Hypersurfaces with prescribed curvature in warped products

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Problems

- Dirichlet problems for the prescribed mean curvature equation
- Evolution of graphs by mean curvature flow under Neumann boundary conditions
- Extension to higher order mean curvatures and anisotropic mean curvature
- Further developments

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Killing graphs

Let Y be a given conformal Killing vector field in a Riemannian manifold $\bar{M}^{n+1}$.

Thus, there exists a function $\varrho \in \mathcal{C}^\infty(\bar{M})$ such that

$$\mathcal{L}_Y \bar{g} = 2\varrho \bar{g}, \tag{1}$$

where $\bar{g}$ is the metric in $\bar{M}^{n+1}$.

From this we deduce the conformal Killing equation

$$\langle \bar{\nabla}_X Y, Z\rangle + \langle \bar{\nabla}_Z Y, X\rangle = 2\varrho\langle X, Z\rangle, \qquad X, Z \in \Gamma(T\bar{M}). \tag{2}$$

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The Killing cylinder K over $\Gamma = \partial\Omega$ is the hypersurface ruled by the flow lines of Y through Γ, that is,

$$K = \{\Phi(t, x) : x \in \Gamma\}. \tag{4}$$

Geometric facts

Since $\Phi_t = \Phi(t, \cdot)$ is a conformal map for any fixed $t \in \mathbb{I}$, there is a positive function $\lambda \in \mathcal{C}^\infty(\mathbb{I} \times M)$ such that $\lambda(0, \cdot) = 1$ and

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One also proves that the integral leaves of $\mathcal{D}$ are totally umbilical hypersurfaces with principal curvatures $k = -\frac{\lambda_t}{\lambda^2}\sqrt{\gamma}$.

If in addition Y is closed, then these leaves are spherical hypersurfaces, i.e., they have constant mean curvature.

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Theorem (Dajczer, –, Annales de l'IHP, 2009)

Let $\Omega \subset M^n$ be a bounded domain with $\mathcal{C}^{2,\alpha}$ boundary Γ. Assume that $H_K \geq 0$ and

$$\mathrm{Ric}_{\bar{M}} \geq -n \inf_\Gamma H_K^2.$$

Let $H \in \mathcal{C}^\alpha(\bar{\Omega})$ and $\phi \in \mathcal{C}^{2,\alpha}(\Gamma)$ be given. Assume that there exists a $\mathcal{C}^{2,\alpha}$ immersion $\iota : \bar{\Omega} \to \bar{M}$ transverse to the vertical fibers in $\pi^{-1}(\bar{\Omega})$. If

$$|H| \leq \inf_\Gamma H_K,$$

then there exists a unique function $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ satisfying $u|_\Gamma = \phi$ whose graph has mean curvature function H.

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- Helicoidal graphs in Euclidean space, an interesting case treated earlier in

  Dajczer and Lira, *Helicoidal graphs with prescribed mean curvature*, Proc. AMS **137**, 7, 2009.
- For proving the theorem above, we use a method of obtaining interior estimates due to Korevaar, based on comparing graphical and normal perturbations of a graph.

Conformal Killing graphs

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In this case the Riemannian ambient manifold is $\bar{M}^{n+1} = \mathbb{I} \times M^n$ endowed with a metric of the form

$$\mathrm{d}s^2 = \lambda^2(t)(\rho^2(x)\mathrm{d}t^2 + \sigma_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j), \tag{10}$$

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$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{\gamma + |\nabla u|^2}}\right) - \frac{1}{\sqrt{\gamma + |\nabla u|^2}}\left(\frac{\langle \nabla\gamma, \nabla u\rangle}{2\gamma} + \frac{n\gamma\lambda_t}{\lambda}\right) - n\lambda H = 0,$$

where the divergence and gradient are taken in M^n and $\gamma = \rho^{-2}$.

Local geometry of the graphs

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$$a_{ij} = \frac{u_{i;j}}{W} - \frac{u_i}{W}\frac{\gamma_j}{2\gamma} - \frac{u_j}{W}\frac{\gamma_i}{2\gamma} - \frac{u_i u_j}{2W} u^k \frac{\gamma_k}{\gamma^2} - \frac{\lambda_t}{\lambda} u_i u_j - \frac{\lambda_t}{\lambda}\gamma\sigma_{ij} \tag{13}$$

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Taking traces we deduce the equation above.

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- If we assume that $H = H(x)$, the first condition says that that the mean curvature of the leaves and of the graph have opposite signs.
- If the leaves are mean convex, that is, if $\lambda_t \geq 0$, then we fix $H \geq 0$.
- The second condition is related to the growing rate of the mean curvature of the leaves.

Existence results

Theorem (Dajczer, –, JGA, 2012)

Let $\Omega \subset M^n$ be a $\mathcal{C}^{2,\alpha}$ bounded domain such that

$$\mathrm{Ric}^{\mathrm{rad}}_{\bar{M}} \geq -n \inf_{\Gamma} H_K^2.$$

Assume $\lambda_t \geq 0$ and $(\lambda_t/\lambda)_t \geq 0$. Let $H \in \mathcal{C}^{\alpha}(\Omega)$ and $\phi \in \mathcal{C}^{2,\alpha}(\Gamma)$ be such that

$$\inf_{\Gamma} H_K > H \geq 0$$

and $\phi \leq 0$. Then, there exists a unique function $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ whose conformal Killing graph has mean curvature function H and boundary data ϕ.

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Corollary

When the conformal Killing field Y is closed the result holds with the assumption on the Ricci curvature replaced by

$$n \, \mathrm{Ric}_M^{\mathrm{rad}} \geq -(n-1)^2 \inf_{\Gamma} H_\Gamma^2.$$

Moreover, if Y is a Killing field we may assume

$$\inf_{\Gamma} H_K \geq H.$$

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$$\mathrm{Ric}^{\mathrm{rad}}_{\bar{M}}(x) = \mathrm{Ric}_{\bar{M}}(\eta, \eta), \tag{15}$$

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where $\eta \in T_x M$ is an unit vector tangent to the unique minimizing geodesic from $x \in \Omega_0$ to Γ.

It was shown by Y.-Y. Li and L. Nirenberg (2005) that the distance function $d(x) = \mathrm{dist}(x, \Gamma)$ in Ω_0 has the same regularity as Γ.

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The hypothesis on Ricci assures that Riccati's equation yields the "nice" behavior for H_{K_ϵ}:

Proposition

Assume that the Ricci curvature tensor of $\bar{M}^{n+1}$ satisfies

$$\inf_{\Omega_0} Ric_{\bar{M}}^{rad} \geq -n \inf_{\Gamma} H_K^2.$$

Then, $H_{K_\epsilon}|_{x_0} \geq H_K|_{y_0}$ if $y_0 \in \Gamma$ is the closest point to $x_0 \in \Gamma_\epsilon \subset \Omega_0$.

Existence results

- The statements may be rewritten in terms of the Ricci curvature of the leaf M using the following relation

$$\mathrm{Ric}_{\bar{M}}(\eta,\eta) + k' + nk^2 = \mathrm{Ric}_M(\eta,\eta) + \eta(\kappa) + \kappa^2 \tag{16}$$

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- Since

$$nH_K = \kappa + \frac{n-1}{\lambda} H_\Gamma \tag{17}$$

the mean convexity of the cylinder does not necessarily imply the mean convexity of the boundary of the domain.

Proof of the results

We apply the well-known continuity method to the family parametrized by $\tau \in [0,1]$ of Dirichlet problems

$$\begin{cases} \mathcal{Q}_\tau[u] = 0, \\ u|_\Gamma = \tau\phi \end{cases}$$

where

$$\mathcal{Q}_\tau[u] = \operatorname{div}\left(\frac{\nabla u}{\sqrt{\gamma + |\nabla u|^2}}\right) - \frac{\langle \nabla\gamma, \nabla u\rangle}{2\gamma\sqrt{\gamma + |\nabla u|^2}} - \tau\left(\frac{n\gamma\rho}{\sqrt{\gamma + |\nabla u|^2}} + n\lambda H\right).$$

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Let $\mathcal{I}$ be the subset of $[0,1]$ consisting of values of τ for which the Dirichlet problem has a $C^{2,\alpha}$ solution. Then, the proof reduces to show that $\mathcal{I} = [0,1]$.

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- The set $\mathcal{I}$ of $[0, 1]$ consisting of values of τ for which the above Dirichlet problem has a $\mathcal{C}^{2,\alpha}$ solution is open since the maximum principle holds.
- That $\mathcal{I}$ is closed follows from standard theory of quasilinear elliptic differential equations provided we have a priori estimates for solutions.

A priori estimates

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Proposition

There exists a positive constant $C = C(\Omega, H)$ such that

$$|u|_0 \leq C + |\phi|_0$$

for any solution u of the Dirichlet problem.

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We construct barriers on Ω_0 which are subsolutions to the PDE of the form

$$\varphi(x) = \inf_{\Gamma} \phi + f(d(x))$$

where $d(x) = \mathrm{dist}(x, \Gamma)$ and

$$f = \frac{e^{DB}}{D}\left(e^{-Dd} - 1\right)$$

where $B > \mathrm{diam}(\Omega)$ and $D > 0$ is a constant to be chosen.

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We use barriers defined on a tubular neighborhood Ω_ϵ of Γ of the form

$$f = -A \ln(1 + Bd) + \phi$$

where A and B are positive constants.

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where $v = |\nabla u|^2$.

If χ achieves its maximum on Γ we already have the desired bound. Thus, we may assume that χ attains maximum value at an interior point $x_0 \in \Omega$ where $|\nabla u| \neq 0$.

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On the other hand using the fact that χ has a maximum at x_0 and the particular choice of coordinates above we have

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The third derivatives are ruled out by Ricci commutation formula and we obtain at the end

$$\begin{aligned}(K^2\gamma - R + C(n,H,\gamma))v^3 - (K^2\gamma^2 + 2R\gamma + C(n,H,\lambda,\gamma))v^2 \\ + C(n,H,\lambda,\gamma)v \leq 0.\end{aligned}$$

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One concludes that $|\nabla^M u|$ is bounded in terms of $|u|_0$ and of the distance to the boundary of the domain.

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This assures the existence of such graphs in $\mathbb{H}^{n+1}$ (joint work with M. Dajczer and J. Ripoll).

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for prescribed functions $u_0 : \bar{\Omega} \to \mathbb{R}$ and $\mathcal{H} : [0,T] \times \bar{\Omega} \to \mathbb{R}$, with Neumann condition of the form

$$\langle N, \nabla d \rangle = \varphi \quad \text{in} \quad \partial\Omega \times [0,\infty), \tag{27}$$

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For Killing graphs the problem is rewritten in nonparametric terms as

$$\frac{\partial u}{\partial s} = W \mathrm{div} \frac{\nabla u}{W} - \gamma \langle \bar{\nabla}_Y Y, \nabla u \rangle - \mathcal{H} W \tag{28}$$

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Consider the linear parabolic operator

$$Lv = g^{ij} v_{i;j} - \left(\frac{1}{2\gamma} + \frac{1}{2W^2}\right)\gamma^i v_i - v_t. \tag{32}$$

Estimates

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Hence we have

$$\begin{aligned} LW - \frac{2}{W}|\nabla W|^2_\Sigma = {} & |A|^2 W + nHW^3\langle AY^T, Y^T\rangle - nHW^3\langle \frac{\nabla\gamma}{2\gamma^2}, N\rangle \\ & -3\gamma\langle AY^T, X_* \frac{\nabla\gamma}{2\gamma}\rangle + g^{ij}\frac{\gamma_{i;j}}{2\gamma} W - \frac{3}{4}\frac{|\nabla\gamma|^2}{4\gamma^2} W - \frac{1}{4}\langle \frac{\nabla\gamma}{2\gamma}, N\rangle^2 W \\ & +\gamma W\langle \bar{\nabla}_N \frac{\bar{\nabla}\gamma}{2\gamma^2}, N\rangle - W\langle \nabla^\Sigma \mathcal{H}, N\rangle - \frac{|\nabla\gamma|^2}{4\gamma}\frac{1}{W} - W_t. \end{aligned}$$

Estimates

On the other hand

$$\frac{1}{\eta}L\eta = \frac{1}{h}|A|^2\langle N, \nabla d\rangle + K^2\frac{\gamma|\nabla u|^2}{W^2} + K\mathcal{H}W + \ldots$$

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Using the fact that ηW attains maximum value at x_0, we proceed as before for obtaining an estimate of the form

$$W(s,x) \leq W(s_0,x_0)\frac{\eta(s_0,x_0)}{\eta(s,x)} \leq C_1 e^{C_2|u-u_0|_{\bar{\Omega}\times[0,T)}} \tag{33}$$

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for $s \in [0, T)$.

Existence result

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This result extends former ones due to B. Guan (Euclidean case) and M. Calle and L. Shahriyari ($M \times \mathbb{R}$).

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Existence results for the Dirichlet problem

Theorem (Flávio Cruz, –)

Suppose that $Ric_M \geq 0$. Assume that

- $\Psi > 0$, $\frac{\partial \Psi}{\partial u} \geq 0$
- *Ω is f-convex and satisfies*

$$\Psi(x,0) \leq f(\lambda', 0), \tag{34}$$

$$f_n(\lambda', 0) \geq 0, \tag{35}$$

where λ' are the principal curvatures of $\partial\Omega$.

Then, provided that there exists any bonded admissible subsolution of the equation $F = \Psi$ in Ω, there exists a unique admissible solution u of the Dirichlet problem with $\varphi = 0$.

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Assume that there exists a subsolution $\underline{u}$ *of* $F = \Psi$ *with* $\underline{u} = \varphi$ *on* $\partial\Omega$. *Suppose that* $\underline{u}$ *is* C^2 *and locally strictly convex in a neighborhood of* $\partial\Omega$.

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Assume that there exists a subsolution $\underline{u}$ of $F = \Psi$ with $\underline{u} = \varphi$ on $\partial\Omega$. Suppose that $\underline{u}$ is C^2 and locally strictly convex in a neighborhood of $\partial\Omega$. Then there exists a unique admissible solution u of the Dirichlet problem for any positive Ψ satisfying $\frac{\partial \Psi}{\partial z} \geq 0$ and for any boundary data φ.

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The second theorem extend contributions by Trudinger, Lin and Ivochkina to Riemannian ambients and for general curvature functions.

Thanks for your attention!