Notes on stable minimal surfaces

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1 Introduction

We will assume that the reader is familiar with the tools employed in a course of *Riemannian Geometry* [9] and *Partial Differential Equations* [16].

1.1 Motivation

A minimal submanifold in a Riemmanian manifold is an immersed submanifold, Σ, which is stationary for the volume functional of the induced metric, i.e., infinitesimal normal variations of Σ do not change its volume. From the Euler-Lagrange equations associated to this variational problem one sees that Σ is a minimal submanifold provided its mean curvature, $\vec{H}$, vanishes identically. The natural geometric quantity $\vec{H}$ is a section of the normal bundle of Σ and is given as the trace of the second fundamental form, $\vec{A}$, of Σ. For an oriented surface in an orientable three-manifold, $\vec{H}$ is a normal vector field to Σ whose length is half of the sum of the principal curvatures.

The study of minimal surfaces started in the 18th century by the works of Euler and Lagrange what represented the birth of the calculus of variations. Later, the physicist J. Plateau (problem raised by Lagrange) conducted classical experiments in this field, dipping wires bent into assorted shapes in tubs of soapy water. Plateau concluded that the soap films that formed were always minimal surfaces. Plateau hypothesized that:

> *For any given closed curve, you can always produce a minimal surface with the same boundary.*

Plateau's problem was solved simultaneously by Radó [24] and Douglas [10]. The kind of minimal surfaces considered by Plateau were area-minimizing surfaces, in particular, critical points of the area functional and hence minimal. Nevertheless, there is a weaker condition for a minimal surface than to be area-minimizer, and it is to be stable. A minimal surface is said to be stable if it does not decrease area up to second order, that is, the second variation of the area functional is nonnegative for any compactly supported variation fixing the boundary.

Stable and area-minimizing surfaces are the main focus along these notes.

1.2 Notation

Along these notes we denote by $\mathcal{N}$ a three-dimensional connected Riemannian manifold, and let $\Sigma \subset \mathcal{N}$ be an immersed, compact, two-sided surface in $\mathcal{N}$ with boundary $\partial\Sigma$ (possibly empty). Let us denote by $\vec{N}$ the unit normal vector field along Σ. Moreover, g is the metric on $\mathcal{N}$ and $\langle , \rangle$ is the induced metric on Σ. Set $\overline{\nabla}$ and ∇ the Levi-Civita connetion associated to g and $\langle , \rangle$ respectively. Denote by $\mathfrak{X}(\Sigma)$ and $\mathfrak{X}(\mathcal{N})$ the linear spaces of smooth vector fields along Σ and $\mathcal{N}$ respectively.

> **Convention:** *Throughtout these notes the will assume that all submanifolds, Σ, and manifolds, $\mathcal{N}$, where these are immersed are orientable and connected. Moreover, $\mathcal{N}$ will be always complete, even it might not be necessary. Also, we will identify g and $\langle , \rangle$ when no confussion occurs. Actually, we focus here in surfaces immersed in three-manifolds, but the generalization of some results to higher dimension are straightforward.*

First, let us fix the notation. Set

$$\overline{R}(X,Y)Z := \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, \ \ X, Y, Z \in \mathfrak{X}(\mathcal{N}), \tag{1.1}$$

the **Riemann Curvature Tensor** in $\mathcal{N}$. Let $\{e_i\} \in \mathfrak{X}(U)$, $U \subset \mathcal{N}$ open and connected, be a local orthonormal frame of the tangent bundle $TU \subset T\mathcal{N}$, then we denote

$$\overline{R}_{ijkl} = \langle \overline{R}(e_i, e_j)e_k, e_l\rangle$$

and the **sectional curvatures** in $\mathcal{N}$ are given by

$$\overline{K}_{ij} := \langle \overline{R}(e_i, e_j)e_j, e_i\rangle = \overline{R}_{ijji}. \tag{1.2}$$

Let us establish our definition for the **Ricci Curvature** and **Scalar Curvature** (in $\mathcal{N}$), i.e,

$$\begin{aligned} \mathrm{Ric}(u,u) &= \sum_{i=1}^{3} \overline{R}(e_i, u, u, e_i) \\ S &= \sum_{i=1}^{3} \mathrm{Ric}(e_i, e_i), \end{aligned}$$

respectively, here $\{e_i\}$ is a orthonormal frame for $\mathcal{N}$.

Thus, by the Gauss Formula, we obtain

$$\overline{\nabla}_X Y = \nabla_X Y + \langle A(X), Y\rangle \text{ for all } X, Y \in \mathfrak{X}(\Sigma),$$

where $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the **Weingarten (or Shape) operator** and it is given by

$$A(X) := -(\overline{\nabla}_X \vec{N})^T,$$

that is, $A(X)$ is the tangential component of $-\overline{\nabla}_X \vec{N}$. In fact, we do not need to take the tangential part in the above definition when we are dealing with orientable surfaces in orientable three-manifolds, but we use the general definition for the sake of completeness.

Since $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is a self-adjoint endomorphism, we denote the **mean curvature** and **extrinsic curvature** as

$$H = \frac{1}{2}\text{Trace}(A) \quad \text{and} \quad K_e = \det(A),$$

respectively. Let us denote $\vec{A} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to N(\mathcal{N})$ the **Second Fundamental Form** of Σ, that is,

$$\vec{A}(X, Y) := (\overline{\nabla}_X Y)^{\perp}, \; X, Y \in \mathfrak{X}(\Sigma),$$

here $(\cdot)^{\perp}$ means the normal part. Therefore, we can write

$$\langle \vec{A}(X, Y), \vec{N}\rangle = \langle A(X), Y\rangle, \; X, Y \in \mathfrak{X}(\Sigma).$$

So, the mean curvature vector of Σ is given by

$$2\,\vec{H}_p = \text{Tr}\vec{A}_p = \sum_{i=1}^{2} \vec{A}_p(v_i, v_i),$$

where $\{v_1, v_2\}$ is a orthonormal basis of $T_p\Sigma$. Let $\{e_1, e_2\}$ be principal directions, i.e.,

$$\overline{\nabla}_{e_i}\vec{N} = -\kappa_i e_i,$$

where κ_i are the principal curvatures, $i = 1, 2$. Hence, the Second Fundamental Form, i.e.,

$$II(u, v) = \langle -\overline{\nabla}_u \vec{N}, v\rangle, \; u, v \in T_p\Sigma,$$

in this frame of principal directions is given by

$$\mathcal{M}\left(II, \{e_1, e_2\}\right) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$

Let $\overline{R}$ and R denote the Riemann Curvature tensors of $\mathcal{N}$ and Σ respectively. Then, the Gauss Equation says that for all $X, Y, Z, W \in \mathfrak{X}(\Sigma)$ we have

$$\langle R(X,Y)Z, W\rangle = \langle \overline{R}(X,Y)Z, W\rangle + \langle A(X), W\rangle\langle A(Y), Z\rangle - \langle A(X), Z\rangle\langle A(Y), W\rangle.$$

In particular, if $K(X,Y) = \langle R(X,Y)Y, X\rangle$ and $\overline{K}(X,Y) = \langle \overline{R}(X,Y)Y, X\rangle$ denote the sectional curvatures in Σ and $\mathcal{N}$, respectively, of the plane generated by the orthonormal vectors $X, Y \in \mathfrak{X}(\Sigma)$, the Gauss Equation becomes

$$K(X,Y) = \overline{K}(X,Y) + \langle A(X), X\rangle\langle A(Y), Y\rangle - \langle A(X), Y\rangle^2.$$

A straightforward computation shows:

Proposition 1.1. *Under the above notation. Let $\Sigma \subset \mathcal{N}$ be a surface and consider $\{e_1, e_2\}$ a local orthonormal frame in Σ and $\vec{N}$ its unit normal vector field. Then, it holds:*

$$\begin{aligned}
\frac{1}{2}S &= \overline{K}_{12} + \mathrm{Ric}(\vec{N}, \vec{N}), \\
2H^2 - K &= \frac{1}{2}|A|^2 - \overline{K}_{12}, \\
|A|^2 + \mathrm{Ric}(\vec{N}, \vec{N}) &= 2H^2 - K + \frac{1}{2}\left(|A|^2 + S\right),
\end{aligned}$$

here K denotes the Gauss curvature of Σ and $|A|^2$ is the squared norm of the second fundamental form, i.e.,

$$|A|^2 = 4H^2 - 2K_e,$$

or equivalently,

$$|A|^2 = k_1^2 + k_2^2,$$

in a local frame of principal directions in Σ.

2 Variation Formulae for the area

In this section we will derive the first and second variation of a compact minimal surface with boundary in a three-manifold. We begin by obtaining the first variation formula for any compact surface with boundary and we give some applications as the isoperimetric inequality. After, we present the second variation formula for a compact minimal surface with boundary.

2.1 First Variation Formula

Let $\Sigma \subset \mathcal{N}$ be a compact surface with boundary $\partial\Sigma$ (possibly empty), and $0 \in \mathcal{I} \subset \mathbb{R}$ an interval. Let $F : \Sigma \times \mathcal{I} \to \mathcal{N}$ be an immersion defining a deformation of $\Sigma(0) = F(\Sigma \times \{0\})$. Let $X(p) = \frac{d}{dt}\big|_{t=0} F(p,t)$ denote the variation vector field of this variation, then:

Theorem 2.1 (First variation Formula)**.** *In the above conditions, if η denotes the inward unit normal along $\partial\Sigma$ and $\vec{H}$ the mean curvature vector along Σ, then*

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Area}\,(\Sigma(t)) = \int_{\partial\Sigma} \langle X, \eta\rangle ds - 2\int_{\Sigma} \langle X, \vec{H}\rangle dv_g. \tag{2.1}$$

Proof. Let us denote by $X^{\top}$ and $X^{\perp}$ the tangential and normal part of X along Σ respectively.

Then, the Change Variable's Theorem says

$$\mathrm{Area}(\Sigma(t)) = \int_{\Sigma} \mathrm{Jac}\,(dF(\cdot,t))\, dv_g,$$

here $\mathrm{Jac}\,(dF(\cdot,t))$ denotes the Jacobian of $F_t := F(\cdot,t) : \Sigma \to F(\Sigma,t)$, that is,

$$\mathrm{Jac}\,(dF(\cdot,t))\,(p) = \sqrt{\det\left(\langle d(F_t)_p(e_i), d(F_t)_p(e_j)\rangle_{i,j}\right)},$$

where $\{e_1, e_2\}$ denote an orthonormal basis for $T_p\Sigma$.

On one hand, let $\alpha_i : (-\varepsilon, \varepsilon) \to \Sigma$ be curves on Σ so that $\alpha_i(0) = p$ and $\alpha'(0) = e_i$ for $i = 1, 2$. Then

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \langle d(F_t)_p(e_i), d(F_t)_p(e_j)\rangle &= \left.\frac{d}{dt}\right|_{t=0} \langle \left.\frac{d}{ds}\right|_{s=0} F_t(\alpha_i(s)), \left.\frac{d}{ds}\right|_{s=0} F_t(\alpha_j(s))\rangle = \\
&= \langle \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} F_t(\alpha_i(s)), \left.\frac{d}{ds}\right|_{s=0} F_0(\alpha_j(s))\rangle \\
&+ \langle \left.\frac{d}{ds}\right|_{s=0} F_0(\alpha_i(s)), \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} F_t(\alpha_j(s))\rangle = \\
&= \langle \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} F_t(\alpha_i(s)), e_j\rangle + \langle e_i, \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} F_t(\alpha_j(s))\rangle \\
&= \langle \left.\frac{d}{ds}\right|_{s=0} X(\alpha_i(s)), e_j\rangle + \langle e_i, \left.\frac{d}{ds}\right|_{s=0} X(\alpha_j(s))\rangle = \\
&= \langle \nabla_{e_i} X, e_j\rangle + \langle e_i, \nabla_{e_j} X\rangle,
\end{aligned}
$$

where we have used that F_0 is the identity and the very definition of the variation vector field X. Recall, ∇ stands for the Levi-Civita connection of the induced metric on Σ.

On the other hand, bearing in main that just the integrand depends on t,

$$
\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(\Sigma(t)) = \int_\Sigma \left.\frac{d}{dt}\right|_{t=0} \mathrm{Jac}\left(dF(\cdot, t)\right)(p) dv_g.
$$

Now, we need the familiar formula for the derivative of a determinant:

$$
\left.\frac{d}{dt}\right|_{t} \det B(t) = \det B(t) \cdot \mathrm{Tr}(B'(t) B^{-1}(t)), \tag{2.2}
$$

where $B : (-\varepsilon, \varepsilon) \to \mathrm{Gl}(n, \mathbb{R})$ and Tr denotes the trace. Thus,

$$
\left.\frac{d}{dt}\right|_{t=0} \mathrm{Jac}\left(dF(\cdot, t)\right)(p) = \frac{1}{2}\mathrm{Tr}\left((\langle \nabla_{e_i} X, e_j\rangle + \langle \nabla_{e_i} X, e_j\rangle)_{i,j}\right) = \mathrm{div} X,
$$

that is,

$$
\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(\Sigma(t)) = \int_\Sigma \mathrm{div}(X)\, dv_g,
$$

where div is the divergence in Σ.

Now, on one hand

$$
\begin{aligned}
\int_\Sigma \mathrm{div}(X)\, dA &= \int_\Sigma \mathrm{div}(X^\top)\, dv_g + \int_\Sigma \mathrm{div}(X^\perp)\, dv_g = \\
&= \int_{\partial\Sigma} \langle X^\top, \eta\rangle\, ds + \int_\Sigma \mathrm{div}(X^\perp)\, dv_g
\end{aligned}
$$

where we have used the Stoke's Theorem.

On the other hand,

$$\begin{aligned}\operatorname{div}(X^{\perp})(p) &= \sum_{i=1}^{2}\langle \nabla_{e_i} X^{\perp}, e_i\rangle = \\ &= \sum_{i=1}^{2}\left(\nabla_{e_i}\langle e_i, X^{\perp}\rangle - \langle \nabla_{e_i} e_i, X(p)\rangle\right) = \\ &= -2\langle \vec{H}_p, X(p)\rangle\end{aligned}$$

where we have used that $\langle e_i, X^{\perp}(p)\rangle = 0$ and the definition of the mean curvature vector. Thus, joining these equations we are done. □

2.2 Minimal surfaces

Let $\{e_1, e_2\}$ be a base of principal directions, i.e., $\overline{\nabla}_{e_i}\vec{N} = -\kappa_i e_i$, where κ_i are the principal curvatures, $i = 1, 2$. Σ is **minimal** if the mean curvature vector vanishes identically on Σ, i.e., $\vec{H}_p = 0$ for all $p \in \Sigma$. Then Σ is minimal, if and only if, $\sum_{i=1}^{2}\kappa_i = 0$. If Σ is minimal, we have that $\kappa_1 = -\kappa_2 = \kappa$. Hence, the Second Fundamental Form, i.e.,

$$II(u, v) = \langle -\overline{\nabla}_u \vec{N}, v\rangle, \; u, v \in T_p\Sigma,$$

in this frame of principal directions is given by

$$\mathcal{M}\left(II, \{e_1, e_2\}\right) = \begin{pmatrix}\kappa & 0 \\ 0 & -\kappa\end{pmatrix}.$$

As we have pointed out in the Introduction, from the First Variation Formula (2.1), Σ is minimal if it is a **critical point** of the area functional for any compactly supported variation fixing the boundary.

2.3 Isoperimetric Inequality

Let us see some applications of the First Variation Formula. Let us consider $F : \Sigma \times \mathbb{R} \to \mathbb{R}^3$ given by $F(t, p) = e^t p$. Then, the variational vector field of this variation is given by

$$X(p) = p.$$

Thus, we have

Lemma 2.1. *Let $\Sigma \subset \mathbb{R}^3$ a compact minimal submanifold with boundary. Then*

$$2\text{Area}(\Sigma) = \int_{\partial\Sigma}\langle p, \eta\rangle \, ds$$

Proof. It is just apply the First Variation Formula to the above variation. $\square$

Thus, as a consequence of the above result we have:

Theorem 2.2. *Let $\Sigma \subset \mathbb{R}^3$ a compact minimal surface with boundary. Then*

$$\mathrm{Area}(\Sigma) \leq \frac{r}{2}\,\mathrm{Length}(\partial\Sigma)$$

being r such that $\mathcal{M} \subset \mathcal{B}(p,r)$.

Proof. By Lemma 2.1, we have

$$2\mathrm{Area}(\Sigma) = \int_{\partial\Sigma} \langle p, \eta\rangle\, ds \leq \int_{\partial\Sigma} |p|\; ds \leq r\,\mathrm{Length}(\partial\Sigma).$$

$\square$

Then,

Theorem 2.3. *Let $\Sigma \subset \mathbb{R}^3$ a compact minimal surface with connected boundary. Then there exists a positive constant C such that*

$$\mathrm{Area}(\Sigma) \leq C\,\mathrm{Length}(\partial\Sigma)^2.$$

Proof. If $\partial\Sigma$ is connected, then for $2r = \mathrm{Length}(\partial\Sigma)$ and $q \in \partial\Sigma$ (we can assume that q is the origin) we have that $\partial\Sigma \subset \mathcal{B}(0,r)$

$$\int_{\partial\Sigma} |p|\; ds \leq r\,\mathrm{Length}(\partial\Sigma).$$

$\square$

2.4 Second Variation Formula

We continue this section deriving the second variation formula of the area for minimal surfaces. We focus on specially interesting variational vector fields X, those which are normal, i.e., $X(p) = f(p)\vec{N}(p)$ and $f \in C_0^\infty(\Sigma)$, where $C_0^\infty(\Sigma)$ denotes the linear space of piecewise smooth function compactly supported on Σ.

Remark 2.1. *Actually, we only need $f \in H_0^{1,2}(\Sigma) = \left\{ f \in H^{1,2}(\Sigma) :\ f_{|\partial\Sigma} \equiv 0 \right\}$. If Σ were no compact, we will consider f as a compactly supported function.*

We have [28]:

Theorem 2.4 (Second Variation Formula)**.** *In the above conditions, then*

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \text{Area}\,(\Sigma(t)) = \int_\Sigma \left(|\nabla f|^2 - f^2|A|^2 - f^2\text{Ric}_{\mathcal{N}}(\vec{N},\vec{N})\right) dv_g. \tag{2.3}$$

Here $|A|^2$ denotes the square of the length of the second fundamental form of Σ, $\text{Ric}_{\mathcal{N}}(\vec{N},\vec{N})$ is the Ricci curvature of $\mathcal{N}$ in the direction of the normal $\vec{N}$ to Σ and ∇ is the gradient w.r.t. the induced metric.

Proof. Since we are considering normal variations, we can write

$$F(p,t) = \exp_p(tf(p)\vec{N}(p)),\ p \in \Sigma,$$

where exp denotes the exponential map of $\mathcal{N}$. It is easy to check, using the Homogeneity Lemma for geodesic ([9, page 64]), that its variation vector field is $X(p) = f(p)\vec{N}(p)$

$$\left.\frac{d}{dt}\right|_{t=0} F(p,t) = \left.\frac{d}{dt}\right|_{t=0} \exp_p(1,p,tf(p)\vec{N}(p)) = \left.\frac{d}{dt}\right|_{t=0} \gamma(1,p,tf(p)\vec{N}(p)) =$$
$$= \left.\frac{d}{dt}\right|_{t=0} \gamma(t,p,f(p)\vec{N}(p)) = f(p)\vec{N}(p),$$

where $\gamma(t,p,v)$ is the point at time t at $\mathcal{N}$ of the unique geodesic passing through p at time $t=0$ with velocity $v \in T_p\mathcal{N}$.

Now, for a differentiable curve $B : (-\varepsilon,\varepsilon) \to \text{Gl}(n,\mathbb{R})$, using (2.2), we obtain

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \sqrt{\det B(t)} = \frac{\sqrt{\det B(0)}}{4}\left(\left(\text{Tr}\left(B'(0)B^{-1}(0)\right)\right)^2 + 2\text{Tr}\left(B''(0)B^{-1}(0) - B'(0)^2B^{-2}(0)\right)\right).$$

Let $\{e_1,e_2\}$ denote an orthonormal basis for $T_p\Sigma$ so that $\overline{\nabla}_{e_i}\vec{N} = -k_ie_i$, here the k_i's stand for the principal curvatures of Σ with respect to $\vec{N}$. Let $\alpha_i : (-\varepsilon,\varepsilon) \to \Sigma$ be curves on Σ so that $\alpha_i(0) = p$ and $\alpha'(0) = e_i$ for $i = 1,2$.

We want to apply the above formula to

$$G(t) = \left(\langle d(F_t)_p(e_i), d(F_t)_p(e_j)\rangle\right)_{i,j}.$$

Then, $G(0) = \text{Id}$ since F_0 is the identity. Thus,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \sqrt{\det G(t)} = \frac{1}{4}\left(\left(\text{Tr}\left(G'(0)\right)\right)^2 + 2\text{Tr}\left(G''(0) - G'(0)^2\right)\right)$$

Reasoning as in the First Variation Formula we get

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \langle d(F_t)_p(e_i), d(F_t)_p(e_j)\rangle &= \frac{d}{dt}\Big|_{t=0} \langle \frac{d}{ds}\Big|_{s=0} F_t(\alpha_i(s)), \frac{d}{ds}\Big|_{s=0} F_t(\alpha_j(s))\rangle = \\
&= \langle \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} F_t(\alpha_i(s)), \frac{d}{ds}\Big|_{s=0} F_0(\alpha_j(s))\rangle \\
&+ \langle \frac{d}{ds}\Big|_{s=0} F_0(\alpha_i(s)), \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} F_t(\alpha_j(s))\rangle = \\
&= \langle \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} F_t(\alpha_i(s)), e_j\rangle + \langle e_i, \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} F_t(\alpha_j(s))\rangle \\
&= \langle \frac{d}{ds}\Big|_{s=0} X(\alpha_i(s)), e_j\rangle + \langle e_i, \frac{d}{ds}\Big|_{s=0} X(\alpha_j(s))\rangle = \\
&= \langle \overline{\nabla}_{e_i} X, e_j\rangle + \langle e_i, \overline{\nabla}_{e_j} X\rangle = \\
&= f(p)\left(\langle \overline{\nabla}_{e_i} \vec{N}, e_j\rangle + \langle e_i, \overline{\nabla}_{e_j} \vec{N}\rangle\right) = \\
&= -2k_i(p) f(p)\langle e_i, e_j\rangle,
\end{aligned}
$$

where we have used that F_0 is the identity and the very definition of the variation vector field X. Thus,

$$G'(0) = -2f\,(k_i\delta_{i,j})_{i,j}\,,$$

where $\delta_{i,j} = 1$ if $i = j$ and zero otherwise. So,

$$\mathrm{Tr}(G'(0)) = -4Hf = 0,$$

and

$$\mathrm{Tr}(G'(0)^2) = 4f^2|A|^2.$$

We still need to compute $G''(0)$. To do so, we introduce the parametrized surfaces. Set $g_i : \mathcal{I} \times (-\varepsilon, \varepsilon) \to \mathcal{N}$, for $i = 1, 2$, the parametrized surface $g_i(s,t) = F(\alpha_i(s), t)$, then $\frac{\partial g_i}{\partial s}(0,t) = d(F_t)_p(e_i)$.

Therefore, following the usual notation for the covariant derivative (here, the covariant derivative, either $\dfrac{D}{dt}$ or $\dfrac{D}{ds}$, is with respect to the metric on the ambient space $\mathcal{N}$) along a curve (see [9, page 50]), we have

$$
\begin{aligned}
G''(0) &= \frac{d^2}{dt^2}\Big|_{t=0} \langle d(F_t)_p(e_i), d(F_t)_p(e_j)\rangle = \frac{d^2}{dt^2}\Big|_{t=0} \langle \frac{\partial g_i}{\partial s}(0,t), \frac{\partial g_j}{\partial s}(0,t)\rangle = \\
&= \langle \frac{D}{dt}\frac{D}{dt}\frac{\partial g_i}{\partial s}(0,0), \frac{\partial g_j}{\partial s}(0,0)\rangle + 2\langle \frac{D}{dt}\frac{\partial g_i}{\partial s}(0,0), \frac{D}{dt}\frac{\partial g_j}{\partial s}(0,0)\rangle + \langle \frac{\partial g_i}{\partial s}(0,0), \frac{D}{dt}\frac{D}{dt}\frac{\partial g_i}{\partial s}(0,0)\rangle.
\end{aligned}
$$

Since in a parametrized surface $\frac{D}{dt}\frac{\partial g_i}{\partial s}(s,t) = \frac{D}{ds}\frac{\partial g_i}{\partial t}(s,t)$, for any $i \in \{1,2\}$, we have

$$\begin{aligned}
\frac{D}{dt}\frac{\partial g_i}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial g_i}{\partial t}(0,0) &= \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \gamma(t, \alpha(s), f(\alpha(s))\vec{N}(\alpha(s))) = \\
&= \left.\frac{d}{ds}\right|_{s=0} \gamma'(0, \alpha(s), f(\alpha(s))\vec{N}(\alpha(s))) = \\
&= \left.\frac{d}{ds}\right|_{s=0} f(\alpha(s))\vec{N}(\alpha(s)) = (df)_p(e_i)\vec{N}(p) + f(p)\overline{\nabla}_{e_i}\vec{N} = \\
&= (df)_p(e_i)\vec{N}(p) - f(p)k_i(p)e_i,
\end{aligned}$$

and so

$$\langle \frac{D}{dt}\frac{\partial g_i}{\partial s}(0,0), \frac{D}{dt}\frac{\partial g_j}{\partial s}(0,0)\rangle = (df)_p(e_i)(df)_p(e_j) + f(p)^2 k_i(p)^2 \delta_{i,j}.$$

On the other hand, by [9, Lemma 4.1] and since $\frac{\partial g_i}{\partial t}(0,0) = f(p)\vec{N}(p)$ and $\frac{\partial g_i}{\partial s}(0,0) = e_i$,

$$\begin{aligned}
\frac{D}{dt}\frac{D}{dt}\frac{\partial g_i}{\partial s}(0,0) = \frac{D}{dt}\frac{D}{ds}\frac{\partial g_i}{\partial t}(0,0) &= R\left(\frac{\partial g_i}{\partial s}(0,0), \frac{\partial g_i}{\partial t}(0,0)\right)\frac{\partial g_i}{\partial t}(0,0) + \frac{D}{ds}\frac{D}{dt}\frac{\partial g_i}{\partial t}(0,0) = \\
&= R\left(\frac{\partial g_i}{\partial s}(0,0), \frac{\partial g_i}{\partial t}(0,0)\right)\frac{\partial g_i}{\partial t}(0,0) = f(p)^2 R\left(e_i, \vec{N}(p)\right)\vec{N}(p)
\end{aligned}$$

where we have used that $\frac{D}{dt}\frac{\partial g_i}{\partial t}(s_0, t) = 0$, since the curve $t \longmapsto g_i(s_0, t)$ is a geodesic (in the ambient space) for each fixed value s_0. Here R denote the Riemann curvature tensor of the ambient space $\mathcal{N}$.

With all these equations and the definition of the Ricci curvature in mind, we obtain

$$\mathrm{Tr}(G''(0)) = 2\left\|\nabla f\right\|^2 + 2f^2|A|^2 - 2f^2\mathrm{Ric}_{\mathcal{N}}(\vec{N}, \vec{N}).$$

Summarizing,

$$\begin{aligned}
\mathrm{Tr}(G'(0)) &= 0 \\
\mathrm{Tr}(G'(0)^2) &= 4f^2|A|^2 \\
\mathrm{Tr}(G''(0)) &= 2\left\|\nabla f\right\|^2 + 2f^2|A|^2 - 2f^2\mathrm{Ric}_{\mathcal{N}}(\vec{N}, \vec{N}).
\end{aligned}$$

Thus, coming back to the original equation

$$\begin{aligned}
\left.\frac{d^2}{dt^2}\right|_{t=0} \sqrt{\det G(t)} &= \frac{1}{4}\left(\left(\mathrm{Tr}\left(G'(0)\right)\right)^2 + 2\mathrm{Tr}\left(G''(0) - G'(0)^2\right)\right) = \\
&= \left\|\nabla f\right\|^2 - f^2|A|^2 - f^2\mathrm{Ric}_{\mathcal{N}}(\vec{N}, \vec{N}).
\end{aligned}$$

□

3 Stability via Schrodinger Operators

In this section we will relate geometric properties of the second variation of the area functional and analytic properties of differential operators acting on the space of piecewise smooth functions. Actually, we will restrict ourself here to piecewise smooth functions vanishing at the boundary of the domain, nevertheless the differentiability conditions are weaker than this, we have chosen this for the reader clarity. See [20] for a more accurate development.

3.1 Stability

A **stable** compact domain Σ on a minimal surface in a Riemannian three-manifold $\mathcal{N}$ is one whose area cannot be decreased up to second order by a variation of the domain leaving the boundary $\partial\Sigma$ fixed, in other words, if

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Area}(\Sigma(t)) \geq 0,$$

for any variation of the domain leaving the boundary fixed. In the case that Σ is complete (without boundary), we will say that it is **stable** if for any relatively compact domain $\Omega \subset \Sigma$, the area cannot be decreased up to second order by a variation Ω of the domain leaving the boundary $\partial\Omega$ fixed.

We can write Second Variation Formula (2.3) as

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Area}(\Sigma(t)) = -\int_\Sigma fLf\, dv_g,$$

where L is the linearized operator of mean curvature or Jacobi operator (see [26]), that is,

$$L := \left.\frac{d}{dt}\right|_{t=0} H(t) = \Delta + |A|^2 + \mathrm{Ric}_{\mathcal{N}}(\vec{N}, \vec{N}),$$

where $H(t)$ is the mean curvature of the variational surface $\Sigma(t)$. The above operator L is well known as the **stability operator**.

For people familiar with stable surfaces, such operator is a self-adjoint elliptic operator, so then, by standard PDE Theory, there exists an orthonormal basis of $L^2(\Sigma)$ consisting of smooth **eigenfunctions** of L that vanishes at $\partial\Sigma$ (recall that we consider Σ compact), i.e., $\{u_i\} \subset L^2(\Sigma)$ so that $Lu_i + \lambda_i u_i = 0$. Here, λ_i are the **eigenvalues**. Futhermore, the eigenvalues verify

$$\lambda_1 \leq \lambda_2 \leq \cdots \to +\infty.$$

In terms of L, Σ is said to be **stable** if L is nonnegative, i.e., all its eigenvalues are nonnegatives. Σ is said to have **finite index** if L has only finitely many negative eigenvalues.

So, in this setting

$$\mathrm{Index}(\Sigma) = \max\{k : \lambda_k < 0\},$$

$$\mathrm{Null}(\Sigma) = \sharp\{k : \lambda_k = 0\}.$$

Next, we collect several results on Schrödinger operator for those nonfamiliar with the theory.

3.2 Schrödinger operators

We will follow the references [7] and [16]. Consider (Σ, g) a compact Riemannian surface with boundary $\partial\Sigma$ (possibly empty). Set $q \in C^\infty(\Sigma)$ and consider the differential linear operator, called **Schrödinger operator**, given by

$$\begin{array}{rccc} L: & C_0^\infty(\Sigma) & \to & C^\infty(\Sigma) \\ & u & \to & Lu := \Delta u + q\,u, \end{array}$$

where Δ is the Laplacian with respect to the Riemannian metric g and $C_0^\infty(\Sigma)$ stands for the linear space of compactly supported piecewise smooth functions on Σ.

> **Convention:** *The required differentiability is much lower of that we consider here (see [20]). Moreover, we will identify the Riemannian metric on Σ with $g \equiv \langle \cdot, \cdot \rangle$. When we say Σ is a compact surface it will mean that Σ is a compact surface with boundary $\partial\Sigma$ possibly empty.*

Take $u, v \in C_0^\infty(\Sigma)$, integrating by parts we obtain

$$(-Lu, v)_{L^2} = -\int_\Sigma (\Delta u + q\,u)\,v dv_g = \int_\Sigma (\langle \nabla u, \nabla v \rangle - q\,uv) dv_g,$$

and therefore L is self-adjoint with respect to the L^2-metric, i.e.,

$$(-Lu, v)_{L^2} = \int_\Sigma (\langle \nabla u, \nabla v \rangle - q\,uv) dv_g = (-Lv, u)_{L^2}.$$

Then, we can associated a bilinear quadratic form to L given by

$$\begin{array}{rccc} Q: & C_0^\infty(\Sigma) \times C_0^\infty(\Sigma) & \to & \mathbb{R} \\ & (u, v) & \to & \int_\Sigma (\langle \nabla u, \nabla v \rangle - q\,uv) dv_g, \end{array} \tag{3.1}$$

and so

$$(-Lu, v)_{L^2} = Q(u, v) = (-Lv, u)_{L^2}.$$

Definition 3.1. *We say that the Schrödinger operator L is* **stable** *if $Q(u,u) \geq 0$ for all $u \in C_0^\infty(\Sigma)$.*

A direct observation is the following

Proposition 3.1. *Set $q_1, q_2 \in C^\infty(\Sigma)$ so that $q_1 \geq q_2$ in Σ. If $L_1 := \Delta + q_1$ is stable, then $L_2 := \Delta + q_2$ is stable in Σ.*

Let $\lambda \in \mathbb{R}$ be a real number, if $u \in C_0^\infty(\Sigma)$ satisfies

$$Lu + \lambda u = 0 \text{ on } \Sigma,$$

then for any test function $\psi \in C_0^\infty(\Sigma)$, since L is self-adjoint w.r.t. the L^2-metric, we have

$$0 = (Lu + \lambda u, \psi)_{L^2} = -Q(u, \psi) + \lambda \int_\Sigma u\psi dv_g.$$

Definition 3.2. *We say that $\lambda \in \mathbb{R}$ is an* **eigenvalue** *of L is there exists $u \in C_0^\infty(\Sigma) \setminus \{0\}$ satisfying $Lu + \lambda u = 0$. In this case, u is called the* **eigenfunction** *of L associated to the eigenvalue λ.*

Hence, we can characterize the eigenfunctions associated to the eigenvalue λ as those functions $u \in C_0^\infty(\Sigma) \setminus \{0\}$ so that

$$Q(u, \psi) = \lambda \int_\Sigma u\psi \, dv_g \, , \; \forall \psi \in C_0^\infty(\Sigma), \tag{3.2}$$

If $\lambda \in \mathbb{R}$ is an eigenvalue of L, we denote by $V_\lambda \subset C_0^\infty(\Sigma)$ the set of functions satisfying (3.2). By its definition, $V_\lambda \neq 0$ and it is a vectorial subspace of $C_0^\infty(\Sigma)$ by (3.2) and the linearity of Q. We call V_λ the **eigenspace** of L associated to λ.

Lemma 3.1. *Let Σ be a compact Riemannian surface with boundary $\partial\Sigma$ (possibly empty) and $L := \Delta + q$, $q \in C^\infty(\Sigma)$.*

- *If $\lambda, \mu \in \mathbb{R}$ are distinct eigenvalues of L, then the eigenspaces V_λ and V_μ are L^2-orthogonal.*
- *The dimension of any eigenspace is finite.*

Next, we would like to localized such eigenvalues of L. To do so, we will introduce the **Rayleigh's Quotient**

$$\mathcal{R}(u) = \frac{Q(u,u)}{\|u\|_{L^2}^2} = \frac{\int_\Sigma (\|\nabla u\|^2 - qu^2) dv_g}{\int_\Sigma u^2 \, dv_g}, \; u \in C_0^\infty(\Sigma) \setminus \{0\} \, .$$

On the one hand, note that if $\lambda \in \mathbb{R}$ is an eigenvalue of L and $u \in V_\lambda$, then

$$\mathcal{R}(u) = \lambda.$$

On the other hand, since $\mathcal{R}$ is invariant under homotheties, we only need to take into account functions with L^2-norm equals to 1 to study the possible minimums of $\mathcal{R}$.

Theorem 3.1. *Let Σ be a compact Riemannian surface with boundary $\partial\Sigma$ (possibly empty) and $L := \Delta + q$, $q \in C^\infty(\Sigma)$. Set $B_1 = \{u \in C_0^\infty(\Sigma) : \|u\|_{L^2} = 1\}$, then*

$$\lambda_1 = \inf \{\mathcal{R}(u) \ : \ u \in B_1\}$$

is an eigenvalue for L and there exists $\psi_1 \in B_1 \cap V_{\lambda_1}$ satisfying

$$Q(\psi_1, \psi_1) = \mathcal{R}(\psi_1) = \lambda_1.$$

Moreover, the eigenfunctions of V_{λ_1} can be variationally characterized as

$$\mathcal{R}(u) \geq \lambda_1 \ \textit{for all} \ u \in C_0^\infty(\Sigma) \setminus \{0\},$$

and equality holds if, and only if, $u \in V_{\lambda_1}$.

So, we have found the first eigenvalue and a first eigenfunction. Therefore, we continue this method in the orthogonal complement of this function. In general, we can state:

Theorem 3.2. *Let Σ be a compact Riemannian surface with boundary $\partial\Sigma$ (possibly empty) and $L := \Delta + q$, $q \in C^\infty(\Sigma)$. Set $B_{k+1} = \{u \in C_0^\infty(\Sigma) : \|u\|_{L^2} = 1\} \cap \{\psi_i\}^\perp_{i=1,\ldots,k}$, then*

$$\lambda_{k+1} = \inf \{\mathcal{R}(u) \ : \ u \in B_{k+1}\}$$

is an eigenvalue for L and there exists $\psi_{k+1} \in B_{k+1} \cap V_{\lambda_{k+1}}$ satisfying

$$Q(\psi_{k+1}, \psi_{k+1}) = \mathcal{R}(\psi_{k+1}) = \lambda_{k+1}.$$

Moreover, the eigenfunctions of $V_{\lambda_{k+1}}$ can be variationally characterized as

$$\mathcal{R}(u) \geq \lambda_{k+1} \ \textit{for all} \ u \in \left(C_0^\infty(\Sigma) \cap \{\psi_i\}^\perp_{i=1,\ldots,k}\right) \setminus \{0\},$$

and equality holds if, and only if, $u \in V_{\lambda_{k+1}}$.

The next result gives us the structure of the set of eigenvalues and eigenfunctions associated to a Schrödinger operator L:

Theorem 3.3. *Let Σ be a compact Riemannian surface with boundary $\partial\Sigma$ (possibly empty) and $L := \Delta + q$, $q \in C^\infty(\Sigma)$. Denote by $\{\lambda_k\}$ and $\{\psi_k\}$ the sequence of eigenvalues and eigenfuntions respectively, then*

- *$\{\lambda_k\}$ diverges to $+\infty$.*
- *If $u \in C_0^\infty(\Sigma)$ and u is L^2-orthogonal to $\{\psi_k\}$, then $u \equiv 0$.*
- *There are no eigenvalues of L in $\mathbb{R} \setminus \{\lambda_k\}$.*
- *For any k, the multiplicity of λ_k equals to the dimension of V_{λ_k}.*
- *$\{\psi_k\}$ is a Hilbert base of $L^2(\Sigma)$.*

3.3 Characterization: Fischer-Colbrie Criteria

We are interested on the study of stable Schrödinger operators on complete (without boundary) surfaces. We know how to define a stable Schrödinger operator on compact (with boundary) domains in terms of its first eigenvalue, when Σ is complete with no boundary, we will say that $L = \Delta + q$ is **stable** if

$$\lambda_1(\Sigma) = \inf\{\lambda_1(\Omega) \,:\, \Omega \subset \Sigma\} \geq 0,$$

for all $\Omega \subset \Sigma$ relatively compact domain with boundary, at least, C^1.

So, a first task to do is to characterize this property in terms of solutions of the differential equation $Lu = 0$, this is the Fischer-Colbrie criteria. Before we state the Fischer-Colbrie Criteria, we remind the following Maximum Principle for Schrödinger operators:

Proposition 3.2 (Maximum Principle)**.** *Let $q \in C^\infty(\Sigma)$, $\Omega \subset \Sigma$ a relatively compact subdomain and $u \in C^\infty(\Sigma)$ such that*

$$\begin{cases} \Delta u + qu \leq 0 & in \quad \Omega \\ \quad u \geq 0 & in \quad \Omega. \end{cases}$$

Then either $u > 0$ or u vanishes identically on Ω.

Proof. Suppose that there exists $p_0 \in \Omega$ so that $u(p_0) = 0$. Set $q_0 := \inf q(p) \,:\, p \in \Omega$ and $c := \min\{0, q_0\} \leq 0$. Then, the function $v = -u$ satisfies:

$$\begin{cases} \Delta v + cv \geq 0 & \text{in} \quad \Omega \\ \quad v \leq 0 & \text{in} \quad \Omega. \end{cases}$$

Since $c \leq 0$ and v attains a maximum at an interior point, the Maximum Principle [16, Theorem 3.5] implies that v is constant in Ω and hence, u vanishes identically in Ω, proving the result. □

Now, we are ready to establish the Fischer-Colbrie criteria (see [14]):

Lemma 3.2 (Fischer-Colbrie Criteria)**.** *Let Σ be a complete (or compact with boundary) surface and $L = \Delta + q$ be a Schrödinger operator acting on $C_0^\infty(\Sigma)$. The following statements are equivalent:*

(1) The operator L is stable on Σ.

(2) There is a smooth positive function u on Σ and $u = 0$ on $\partial\Sigma$ such that $\Delta u + qu = 0$.

(3) There is a smooth positive function u on Σ and $u = 0$ on $\partial\Sigma$ such that $\Delta u + qu \leq 0$.

Proof. **(1) implies (2):** By the variational characterization of the stability of L using Rayleigh's quotient, we have

$$\lambda_1(\Sigma) = \inf\left\{\frac{Q(f,f)}{\int_\Sigma f^2 dv_g} \,:\, f \in C_0^\infty(\Sigma),\, f \neq 0\right\} \geq 0.$$

The monotonicity of the first eigenvalue under inclusion, that is, $\lambda_1(\Omega) > \lambda_1(\Omega')$ if $\Omega \subset \overline{\Omega} \subset \Omega'$, implies that $\lambda_1(\Omega) > 0$ for any relatively compact domain $\Omega \subset \Sigma$ whose closure does not touch the boundary $\partial\Sigma$.

Classical elliptic theory [16, Chapter 8] ensures us the existence of an unique solution $v \in C_0^\infty(\Omega)$ to the Dirichlet problem

$$\begin{cases} \Delta v + qv = -q & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

if $\partial\Omega$ is at least of class C^1. Set $w = v + 1$, then w satisfies

$$\begin{cases} \Delta w + qw = 0 & \text{in} \quad \Omega \\ w = 1 & \text{on} \quad \partial\Omega, \end{cases}$$

Claim A: *$w > 0$ in Ω.*

Proof of Claim A:. Assume there exists $x_0 \in \Omega$ so that $w(p_0) < 0$. Consider the connected component U of $w^{-1}(-\infty, 0)$ containing p_0, i.e., $p_0 \in U \subset w^{-1}(-\infty, 0) \subset \Omega$. Then

$$\begin{cases} \Delta w + qw = 0 & \text{in} \quad U \\ w = 0 & \text{on} \quad \partial U, \end{cases}$$

that is, $\lambda_1(U) = 0$ by the Rayleigh's variational characterization, a contradiction since the first eigenvalue increases under the inclusion.

Hence $w \geq 0$ in Ω. Now, by the Maximum Principle (see Proposition 3.2), we have $w > 0$ in Ω and Claim A is proved. □

Set $p \in \Sigma$ a point, and consider a smooth compact exhaustion by relatively compact subdomains $\Omega_n \subset \Sigma$ so that $p \in \Omega_1$ and $\overline{\Omega_n} \subset \Omega_{n+1}$ for all n. For each n, consider $w_n \in C_0^\infty(\Sigma)$ the positive smooth function constructed above associated to the relatively compact domain Ω_n. Set $u_n := \frac{w_n}{w_n(p)}$ for each n.

Then, the sequence $\{u_n\}$ is uniformly bounded on compact sets of Σ by the Harnack Inequality [16, Theorem 8.20]. Also, $\{u_n\}$ has all its derivatives uniformly bounded on compact subsets of Σ by Schauder estimates [16, Theorem 6.2]. Therefore, Arzela-Ascoli's Theorem and a diagonal argument give us that a subsequence (that we still denote by $\{u_n\}$) converges on compact subsets of Σ to a function $u \in C^\infty(\Sigma)$ which satisfies

$$\begin{cases} \Delta u + qu = 0 & \text{in} \quad \Sigma \\ u \geq 0 & \text{in} \quad \Sigma \\ u(p) = 1 & \text{at} \quad p \in \Sigma, \end{cases}$$

again, by the Maximum Principle (Proposition 3.2), we obtain that $u > 0$ in Σ. Hence, (1) implies (2) is proved.

(2) implies (1): Assume there exists a positive solution $u > 0$ in Σ to $\Delta u + qu = 0$. Then, the function $w := \ln u \in C^\infty(\Sigma)$ satisfies:

- $\nabla w = u^{-1}\nabla u$,
- $\Delta w = u^{-2}(u\Delta u - |\nabla u|^2) = -q - |\nabla w|^2$.

Take $f \in C_0^\infty(\Sigma)$, then integrating by parts we have

$$\begin{aligned} \int_\Sigma f^2(|\nabla w|^2 + q)\, dv_g &= -\int_\Sigma f^2 \Delta w \, dv_g = \int_\Sigma \langle \nabla f^2, \nabla w\rangle \, dv_g = \int_\Sigma 2f\langle \nabla f, \nabla w\rangle \, dv_g \\ &\leq \int_\Sigma 2|f||\nabla f||\nabla w| \, dv_g \leq \int_\Sigma (f^2|\nabla w|^2 + |\nabla f|^2)\, dv_g, \end{aligned}$$

from where we obtain

$$Q(f,f) = \int_\Sigma |\nabla f|^2 - qf^2 \, dv_g \geq 0,$$

that is, L is stable.

(2) implies (3): It is obvious.

(3) implies (1): Assume there exists a positive solution $u > 0$ in Σ to $\Delta u + qu \leq 0$. Take $f \in C_0^\infty(\Sigma)$ and set $\psi = f/u \in C_0^\infty(\Sigma)$, then integrating by parts we have

$$
\begin{aligned}
Q(f,f) &= \int_\Sigma |\nabla f|^2 - qf^2 \, dv_g = \int_\Sigma |\nabla u\psi|^2 - qu^2\psi^2 \, dv_g = \int_\Sigma -u\psi\Delta(u\psi) - qu^2\psi^2 \\
&= \int_\Sigma -u\psi^2\Delta u - u^2\psi^2 q - 2u\psi\langle \nabla u, \nabla\psi\rangle - u^2\psi\Delta\psi \, dv_g \\
&= \int_\Sigma -u\psi^2(\Delta u - qu) - 2u\psi\langle \nabla u, \nabla\psi\rangle - u^2\psi\Delta\psi \, dv_g \geq \int_\Sigma \langle \nabla\psi^2, \nabla u^2\rangle - u^2\psi\Delta\psi \, dv_g \\
&= \int_\Sigma \langle \nabla\psi^2, \nabla u^2\rangle + \langle \nabla u\psi^2, \nabla u\rangle \, dv_g = \int_\Sigma |\nabla\psi|^2 u^2 \, dv_g \geq 0,
\end{aligned}
$$

that is, $Q(f,f) \geq 0$, therefore L is stable as wished.

$\square$

4 Colding-Minicozzi Inequality

The aim here is to establish a general inequality for the quadratic bilinear form $Q(f, f)$ (see (3.1)) associated to a Schrödinger operator L when f is a radial function defined on a geodesic disk, following the method used by T. Colding and W. Minicozzi in [8]. The proof of this can be found in [6, 13].

As above, we denote by Σ a connected Riemannian surface, with Riemannian metric g, and possibly with boundary $\partial\Sigma$. Let $p_0 \in \Sigma$ be a point of the surface and $D(p_0, s)$, for $s > 0$, denote the geodesic disk centered at p_0 of radius s. We assume that $\overline{D(p_0, s)} \cap \partial\Sigma = \emptyset$. Moreover, let r be the radial distance of a point p in $D(p_0, s)$ to p_0. We write $D(s) = D(p_0, s)$ if no confussion occurs.

We also denote

$$\begin{aligned} l(s) &= \text{Length}(\partial D(s)), \\ a(s) &= \text{Area}(D(s)), \\ K(s) &= \int_{D(s)} K dv_g, \\ \chi(s) &= \text{Euler characteristic of } D(s), \end{aligned}$$

where length and area are measured with respect to the metric g.

First, we will need the following result due to K. Shiohama and M. Tanaka (see [30]) which follows from the first variation formula for length and the Gauss-Bonnet formula.

Theorem 4.1. *The function l is differentiable almost everywhere, and we have*

1. *for almost all $r \in \mathbb{R}$,*
$$l'(r) \leq 2\pi\chi(r) - K(r), \tag{4.1}$$
where $'$ denotes the derivative with respect to r.

2. *for all $0 \leq a < b$,*
$$l(b) - l(a) \leq \int_a^b l'(r). \tag{4.2}$$

In this section we will consider Schrödinger operators on the form

$$L = \Delta + a\,K + V,$$

where a is a real constant, K is the Gaussian curvature of g and $V \in C^\infty(\Sigma)$. In what follows, V is called the **potential**.

Lemma 4.1 (Colding-Minicozzi stability inequality)**.** *Let Σ be a Riemannian surface possibly with boundary. Let us fix a point $p_0 \in \Sigma$ and positive numbers $0 \le \varepsilon < s$ such that $\overline{D(s)} \cap \partial\Sigma = \emptyset$. Let us consider the differential operator $L = \Delta + V - aK$, where $V \in C^\infty(\Sigma)$ and a is a positive constant, acting on $f \in H_0^{1,2}(\Sigma)$. Let $f : D(s) \to \mathbb{R}$ be a nonnegative radial function, i.e., $f \equiv f(r)$, such that*

$$\begin{array}{ll} f(r) \equiv 1, & \text{for } r \le \varepsilon, \\ f(r) \equiv 0, & \text{for } r \ge s, \\ f'(r) \le 0, & \text{for } \varepsilon < r < s. \end{array}$$

Then, the following holds:

$$\begin{aligned} Q(f,f) \le 2a\left(\pi G(s) - f'_-(\varepsilon) l(\varepsilon)\right) - \int_{D(s)} V f(r)^2 \\ + \int_\varepsilon^s \left\{(1-2a) f'(r)^2 - 2a f(r) f''(r)\right\} l(r), \end{aligned} \tag{4.3}$$

where

$$\begin{array}{rcl} G(s) & := & -\displaystyle\int_0^s (f(r)^2)' \chi(r) \le 1, \\ f'_-(\varepsilon) & := & \displaystyle\lim_{\substack{r \to \varepsilon \\ \varepsilon < r}} f'(r). \end{array}$$

Proof. Let us denote

$$\alpha = \int_{D(s)} \|\nabla f\|^2, \ \beta = \int_{D(s)} K f^2.$$

On the one hand, by the Co-Area Formula,

$$\alpha = \int_{D(s)} \|\nabla f\|^2 = \int_\varepsilon^s f'(r)^2 \int_{\partial D(r)} 1 = \int_\varepsilon^s f'(r)^2 l(r).$$

On the other hand, by Fubini's Theorem and integrating by parts, we have

$$\beta = \int_0^s f(r)^2 \int_{\partial D(r)} K = \int_0^s f(r)^2 K'(r) = -\int_0^s (f(r)^2)' K(r).$$

Now, by (4.1) and $(f(r)^2)' = 2f(r)f'(r) \le 0$, we have

$$-(f(r)^2)' K(r) \le (f(r)^2)'(l'(r) - 2\pi\chi(r)).$$

Integrating by parts and taking into account that $\int_0^s (f(r)^2)' = -1$, we obtain

$$\begin{aligned}\beta &\leq \int_0^s (f(r)^2)'(l'(r) - 2\pi\chi(r)) = -2\pi \int_0^s (f(r)^2)'\chi(r) + \int_0^s (f(r)^2)' l'(r) \\ &= 2\pi G(s) + \int_0^s (f(r)^2)' l'(r) = 2\pi G(s) + \int_\varepsilon^s ((f(r)^2)' l(r))' - \int_\varepsilon^s (f(r)^2)'' l(r) \\ &= 2\pi G(s) - 2f'_-(\varepsilon) l(\varepsilon) - \int_\varepsilon^s (f(r)^2)'' l(r).\end{aligned}$$

Thus, putting α and β together,

$$\int_{D(s)} \left\{ \|\nabla f\|^2 + aKf^2 \right\} \leq 2a \left(\pi G(s) - f'_-(\varepsilon) l(\varepsilon) \right) + \int_\varepsilon^s \left\{ (1-2a) f'(r)^2 - 2af(r)f''(r) \right\} l(r).$$

Note that the bound on $G(s)$ follows since the Euler characteristic of $D(s)$ is less than or equal to 1. □

Before we finish this secion, we will see an useful Lemma:

Lemma 4.2. *Under the conditions of Lemma 4.1, if Σ is complete and there exists s_0 such that $\chi(s) \leq -M$, $M \geq 0$, for all $s \geq s_0$, then*

$$G(s) \leq -(M+1) f(s_0)^2 + 1.$$

Proof. Assume there exists s_0 so that for all $s \geq s_0$, we have $\chi(s) \leq -M$. Therefore, following Lemma 4.1, we have

$$\begin{aligned}G(s) &= -\int_0^s (f(r)^2)'\chi(r) = -\int_0^{s_0} (f(r)^2)'\chi(r) - \int_{s_0}^s (f(r)^2)'\chi(r) \\ &\leq -\int_0^{s_0} (f(r)^2)' + M \int_{s_0}^s (f(r)^2)' = -\left(f(s_0)^2 - f(0)^2 \right) + M \left(f(s)^2 - f(s_0)^2 \right) \\ &= -(M+1) f(s_0)^2 + 1,\end{aligned}$$

since $-(f(r)^2)' \geq 0$ and $\chi(r) \leq 1$ for all r. □

4.1 Consequence. The Distance Lemma

Here we will describe the first consequences of Lemma 4.1 when Σ carries a stable Schrödinger operator, the *Distance Lemma*, which states that the intrisic distance of any point to the boundary is bounded. The result is the abstract version of ideas of D. Fischer-Colbrie developed on her seminar paper [14]. Actually, the results on this section can be extended under weaker assumptions on the potential V. To that matter, we refer the reader to [2, 11, 12, 13, 18].

Here, we show the Distance Lemma given by Meeks-Pérez-Ros [20] for stable Schrödinger operators:

Lemma 4.3 (Analytic Version)**.** *Let (Σ, g) be a Riemannian surface. Let $L = \Delta + V - aK$ be a differential operator on Σ acting on compactly supported $f \in C_0^\infty(\Sigma)$, where $a > 1/4$ is constant, $V \geq c > 0$, Δ and K are the Laplacian and Gauss curvature associated to the metric g respectively.*

Assume that L is stable, then the distance from every point $p \in \Sigma$ to the boundary $\partial\Sigma$ is bounded, i.e.,

$$d_\Sigma(p, \partial\Sigma) \leq \pi \sqrt{\left(1 + \frac{1}{4a-1}\right)\frac{a}{c}}$$

where d_Σ denotes the intrinsic distance in Σ. Moreover, if Σ is complete without boundary, then it must be topologically a sphere.

Proof. Since L is stable, from Lemma 3.2, there exists a positive function u such that $Lu = 0$. Set $\alpha := 1/a$. Consider the conformal metric $\tilde{g} := u^{2\alpha} g$, where g is the metric on Σ. Denote by $\tilde{K}$ and K the Gaussian curvature of $\tilde{g}$ and g respectively.

On the one hand, the respective Gaussian curvatures are related by

$$\alpha \Delta \ln u = K - \tilde{K} u^{2\alpha}.$$

On the other hand, since $Lu = 0$, we obtain

$$\Delta \ln u = \frac{u\Delta u - \|\nabla u\|^2}{u^2} = aK - c - \frac{\|\nabla u\|^2}{u^2}.$$

Now, combining the above two equalities, we get

$$\tilde{K} = u^{-2\alpha}\left(\frac{c}{a} + \frac{\|\nabla u\|^2}{au^2}\right). \tag{4.4}$$

Take $p \in \Sigma$ and let γ be a $\tilde{g}-$geodesic ray emanating from p. Denote by $\tilde{l}$ and l the length of γ with respect to $\tilde{g}$ and g respectively. Since γ is a $\tilde{g}-$minimizing geodesic, the Second Variation Formula of the arc-length (see [9]) gives that

$$\int_0^{\tilde{l}} \left(\left(\frac{d\phi}{d\tilde{s}}\right)^2 - \tilde{K}\,\phi^2\right) d\tilde{s} \geq 0, \tag{4.5}$$

for any smooth function $\phi : [0, \tilde{l}] \to \mathbb{R}$ such that $\phi(0) = \phi(\tilde{l})$.

From (4.4), (4.5), $\|\nabla u\| \geq (u \circ \gamma)'(s) = u'(s)$ and changing variables $d\tilde{s} = u^\alpha ds$, we get

$$\int_0^l u(s)^{-\alpha}\left(\phi'(s)^2 - \frac{1}{a}\left(c + \frac{u'(s)^2}{u(s)^2}\right)\phi(s)^2\right) ds \geq 0. \tag{4.6}$$

Take $\phi = u^{1/2a}\psi$, where $\psi : [0, l] \to \mathbb{R}$ is a smooth function such that $\psi(0) = \psi(l) = 0$. Then, the above (4.6) yields

$$\int_0^l \left(\psi'(s)^2 + \frac{1}{a}\left(\frac{1}{4a} - 1\right)\frac{u'(s)^2}{u(s)^2}\psi(s)^2 + \frac{1}{a}\frac{u'(s)}{u(s)}\psi(s)\psi'(s) - \frac{c}{a}\psi(s)^2 \right) ds \geq 0. \tag{4.7}$$

Now, since $4a > 1$, we obtain

$$\frac{1}{4a-1}\psi'(s)^2 \geq \frac{1}{a}\left(\frac{1}{4a} - 1\right)\frac{u'(s)^2}{u(s)^2}\psi(s)^2 + \frac{1}{a}\frac{u'(s)}{u(s)}\psi(s)\psi'(s),$$

therefore, from (4.7), we get

$$\int_0^l \left(\left(1 + \frac{1}{4a-1}\right)\psi'(s)^2 - \frac{c}{a}\psi(s)^2 \right) ds \geq 0,$$

and integrating by parts the first term, i.e.,

$$\int_0^l \psi'(s)^2 \, ds = -\int_0^l \psi''(s)\psi(s)\, ds,$$

we finally obtain

$$\int_0^l \left(-\left(1 + \frac{1}{4a-1}\right)\psi''(s)\psi(s) - \frac{c}{a}\psi(s)^2 \right) ds \geq 0, \tag{4.8}$$

for all $\psi \in C_0^\infty([0, l])$. Taking $\psi(s) = \sin\left(\frac{\pi}{l}s\right)$ in (4.8), we get

$$\int_0^l \left(\left(1 + \frac{1}{4a-1}\right)\frac{\pi^2}{l^2} - \frac{c}{a} \right) \sin^2\left(\frac{\pi}{l}s\right) ds \geq 0,$$

which implies

$$\left(1 + \frac{1}{4a-1}\right)\frac{\pi^2}{l^2} - \frac{c}{a} \geq 0,$$

or equivalently,

$$l \leq \pi\sqrt{\left(1 + \frac{1}{4a-1}\right)\frac{a}{c}}, \tag{4.9}$$

which gives the desired estimate.

Now, if Σ is complete, then (4.9) and the Hopf-Rinow Theorem imply that Σ must be compact. Moreover, applying the operator L to the test function 1, we have

$$a\int_\Sigma K \geq c\,\mathrm{Area}(\Sigma),$$

which implies, by the Gauss-Bonnet Theorem, that $\chi(\Sigma) > 0$. $\square$

Remark 4.1. *L. Mazet [19] gave a sharp "Distance Lemma" for compact stable $H-$surfaces with boundary. Latter, P. Bérard and P. Castillon [3] provided an intrisic version of the optimal length estimate of L. Mazet to the realm of stable Schrodinger operators.*

Now, we obtain a geometric version, which is not sharp, of the Distance Lemma [12]:

Lemma 4.4 (Geometric Version)**.** *Let Σ be a Riemannian surface possibly with boundary. Suppose that $L = \Delta + V - aK$ is nonpositive acting on $f \in C_0^\infty(\Sigma)$, with $V \geq c > 0$ and $a > 1/4$. Then, if the area of the geodesic disks goes to infinity as its radius goes to infinity, there exists a positive constant C (depending only on a and c) such that*

$$\mathrm{dist}_\Sigma(p, \partial\Sigma) \leq C, \ \forall p \in \Sigma.$$

In particular, if Σ is complete with $\partial\Sigma = \emptyset$, then it must be either compact with $\chi(\Sigma) > 0$ or parabolic with finite area. Here, $\chi(\Sigma)$ denotes Euler characteristic of Σ.

Proof. Since $a > 1/4$, take $b \geq 1$ so that

$$-\alpha := b(b(1-4a) + 2a) < 0.$$

let us consider the radial function

$$f(r) := \begin{cases} (1 - r/s)^b & r \leq s \\ 0 & r > s \end{cases},$$

where r denotes the radial distance from a point $p_0 \in \Sigma$. Then, from Lemma 4.1, we have

$$\int_{D(s)} (1 - r/s)^{2b} V \leq 2a\pi G(s) - \frac{\alpha}{s^2} \int_0^s l(r),$$

where

$$G(s) := -\int_0^s (f(r)^2)' \chi(r) \leq 1.$$

Therefore, using that $V \geq c > 0$, we get

$$\frac{c}{2^b} a(s/2) \leq 2a\pi. \tag{4.10}$$

Let us suppose that the distance to the boundary were not bounded. Then there exists a sequence of points $\{p_i\} \in \Sigma$ such that $\mathrm{dist}_\Sigma(p_i, \partial\Sigma) \to +\infty$. So, for each p_i we can choose a real number s_i such that $s_i \to +\infty$ and $\overline{D(p_i, s_i)} \cap \partial\Sigma = \emptyset$. For each disk $D(p_i, s_i)$, we have from

$$a(p_i, s_i/2) \leq C,$$

where C is constant independing only on a and c.

Now, bearing in mind that the left hand side of the above inequality goes to infinity and the right hand side remains bounded, we obtain a contradiction.

Also, if Σ is complete and has not finite area, the above estimate and the Hopf-Rinow Theorem imply that Σ must be compact.

When Σ is compact, taking the constant function $f \equiv 1$ on Σ, we get

$$a \int_{\Sigma} K \, dv_{\Sigma} \geq c\text{Area}(\Sigma) > 0,$$

which finishes the proof. $\square$

5 Classification of complete stable minimal surfaces in 3-manifolds with nonnegative scalar curvature

In this Section, we will study complete stable $H-$stable surfaces in three-manifolds of nonnegative scalar curvature $S \geq 0$ as a direct application of the previous Section 4.

Let Σ be a two-sided surface with constant mean curvature H (in short, $H-$surface) in a Riemannian three-manifold $\mathcal{N}$. Σ is stable if (see [29] for the minimal case or [1] for the constant mean curvature case)

$$\int_\Sigma \left(f^2 |A|^2 + \int_\Sigma f^2 \mathrm{Ric}(\vec{N}, \vec{N}) \right) dv_\Sigma \leq \int_\Sigma |\nabla f|^2 \, dv_\Sigma$$

for all compactly supported functions $f \in C_0^\infty(\Sigma)$. Here $|A|^2$ denotes the the square of the length of the second fundamental form of Σ, $\mathrm{Ric}(\vec{N}, \vec{N})$ is the Ricci curvature of $\mathcal{N}$ in the direction of the normal $\vec{N}$ to Σ and ∇ is the gradient w.r.t. the induced metric.

One writes the stability inequality in the form

$$\left. \frac{d^2}{dt^2} \right|_{t=0} (\mathrm{Area}(\Sigma(t)) - 2H\mathrm{Volume}(\Sigma(t))) = - \int_\Sigma f L f \, dv_\Sigma \geq 0,$$

where L is the linearized operator of the mean curvature

$$L = \Delta + |A|^2 + \mathrm{Ric}.$$

As we have said in Section 3, in terms of L, stability means that $-L$ is nonnegative, i.e., all its eigenvalues are nonnegative. It is well known (see Proposition 1.1) that the stability operator L can be written as

$$L = \Delta - K + (4H^2 - K_e + \frac{S}{2}),$$

where K and K_e are the Gaussian curvature and extrinsic curvature (i.e., the product of the principal curvatures) of Σ, and S is the scalar curvature of $\mathcal{N}$.

The first thing we shall observe is when we can have a complete noncompact $H-$stable surface under conditions on the scalar curvature. The following result is the diameter estimate for stable $H-$surfaces given by H. Rosenberg [27]:

Theorem 5.1. *Let $\Sigma \subset (\mathcal{N}, g)$ be a $H-$stable surface with boundary $\partial\Sigma$. If $\frac{S}{2} + 3H^2 \geq c > 0$ on Σ, then*

$$d_\Sigma(p, \partial\Sigma) \leq \frac{2\pi}{\sqrt{3c}} \text{ for all } p \in \Sigma,$$

where d_Σ denotes the intrinsic distance in Σ. Moreover, if Σ is complete without boundary, then it must be topologically a sphere.

Proof. Since Σ is a $H-$surface, by the Gauss Equation, the Jacobi operator is given by

$$L := \Delta - K + V,$$

where

$$V := 4H^2 - K_e + \frac{S}{2} \geq 3H^2 + \frac{S}{2} \geq c,$$

since $H^2 - K_e \geq 0$, here K_e denotes the extrinsic curvature of Σ. Moreover, since Σ is stable, L is stable in the sense of operators with $a = 1$ and $V \geq c$. Then, Lemma 4.3 finishes the proof. □

Remark 5.1. *L. Mazet [19] gave a sharp "Distance Lemma" for compact stable $H-$surfaces with boundary immersed in a space form.*

The above result says that, in a manifold $(\mathcal{N}, g)$ of nonnegative scalar curvture $S \geq 0$, the only complete noncompact $H-$stable surface we shall consider are the minimal ones. So, the next step is to extend the wellknow result of D. Fischer-Colbrie and R. Schoen [15] on the topology and conformal type of complete noncompact stable minimal surfaces.

Theorem 5.2. *Let $\Sigma \subset (\mathcal{N}, g)$ be a complete (noncompact) stable minimal surface where $S \geq 0$. Then, Σ is conformally equivalent either to the complex plane $\mathbb{C}$ or to the cylinder $\mathbb{S}^1 \times \mathbb{R}$. In the latter case, Σ is totally geodesic, flat and $S \equiv 0$ along Σ.*

Proof. Let Σ be a complete minimal surface then, by the Gauss Equation, the Jacobi operator is given by

$$L := \Delta - K + V,$$

where

$$V := -K_e + \frac{S}{2} \geq 0,$$

here K_e denotes the extrinsic curvature of Σ. Moreover, since Σ is stable, L is stable in the sense of operators with $a = 1$ and $V \geq 0$.

Let us consider the radial function

$$f(r) := \begin{cases} (1 - r/s) & r \leq s \\ 0 & r > s \end{cases},$$

where r denotes the radial distance from a point $p_0 \in \Sigma$. Then, from Lemma 4.1, we have

$$\int_{D(s)} (1 - r/s)^2 V \leq 2\pi G(s) - \frac{1}{s^2} \int_0^s l(r), \tag{5.1}$$

where

$$G(s) := -\int_0^s (f(r)^2)' \chi(r).$$

First, since Σ is complete, from (5.1) and Lemma 4.2, we get that $\chi(\Sigma)$ equals either to 0 or 1, that is, Σ is topologically either the plane or the cylinder.

Second, again from (5.1), we obtain

$$\frac{a(s)}{s^2} \leq 2\pi,$$

which yields that Σ has quadratic area growth. Thus, since Σ has finite topology and quadratic area growth, each end of Σ is parabolic (see [8]). Therefore Σ is conformally equivalent either to the plane or to the cylinder.

Now, suppose that Σ is conformally equivalent to the cylinder. We will show that Σ is flat, totally geodesic and the scalar curvature S vanishes along Σ.

- **Step 1:** V vanishes identically on Σ. That is, Σ is totally geodesic and S vanishes along Σ.

  Suppose there exists a point $p_0 \in \Sigma$ so that $V(p_0) > 0$. From now on, we fix the point p_0. Then, there exists $\epsilon > 0$ so that $V(q) \geq \delta$ for all $q \in D(\epsilon) = D(p_0, \epsilon)$. Since Σ is topollogically a cylinder, there exists $s_0 > 0$ so that for all $s > s_0$ we have $\chi(s) \leq 0$ (see [6, Lemma 1.4]).

  Now, from the above considerations and (5.1), there exists $\beta > 0$ so that

  $$0 < \beta \leq 2\pi G(s).$$

  But, from Lemma 4.2, we can see that

  $$G(s) = -f(s_0)^2 + 1 = -\left(1 - s_0/s\right)^2 + 1,$$

  therefore,

  $$G(s) \leq 1 - (1 - s_0/s)^2 \to 0, \text{ as } s \to +\infty,$$

  which is a contradiction. Thus, V vanishes identically along Σ.

- **Step 2:** K vanishes identically on Σ.

  First, note that $L := \Delta - K$. From Lemma 3.2, there is a smooth positive function u on Σ such that $Lu = 0$. Then, following ideas of Lemma 4.3, the conformal metric $\tilde{g} := u^2 g$, where g is the metric on Σ, has Gaussian curvature $\tilde{K}$ of is non-negative, i.e. $\tilde{K} \geq 0$.

  Claim: *$\tilde{g}$ is complete if g is complete.*

Proof of the Claim: We argue as in Lemma 4.3. Take $p \in \Sigma$ and let γ be a complete $\tilde{g}-$geodesic ray emanating from p. Since γ is a $\tilde{g}-$minimizing geodesic, the Second Variation Formula of the arc-length (see [9]) gives that

$$\int_0^\infty \left(\left(\frac{d\phi}{d\tilde{s}} \right)^2 - \tilde{K}\,\phi^2 \right) d\tilde{s} \geq 0,$$

for any smooth function $\phi : [0, \infty) \to \mathbb{R}$ of compact support.
Now, reasoning as in Lemma 4.3 and changing variables $d\tilde{s} = u ds$, we get

$$\int_0^\infty u(s)^{-1} \left(\phi'(s)^2 - \left(\frac{u'(s)^2}{u(s)^2} \right) \phi(s)^2 \right) ds \geq 0.$$

Take $\phi = u\psi$ in the above equation, where $\psi : [0, \infty) \to \mathbb{R}$ is a smooth function of compact support. Then, it yields

$$\int_0^\infty \left(u(s)\psi'(s)^2 + 2u'(s)\psi(s)\psi'(s) \right) ds \geq 0.$$

Integrating by part the second term in the above equation we obtain

$$\int_0^\infty u(s) \left(-\psi'(s)^2 - 2\psi(s)\psi''(s) \right) ds \geq 0 \text{ for all } \psi \in C_0^\infty([0, \infty)). \tag{5.2}$$

Taking $\psi(s) := s\xi(s)$, where $\xi : [0, \infty) \to \mathbb{R}$ a smooth function of compact support, we obtain from 5.2:

$$\int_0^\infty u(s)\xi(s)^2 ds \leq \int_0^\infty u(s) \left(-6s\xi(s)\xi'(s) - 2s^2\xi\xi''(s) - s^2\xi'(s)^2 \right) ds. \tag{5.3}$$

Consider ξ so that $\xi(s) = 1$ for $s \leq R$, $\xi(s) = 0$ for $s > 2R$ and $\xi'(s)$ and $\xi''(s)$ are bounded by c/R and c/R^2 respectively for $R \leq s \leq 2R$, for some uniform constant c. Therefore

$$|s\xi'(s)| \leq c \text{ and } |s^2\xi''(s)| \leq c.$$

So, from (5.3), we obtain:

$$\int_0^R u(s)\, ds \leq \int_0^\infty u\xi(s)^2\, ds \leq C \int_R^{2R} u(s)\, ds \leq C \int_R^\infty u(s)\, ds,$$

that is,

$$\int_0^R u(s)\,ds \leq C \int_R^\infty u(s)\,ds,$$

where C is a constant independent of R. This inequality implies that

$$\int_0^\infty u(s)\,ds = \infty,$$

which yields that the conformal metric $\tilde{g}$ is complete.

Since Σ is topologically a cylinder, the Cohn-Vossen inequality says

$$\int_\Sigma \tilde{K} \leq 0,$$

that is, $\tilde{K}$ vanishes identically.

Thus, $K = \Delta \ln u$. From this last equation, we get:

$$K = \frac{1}{u}\Delta u - \frac{|\nabla u|^2}{u^2},$$

that is,

$$\frac{|\nabla u|^2}{u} = \Delta u - Ku = 0.$$

This last equation implies that u is constant, and since u satisfies $Lu = 0$, we have that K vanishes identically on Σ. This finishes the proof of Theorem 5.2.

$\square$

Remark 5.2. *In [3, 12] we can find extensions of the above result under weaker conditions.*

6 Area Minimizing surfaces and the structure of 3-manifolds

In this section we study the topology and geometry of complete manifold $(\mathcal{N}, g)$ verifying some lower bound on its scalar curvature S. The idea is to show the splitting theorems developed by Cai-Galloway [5], Bray-Brendle-Neves [4] and I. Nunes [23]. Here, we will take the unified point of view considered by Micallef-Moraru [22].

First, we will give an area estimate for compact stable minimal surfaces in three-dimensional manifolds with a lower bound on its scalar curvature:

Proposition 6.1. *Let $\Sigma \subset (\mathcal{N}, g)$ be a compact stable minimal surface.*

1. *If $S \geq \lambda$, for some positive constant λ, then $A(\Sigma) \leq \frac{8\pi}{\lambda}$. Moreover, if $A(\Sigma) = \frac{8\pi}{\lambda}$, then Σ is totally geodesic and the normal Ricci curvature of $(\mathcal{N}, g)$ vanishes along Σ. Moreover, $S = 2K = \lambda$ along Σ.*

2. *If $S \geq 0$ and Σ has genus one, then Σ is totally geodesic and the normal Ricci curvature of $(\mathcal{N}, g)$ vanishes along Σ. Moreover, $S = 2K = 0$ along Σ.*

3. *If $S \geq -\lambda$, for some positive constant λ, and $\gamma = \mathrm{genus}(\Sigma) \geq 2$, then $A(\Sigma) \geq \frac{4\pi(\gamma-1)}{\lambda}$. Moreover, if $A(\Sigma) = \frac{4\pi(\gamma-1)}{\lambda}$, then Σ is totally geodesic and the normal Ricci curvature of $(\mathcal{N}, g)$ vanishes along Σ. Moreover, $S = 2K = -\lambda$ along Σ.*

Proof. First, its Jacobi operator (see Proposition 1.1) is given by

$$L := \Delta - K + \frac{1}{2}\left(S + |A|^2\right),$$

since Σ is minimal.

Since Σ is compact and stable, L is stable in the sense of operators and taking $f \equiv 1$ as a test function, we get

$$\frac{1}{2}\int_\Sigma \left(S + |A|^2\right) dv_\Sigma \leq \int_\Sigma K \, dv_\Sigma \tag{6.1}$$

which yields, from Gauss-Bonnet formula, the following

$$\frac{1}{2}\int_\Sigma S \, dv_\Sigma \leq 4\pi(1-\gamma), \tag{6.2}$$

where $\gamma := \mathrm{genus}(\Sigma)$.

1. If $S \geq \lambda$, for some constant $\lambda > 0$.

   From (6.2), we have

$$\frac{\lambda}{2} A(\Sigma) = \frac{\lambda}{2} \int_{\Sigma} dv_{\Sigma} \leq \frac{1}{2} \int_{\Sigma} S \, dv_{\Sigma} \leq 4\pi,$$

   note that Σ must be a topological sphere. Therefore, from this last equation, the inequality follows.

   Moreover, in the case equality holds, (6.1) implies that Σ is totally geodesic. This implies that the Jacobi operator associated to Σ reads as

$$L := \Delta + \frac{1}{2} (S - 2K) .$$

   Therefore, the equality implies that

$$\int_{\Sigma} (S - 2K) \, dv_{\Sigma} = 0,$$

   which yields that the first eigenvalue of L is zero, i.e., $\lambda_1(-L) = 0$, and so, the constant functions are in the kernel of L, therefore

$$S = 2K \text{ along } \Sigma.$$

   Thus, from Proposition 1.1 and the fact that $|A|^2 \equiv 0$, we obtain

$$\mathrm{Ric}(\vec{N}, \vec{N}) = \mathrm{Ric}(\vec{N}, \vec{N}) + |A|^2 = \frac{1}{2} S - K = 0 \text{ along } \Sigma,$$

   which finishes the proof of item 1.

2. If $S \geq 0$ and Σ has genus one.

   As above, taking $f \equiv 1$ in (6.1), we get

$$\frac{1}{2} \int_{\Sigma} \left(S + |A|^2 \right) dv_{\Sigma} \leq \int_{\Sigma} K \, dv_{\Sigma} = 0,$$

   which implies that Σ is totally geodesic and $S \equiv 0$ along Σ. So, arguing as above, we get $K \equiv 0$ and $\mathrm{Ric}(\vec{N}, \vec{N}) = 0$ along Σ as well.

3. If $S \geq -\lambda$, for some constant $\lambda > 0$.

   This case is completely analogous to item 1.

□

Next, we shall prove the existence of a one parameter family of constant weighted mean curvature surfaces in a neighborhood of a totally geodesic compact surface verifying the conditions on Proposition 6.1.

Proposition 6.2. *Let $\Sigma \subset (\mathcal{N}, g)$ be a compact immersed surface with unit normal vector field N. Assume that*

- *Σ is totally geodesic,*
- *the normal Ricci curvature of $(\mathcal{N}, g)$ vanishes along Σ.*

Then, there exists $\varepsilon > 0$ and a smooth function $w : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that, for all $t \in (-\varepsilon, \varepsilon)$, the surfaces

$$\Sigma_t := \left\{ \exp_p(w(p,t)\vec{N}(p)) \ : \ p \in \Sigma \right\}$$

have constant mean curvature $H(t)$. Moreover, we have

$$w(p,0) = 0, \ \left.\frac{\partial}{\partial t}\right|_{t=0} w(p,t) = 1 \ \text{and} \ \int_\Sigma (w(\cdot,t) - t)\, dv_\Sigma = 0,$$

for all $p \in \Sigma$ and $t \in (-\varepsilon, \varepsilon)$.

Proof. We follow [23]. Fix $\alpha \in (0,1)$ and consider the Banach spaces

$$\mathcal{X}(n) := \left\{ u \in C^{n,\alpha}(\Sigma) \ : \ \int_\Sigma u\, dv_\Sigma = 0 \right\} \text{ for each } n \in \mathbb{N}.$$

For each $u \in C^{2,\alpha}(\Sigma)$ we define

$$\Sigma(u) := \left\{ \exp_p(u(p)\vec{N}(p)) \ : \ p \in \Sigma \right\}.$$

Choose $\varepsilon_1 > 0$ and $\delta > 0$ so that $\Sigma(u+t)$ is a compact surface of class $C^{2,\alpha}$ for all $(t,u) \in (-\varepsilon_1, \varepsilon_1) \times B(0,\delta)$, here $B(0,\delta) := \{u \in C^{2,\alpha}(\Sigma) \ : \ \|u\|_{C^{2,\alpha}} < \delta\}$. Denote by $H(u+t)$ the mean curvature of $\Sigma(u+t)$.

Consider the map $\Phi : (-\varepsilon_1, \varepsilon_1) \times (B(0,\delta) \cap \mathcal{X}(2)) \to \mathcal{X}(0)$ given by

$$\Phi(t,u) := H(u+t) - \frac{1}{A(\Sigma)} \int_\Sigma H(u+t)\, dv_\Sigma.$$

First, note that $\Phi(0,0) = 0$ since $\Sigma(0) = \Sigma$. Second, we compute $D_2\Phi_{(0,0)}(v)$ for any $v \in \mathcal{X}(2)$. We have

$$
\begin{aligned}
D_2\Phi_{(0,0)}(v) &= \frac{d}{ds}\bigg|_{s=0} \Phi(0, sv) = \frac{d}{ds}\bigg|_{s=0} H(sv) - \frac{1}{A(\Sigma)} \int_\Sigma \frac{d}{ds}\bigg|_{s=0} H(sv)\, dv_\Sigma \\
&= Lv - \frac{1}{A(\Sigma)} \int_\Sigma (Lv)\, dv_\Sigma = \Delta v - \frac{1}{A(\Sigma)} \int_\Sigma \Delta v\, dv_\Sigma \\
&= \Delta v
\end{aligned}
$$

since

$$
\frac{d}{ds}\bigg|_{s=0} H(sv) = Lv = \Delta v
$$

from $L := \Delta + |A|^2 + \mathrm{Ric}(\vec{N}, \vec{N})$ and the hypothesis.

So, since $\Delta : \mathcal{X}(2) \to \mathcal{X}(0)$ is a linear isomorphism, by the Implicit Function Theorem, there exists $0 < \varepsilon < \varepsilon_1$ and $u(t) := u(t, \cdot) \in B(0, \delta)$ for $t \in (-\varepsilon, \varepsilon)$ such that

$$
u(0) = 0 \text{ and } \Phi(t, u(t)) = 0 \text{ for all } t \in (-\varepsilon, \varepsilon).
$$

Finally, defining

$$
w(t, p) = u(t, p) + t, \;\; (t, p) \in (-\varepsilon, \varepsilon) \times \Sigma,
$$

we obtain the result. □

6.1 Local splitting

Now, we are ready to prove the local splitting result:

Theorem 6.1. *Let $(\mathcal{N}, g)$ be a complete manifold containing a compact, embedded area minimizing surface Σ.*

1. *Suppose that $S \geq \lambda$, for some positive constant λ, and $A(\Sigma) = \frac{8\pi}{\lambda}$. Then Σ has genus zero and it has a neighborhood in $\mathcal{N}$ which is isometric to the product $\mathbb{S}^2 \times (-\varepsilon, \varepsilon)$ with the product metric $g_{+1} + dt^2$ (up to scaling the metric g), here g_{+1} is the metric of constant Guassian curvature $+1$.*

2. *Suppose that $S \geq 0$ and Σ has genus one. Then Σ has a neighborhood in $\mathcal{N}$ which is flat and isometric to the product $\mathbb{T}^2 \times (-\varepsilon, \varepsilon)$ with the product metric $g_0 + dt^2$, here g_0 is the metric of constant Guassian curvature 0.*

3. *Suppose that $S \geq -\lambda$, for some positive constant λ, Σ has genus $\gamma \geq 2$ and $A(\Sigma) = \frac{4\pi(\gamma-1)}{\lambda}$. Then Σ has a neighborhood in $\mathcal{N}$ which is isometric to the product $\Sigma \times (-\varepsilon, \varepsilon)$ with the product metric $g_{-1} + dt^2$ (up to scaling the metric g), here g_{-1} is the metric of constant Guassian curvature -1.*

Proof. Let $\Sigma \subset (\mathcal{N}, g)$ be a compact embedded area minimizing surface under any of the conditions above. From Proposition 6.1 and Proposition 6.2, there exists $\varepsilon > 0$ and a smooth function $w : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that, for all $t \in (-\varepsilon, \varepsilon)$, the surfaces

$$\Sigma_t := \left\{ \exp_p(w(p,t)\vec{N}(p)) \ : \ p \in \Sigma \right\}$$

have constant mean curvature $H(t)$. Moreover, we have

$$w(p,0) = 0, \ \left.\frac{\partial}{\partial t}\right|_{t=0} w(p,t) = 1 \text{ and } \int_\Sigma (w(\cdot,t) - t)da = 0,$$

for all $p \in \Sigma$ and $t \in (-\varepsilon, \varepsilon)$.

On the one hand, from Proposition 1.1, we have:

$$Lu = \Delta u + (\mathrm{Ric}(N,N) + |A|^2)u = u\Delta \ln u + u\left(\frac{|\nabla u|^2}{u^2}\right) + \left(2H^2 - K + \frac{1}{2}\left(S + |A|^2\right)\right)u$$

that is,

$$\frac{Lu}{u} = \Delta \ln u + \frac{|\nabla u|^2}{u^2} + \left(2H^2 - K + \frac{1}{2}\left(S + |A|^2\right)\right). \tag{6.3}$$

Let

$$F_t(x) = \exp_p(w(p,t)\vec{N}(p)), \ p \in \Sigma, \ t \in (-\varepsilon, \varepsilon),$$

thus $F_t(\Sigma) = \Sigma_t$ for each $t \in (-\varepsilon, \varepsilon)$ being $\Sigma_0 = \Sigma$. Define the lapse function $\rho_t : \Sigma \to \mathbb{R}$ by

$$\rho_t(p) = \langle \vec{N}_t(p), \frac{\partial}{\partial t}F_t(p) \rangle,$$

where $\vec{N}_t$ is a unit normal vector along Σ_t. Set $\xi : \Sigma \times (-\varepsilon, \varepsilon) \to \mathcal{N}$ given by

$$\xi(p,t) = F_t(p), \ (p,t) \in \Sigma \times (-\varepsilon, \varepsilon),$$

and let us denote

$$\mathcal{U} := \xi\left(\Sigma \times (-\varepsilon, \varepsilon)\right).$$

By the First Variation Formula for the Mean Curvature (see [26]), we have

$$H'(t) = L\rho_t.$$

On the other hand, from (6.3), we obtain

$$H'(t)\frac{1}{\rho_t} = \frac{\Delta \rho_t}{\rho_t} + (\mathrm{Ric}(N,N) + |A|^2) \geq \Delta_t \ln \rho_t - K_t + \frac{1}{2}S. \tag{6.4}$$

1. If $S \geq \lambda$, for some positive constant λ, and $A(\Sigma) = \frac{8\pi}{\lambda}$.

   - **Step 1:** *Σ_t is totally geodesic and $S = 2K_t = \lambda$ is constant along Σ for each $t \in (-\varepsilon, \varepsilon)$. Moreover, $A(t) = A(0)$ for all $t \in (-\varepsilon, \varepsilon)$.*

     Since $S \geq \lambda$, integrating (6.4) over Σ_t, we obtain for any $t \in (0, \varepsilon)$

     $$H'(t) \int_{\Sigma_t} \frac{1}{\rho_t} \, dv_{\Sigma_t} = \int_{\Sigma_t} H'(t) \frac{1}{\rho_t} \, dv_{\Sigma_t} \geq -\int_{\Sigma_t} K_t \, dv_{\Sigma_t} + \frac{\lambda}{2} \int_{\Sigma_t} dv_{\Sigma_t} = \frac{\lambda}{2} A(t) - 4\pi,$$

     that is,

     $$H'(t) \int_{\Sigma_t} \frac{1}{\rho_t} \, dv_{\Sigma_t} \geq \frac{\lambda}{2} \left(A(t) - A(0) \right),$$

     where we have used that $H(t)$ is constant on each Σ_t. Since $\rho_0 \equiv 1$ on Σ, by continuity, there exists $\varepsilon > 0$ so that $1/2 < \rho_t < 2$ for all $t \in (-\varepsilon, \varepsilon)$. This yields that

     $$H'(t) \int_{\Sigma_t} \frac{1}{\rho_t} \geq \frac{\lambda}{2} \left(A(t) - A(0) \right) \geq 0$$

     since Σ is a minimizer for the area funtional. Therefore, $H'(t) \geq 0$ for all $t \in (0, \varepsilon)$. So, by the First Variation Formula for the area, we have

     $$A'(t) = -\int_{\Sigma_t} H(t) \rho_t \leq 0 \text{ for all } t \in (0, \varepsilon),$$

     and using that Σ is a miminizer for the area, we obtain $A(t) = A(0)$ for all $t \in (0, \varepsilon)$. Analogously, we can prove that $A(t) = A(0)$ for all $t \in (-\varepsilon, 0)$.

     Thus, Proposition 6.1 implies that Σ_t is totally geodesic and $S = 2K_t = \lambda$ is constant along Σ for each $t \in (-\varepsilon, \varepsilon)$.

   - **Step 2:** *ρ_t is constant at each Σ_t.*

     Since Σ_t is totally geodesic and $S = 2K_t$ for each $t \in (-\varepsilon, \varepsilon)$, From (6.4) we obtain

     $$\Delta \rho_t = 0 \text{ on } \Sigma_t \text{ for each } t \in (-\varepsilon, \varepsilon),$$

     that is, ρ_t is constant along each Σ_t, $t \in (-\varepsilon, \varepsilon)$, or equivalently, ρ_t is a function of t only.

   - **Step 3:** *For each $p \in \Sigma$, the vector field $\vec{N}_t(p)$ is parallel along the curve $\alpha_p : (-\varepsilon, \varepsilon) \to \mathcal{U} \subset \mathcal{N}$ given by*

     $$\alpha_p(t) = F_t(p) = \exp_p(w(p,t)\vec{N}(p)).$$

Let $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$ be a local chart on Σ, then

$$\overline{\nabla}_{\frac{\partial F_t}{\partial x_i}} \vec{N}_t = 0,\ i = 1, 2,$$

since Σ_t is totally geodesic. Thus, using that ρ_t is constant on each Σ we obtain

$$\begin{aligned} 0 = \frac{\partial}{\partial x_i}\rho_t &= \langle \overline{\nabla}_{\frac{\partial F_t}{\partial x_i}} \vec{N}_t, \frac{\partial F_t}{\partial t}\rangle + \langle \vec{N}_t, \overline{\nabla}_{\frac{\partial F_t}{\partial x_i}} \frac{\partial F_t}{\partial t}\rangle = \langle \vec{N}_t, \overline{\nabla}_{\frac{\partial F_t}{\partial x_i}} \frac{\partial F_t}{\partial t}\rangle = \\ &= \langle \vec{N}_t, \overline{\nabla}_{\frac{\partial F_t}{\partial t}} \frac{\partial F_t}{\partial x_i}\rangle = \frac{\partial}{\partial t}\langle \vec{N}_t, \frac{\partial F_t}{\partial x_i}\rangle - \langle \overline{\nabla}_{\frac{\partial F_t}{\partial t}} \vec{N}_t, \frac{\partial F_t}{\partial x_i}\rangle = \\ &= -\langle \overline{\nabla}_{\frac{\partial F_t}{\partial t}} \vec{N}_t, \frac{\partial F_t}{\partial x_i}\rangle, \end{aligned}$$

that is,

$$\langle \overline{\nabla}_{\frac{\partial F_t}{\partial t}} \vec{N}_t, \frac{\partial F_t}{\partial x_i}\rangle = 0,\ i = 1, 2.$$

Now, since $\vec{N}_t$ is unitary, we get,

$$\langle \overline{\nabla}_{\frac{\partial F_t}{\partial t}} \vec{N}_t, \vec{N}_t\rangle = 0,$$

therefore, $\overline{\nabla}_{\frac{\partial F_t}{\partial t}} \vec{N}_t = 0$ on $\mathcal{U}$. This proves the assertion.

- **Step 4:** *$\rho_t(p) = \frac{\partial}{\partial t} w(p,t)$ for all $(p,t) \in \Sigma \times (-\varepsilon, \varepsilon)$.*
Set

$$V(t) := d\left(\exp_p\right)_{w(p,t)\vec{N}(p)}\left(\vec{N}(p)\right),$$

and note that $\|V(t)\| = 1$ for all $t \in (-\varepsilon, \varepsilon)$ by the Gauss Lemma.
Consider the curve $\beta : (-\delta, \delta) \to T_p\mathcal{N}$ given by $\beta(s) = (w(p,t) + s)\vec{N}$. Then, $\beta(0) = w(p,t)\vec{N}$ and $\beta'(0) = \vec{N}$, and therefore

$$\begin{aligned} V(t) = d\left(\exp_p\right)_{w(p,t)\vec{N}(p)}\left(\vec{N}(p)\right) &= \left.\frac{d}{ds}\right|_{s=0} \exp_p(\beta(s)) = \\ \left.\frac{d}{ds}\right|_{s=0} &\gamma(w(p,t) + s, p, \vec{N}) = \gamma'(w(p,t), p, \vec{N}), \end{aligned}$$

where $\gamma(\cdot, p, \vec{N})$ is the unique geodesic in $\mathcal{N}$ with initial conditions $\gamma(0, p, \vec{N}) = p$ and $\gamma'(0, p, \vec{N}) = \vec{N}$.
Now, since

$$\frac{\partial F_t}{\partial t}(p) = \frac{d}{dt}\gamma(w(p,t), p, \vec{N}) = \gamma'(w(p,t), p, \vec{N}) \cdot \frac{\partial w}{\partial t}(p,t) = \frac{\partial w}{\partial t}(p,t)V(t), \tag{6.5}$$

we have

$$\overline{\nabla}_{\frac{\partial F_t}{\partial t}(p)} V(t) = \frac{\partial w}{\partial t}(p,t) \overline{\nabla}_{\gamma'(w(p,t),p,\vec{N})} \gamma'(w(p,t),p,\vec{N}) = 0,$$

since γ is a geodesic.

Therefore,

$$\frac{d}{dt}\langle V(t), \vec{N}_t \rangle = 0$$

along α_p, which implies that $V(t)$ is parallel along α_p. Since

$$\|V(t)\| = \left\|\vec{N}_t\right\| \text{ and } V(0) = \vec{N} = \vec{N}_0,$$

we obtain $\vec{N}_t = V(t)$ along α_p for each $t \in (-\varepsilon, \varepsilon)$.

Thus, from (6.5), we finally get

$$\rho_t = \langle \vec{N}_t, \frac{\partial F_t}{\partial t} \rangle = \frac{\partial w}{\partial t}(p,t) \langle \vec{N}_t, \vec{N}_t \rangle = \frac{\partial w}{\partial t}(p,t).$$

- **Step 5:** $w(p,t) = t$ *on* $\Sigma \times (-\varepsilon, \varepsilon)$.

  From Proposition 6.2, we know

  $$\int_\Sigma (w(p,t) - t)\, dv_\Sigma = 0,$$

  which implies that

  $$\int_\Sigma \frac{\partial w}{\partial t}(p,t)\, dv_\Sigma = \text{Area}(\Sigma),$$

  but $\frac{\partial w}{\partial t}(p,t) = \rho_t(p)$ does not depend on p, then

  $$\frac{\partial w}{\partial t}(p,1) = 1$$

  and using that $w(p,0) = 0$, we obtain $w(p,t) = t$ for all $(p,t) \in \Sigma \times (-\varepsilon, \varepsilon)$.

- **Step 6:** *The map*

  $$\xi(t,p) = \exp_p(t\vec{N}(p)) \text{ for all } p \in \Sigma,$$

  is an isometry from $\Sigma \times (-\varepsilon, \varepsilon)$ *to* $\mathcal{U}$, *where we consider the product metric* $g_\Sigma + dt^2$ *in* $\Sigma \times (-\varepsilon, \varepsilon)$.

So, scaling g, we can assume that $K_t \equiv 1$ for all Σ_t, which finishes item 1.

2. If $S \geq 0$ and Σ has genus one.

   The proof is completely analogous to the one on item 1.

3. If $S \geq -\lambda$, for some positive constant λ, Σ has genus $\gamma \geq 2$ and $A(\Sigma) = \frac{4\pi(\gamma)}{\lambda}$.

   As we did in item 1, integrating (6.4) over Σ_t, we obtain for any $t \in (0, \varepsilon)$

$$\int_{\Sigma_t} H'(t) \frac{1}{\rho_t}\, dv_{\Sigma_t} \geq - \int_{\Sigma_t} K_t\, dv_{\Sigma_t} - \frac{\lambda}{2} \int_{\Sigma_t} dv_{\Sigma_t} = 2\pi(\gamma - 1) - \frac{\lambda}{2} A(t),$$

   that is,

$$H'(t) \int_{\Sigma_t} \frac{1}{\rho_t}\, dv_{\Sigma_t} \geq \frac{\lambda}{2} \left(A(0) - A(t)\right) = -\frac{\lambda}{2} \int_0^t A'(s)\, ds,$$

   so, using the first variation formula for the area and that $H(t)$ is constant on each Σ_t we get

$$H'(t) \int_{\Sigma_t} \frac{1}{\rho_t}\, dv_{\Sigma_t} \geq \int_0^t H(s) \left(\int_{\Sigma_s} \rho_s\, dv_{\Sigma_t} \right) ds. \tag{6.6}$$

   Assume there exists $t_0 \in (0, \varepsilon)$ so that $H(t_0) < 0$. Define

$$I := \left\{ t \in [0, t_0] \ : \ H(t) \leq H(t_0) \right\}.$$

   We will show that $\inf I = 0$. Assume by contradiction that $\inf I = \bar{t} > 0$. First, rewrite (6.6) as

$$H'(t) \geq \frac{1}{\psi(t)} \int_0^t H(s) \mu(s)\, ds, \tag{6.7}$$

   where

$$\psi(t) := \int_{\Sigma_t} \frac{1}{\rho_t}\, dv_{\Sigma_t} \text{ and } \mu(t) = \int_{\Sigma_t} \rho_t\, dv_{\Sigma_t}.$$

   Since $\rho_0 \equiv 1$ on Σ, by continuity, there exists $\varepsilon > 0$ so that $1/2 < \rho_t < 2$. This yields that

$$\frac{A(t)}{2} < \mu(t) < 2A(t) \text{ , for all } t \in (-\varepsilon, \varepsilon)$$

   and

$$\frac{A(t)}{2} < \psi(t) < 2A(t) \text{ , for all } t \in (-\varepsilon, \varepsilon)$$

Shrinking ε if necessary, we can assume $A(0)/2 < A(t) < 2A(0)$ for all $t \in (-\varepsilon, \varepsilon)$. Therefore, we obtain

$$\frac{1}{4A(0)} < \frac{1}{\psi(t)} < \frac{4}{A(0)} \text{ and } \frac{A(0)}{4} < \mu(t) < 4A(0) \text{ for all } t \in (-\varepsilon, \varepsilon)$$

Second, by the Mean Value Theorem, there exists $t' \in (0, \bar{t})$ so that

$$H(\bar{t}) = H'(t')\bar{t}. \tag{6.8}$$

Therefore, from (6.7) and (6.8), we get

$$H(t') \geq \frac{t'}{\psi(\bar{t})} \int_0^{\bar{t}} H(s)\mu(s)\, ds \geq 16H(t')t'\bar{t} \geq 16\varepsilon^2 H(t'),$$

since $H(s) \leq 0$ for all $s \in (0, \bar{t})$. So, if $\varepsilon < \frac{1}{\sqrt{16}}$, we get the desired contradiction (recall that $H(t') < 0$).

Since $\inf I = 0$, it follows that $H(0) \leq H(t_0) < 0$, which contradicts that Σ is minimal. Therefore, $H(t) \geq 0$ for all $t \in (0, \varepsilon)$. Now, we can do the same for $t \in (-\varepsilon, 0)$. And we can finish as in item 1.

□

6.2 Topology of three-manifolds with scalar curvature bounded below

This section is mostly based on previous results. First, we give the rigidity result of Bray-Brendle-Neves [4]:

Theorem 6.2. *Let $(\mathcal{N}, g)$ be a compact manifold so that S is positive and $\pi_2(\mathcal{N}) \neq 0$. Define*

$$\mathcal{A}(\mathcal{N}, g) := \inf \left\{ A(f(\mathbb{S}^2)) \ : \ f \in \mathcal{F} \right\},$$

where $\mathcal{F}$ is the set of all smooth maps $f : \mathbb{S}^2 \to \mathcal{N}$ which represent a nontrivial element in $\pi_2(\mathcal{N})$.

Then, we have

$$\mathcal{A}(\mathcal{N}, g) \cdot \min \{S(x) : \ x \in \mathcal{N}\} \leq 8\pi. \tag{6.9}$$

Moreover, if the equality holds, the universal cover of $(\mathcal{N}, g)$ is isometric (up to scaling) to the standard cylinder $\mathbb{S}^2 \times \mathbb{R}$.

Proof. Set
$$\lambda = \min \{S(x) : x \in \mathcal{N}\}.$$

Since S is positive and $\mathcal{N}$ is compact, we get $S \geq \lambda$. So, from Theorem 6.1, we get the upper bound.

Now, assume the equality holds in (6.9). From results of Meeks-Yau [21], there exists a smooth inmersion $F \in \mathcal{F}$ so that

$$A(F(\mathbb{S}^2)) = \mathcal{A}(\mathcal{N}, g).$$

Denote $\Sigma = F(\mathbb{S}^2)$. Since we are assuming equality, Theorem 6.1 asserts that locally, in a neighborhood of Σ, $\mathcal{N}$ splits as a product manifold. We can also see that each leaf of the product structure is area minimizer, so we can continue the process and we construct a foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$,
$$\Sigma_t := \left\{ \exp_{F(p)}(t\vec{N}(p)) \ : \ p \in \Sigma \right\},$$
of embedded spheres which are totally geodesic, has the same constant Gaussian curvature and S is constant.

Now, it is not hard to see that the map

$$\xi : \mathbb{S}^2 \times \mathbb{R} \to \mathcal{N},$$

given by
$$\xi(p,t) = \exp_{F(p)}(t\vec{N}(p))$$
is a local isometry (see [4, Proposition 11]). From here, it follows that ξ is a covering map and therefore, the universal cover of $(\mathcal{N}, g)$ is isometric to $\mathbb{S}^2 \times \mathbb{R}$ equiped with the standard metric. This finishes the proof of Theorem 6.2 $\square$

Second, the rigidity result of Cai-Galloway [5]. The proof is completely similar to Theorem 6.2 so, we omit the proof.

Theorem 6.3. *Let $(\mathcal{N}, g)$ be a complete manifold with density so that S is nonnegative. If $(\mathcal{N}, g)$ contains an area miminizing compact surface in its homotopy class of genus greater than or equal to 1, then the product manifold $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}^2$ is a torus equiped with the standard flat metric, is an isometric covering of $(\mathcal{N}, g)$. In particular, $(\mathcal{N}, g)$ is flat.*

And finally, the rigidity result of Nunes [23]:

Theorem 6.4. *Let $(\mathcal{N}, g)$ be a complete manifold with density so that $S \geq -\lambda$ for some positive constant λ. Moreover, suppose that $\Sigma \subset (\mathcal{N}, g)$ is a two-sided compact embedded Riemannian surface of genus $\gamma \geq 2$ which minimizes area in its homotopy class. Then,*

$$A(\Sigma) \geq \frac{4\pi(\gamma - 1)}{\lambda}.$$

Moreover, if the equality holds, the universal cover of $(\mathcal{N}, g)$ *is isometric (up to scaling) to the product manifold* $\Sigma_\gamma \times \mathbb{R}$*, where* Σ_γ *is a compact surface of genus* γ *equiped with a metric of constant Gaussian curvature* -1*.*

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