r_m \sigma_{g'},$$

which is equivariant by diffeomorphisms. The result easily follows.

In particular, the scalar curvature of the metric g', which is a second order invariant, turns out to be a conformal invariant of fourth order that we call the *conformal scalar curvature (CSC)* of $[g]$.

Conformal scalar curvature

- I have calculated, for several spacetime metrics g, the scalar curvature of the distinctive metric g' of $[g]$; I mean the *CSC* of $[g]$. To do this I used the package *xAct`xCoba`* for the *Mathematica* software program.
- In the Table below, we can compare the scalar curvature, R_g, and the Kretschmann scalar, K_g, with the CSC of $[g]$, denoted by $S_{[g]}$.

Metrics	R_g	K_g	$S_{[g]}$
Schwarzschild	0	$\frac{48M^2}{r^6}$	$\frac{9\sqrt{3}}{4}(1-\frac{r}{6M})$
Reissner-Nordstrøm	0	$\frac{8(7q^4-12Mq^2r+6M^2r^2)}{r^8}$	$\frac{9\sqrt{3}}{8}(1-\frac{r}{3M}+\frac{q^2}{3M^2})$
Gödel	$-\frac{1}{a^2}$	$\frac{3}{a^4}$	$-\frac{\sqrt{3}}{2}$
Barriola-Vilenkin	$\frac{2-k^2}{k^2r^2}$	$\frac{4(k+1)^2(k-1)^2}{k^4r^4}$	$-\sqrt{3}$

Table: Scalar invariants of some spacetimes

As a conclusion: further research

- Einstein proposed the natural addition of the differential equation $J = J_0$ (constant), free of genericity conditions, to the field equations of General Relativity. What solutions of the field equations of GR verify $J = J_0 \neq 0$?

- The so-called Weyl gravity theory (or conformal gravity) is governed by the Lagrangian $H\,\Omega_g$, with Ω_g being the volume form of g. In four dimensions, $\Omega_{g'} = H\,\Omega_g$; what does this mean for that theory?

- The CSC of Schwarzschild spacetime is 0 when $r = 6M$ which is the radius of the *innermost stable circular orbit* (ISCO), the smallest stable orbit that exists for a test particle around a massive object. Could be related CSC with ISCO?

- The question about uniqueness: What other factors J can be made from g and its derivatives such that Jg is a distinctive metric of the conformal class $[g]$?

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