Horizontal Delaunay surfaces with constant mean curvature in product spaces

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based on

• —, F. Torralbo New examples of constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.
Michigan J. Math. **63** (2014), no. 4, 701–723.

• —, F. Torralbo Compact embedded surfaces with constant mean curvature in $\mathbb{S}^2 \times \mathbb{R}$.
Amer. J. Math. **142** (2020), no. 4, 1981–1994.

• —, F. Torralbo Horizontal Delaunay surfaces with constant mean curvature in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.
Preprint, arXiv:2007.06882.

Constant mean curvature surfaces

Definition

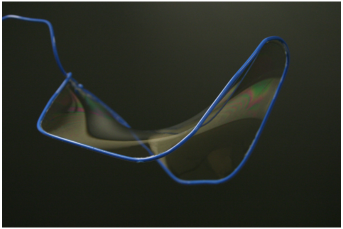
A surface Σ immersed in a 3-manifold N is an H-surface (i.e., it has constant mean curvature H) if:

(i) The second fundamental form σ has constant trace $2H$, or equivalently

(ii) Σ is a critical point of $\mathcal{J} = \text{Area} - 2H \cdot \text{Volume}$.

If $H = 0$, then such a Σ is called a minimal surface.

- They show up in nature as interfaces between fluids (Laplace-Young), motivating the popular isoperimetric and Plateau problems.



- However, nature is only interested in (local) minima.

Compact embedded H-surfaces in $\mathbb{S}^2 \times \mathbb{R}$

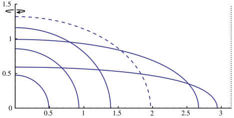
Alexandrov reflection principle

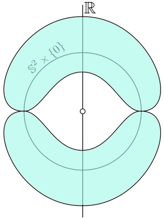
Compact embedded H-surfaces in a product 3-manifold $M \times \mathbb{R}$ are bigraphs over domains $\Omega \subset M$.

Compact embedded H-surfaces in $\mathbb{R}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ must be rotational H-spheres (Alexandrov problem). However, there are many compact embedded H-surfaces in $\mathbb{S}^2 \times \mathbb{R}$.

- The only compact minimal surfaces are the horizontal slices $\mathbb{S}^2 \times \{t_0\}$.
- For any $H > 0$, there are rotationally invariant H-spheres and H-tori (Pedrosa–Ritoré).

The value $H = \frac{1}{2}$ will play an important role ($\frac{1}{2}$-spheres are bigraphs over an hemisphere of $\mathbb{S}^2$).





Theorem (—, 2012)

If $0 < H < \frac{1}{2}$, the complement of the domain of a compact H-bigraph in $\mathbb{S}^2 \times \mathbb{R}$ of genus g consists of $g+1$ convex disks.

Theorem (— & Torralbo, 2019)

For each $0 < H < \frac{1}{2}$ and $g \geq 0$, we find one compact embedded H-surface with genus g and dihedral symmetry in $\mathbb{S}^2 \times \mathbb{R}$.

Theorem (— & Torralbo, 2020)

For each $H > \frac{1}{2}$, we find finitely-many embedded H-tori with dihedral symmetry in $\mathbb{S}^2 \times \mathbb{R}$.

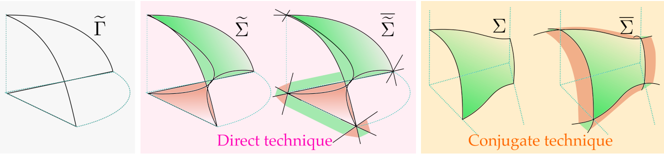
Open questions:

- Are there more embedded H-tori in $\mathbb{S}^2 \times \mathbb{R}$?
- Are there compact embedded H-surfaces in $\mathbb{S}^2 \times \mathbb{R}$ with arbitrary genus if $H \geq \frac{1}{2}$?

Lawson's conjugate technique in $\mathbb{R}^3$

Lawson correspondence

There is an isometric conjugation between 0-surfaces in $\mathbb{S}^3$ and 1-surfaces in $\mathbb{R}^3$.

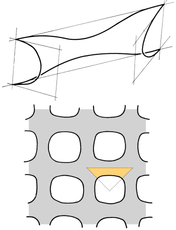


Steps in the construction:

1. Choose a geodesic polygon $\widetilde{\Gamma} \subset \mathbb{S}^3$ whose angles are divisors of π.
2. Make sure that the Plateau problem for $\widetilde{\Gamma}$ has a solution $\widetilde{\Sigma}$.
   $\rightsquigarrow$ Meeks-Yau's solution using mean-convex barriers
3. Consider the conjugate surface Σ in $\mathbb{R}^3$.
   $\rightsquigarrow$ Each component of its boundary is a plane curve
4. Reflect $\widetilde{\Sigma}$ or Σ to obtain complete surfaces.
   $\rightsquigarrow$ Schwarz reflection principle for H-surfaces + absence of isolated singularities.

Difficulty: $\widetilde{\Sigma}$ and Σ are not explicit surfaces. Any desired property of Σ must be deduced from properties of the boundary $\widetilde{\Gamma}$.

Example: As for Lawson's doubly periodic H-surfaces in $\mathbb{R}^3$, the fundamental piece Σ looks like this:



Daniel's sister correspondence in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$

$\mathbb{E}(\kappa,\tau)$-spaces

Simply-connected homogeneous 3-manifolds with 4-dimensional isometry group are given by a 2-parameter family $\mathbb{E}(\kappa,\tau)$ with $\kappa,\tau \in \mathbb{R}$.

	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$\tau = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$\tau \neq 0$	$\mathbb{S}^3_b$	Nil_3	$\widetilde{\mathrm{Sl}}_2(\mathbb{R})$

- Common framework for Thurston geometries except for $\mathbb{H}^3$ and Sol_3.
- $\mathbb{E}(\kappa,\tau)$ admits a Killing submersion over $\mathbb{M}^2(\kappa)$ whose fibers are the integral curves of a unitary Killing vector field.
  - ⇝ The constant τ is the bundle curvature and accounts for the integrability of the horizontal distribution.
  - ⇝ The notions of *vertical* and *horizontal* are natural in $\mathbb{E}(\kappa,\tau)$.

Sister correspondence (Daniel, 2007)

Let $\epsilon \in \{-1,0,1\}$. There is an isometric conjugation between:

1. minimal surfaces in $\mathbb{E}(4H^2+\epsilon, H)$,
2. H-surfaces in $\mathbb{E}(\epsilon,0) = \mathbb{M}^2(\epsilon) \times \mathbb{R}$.

They determine each other up to (positive) isometries.

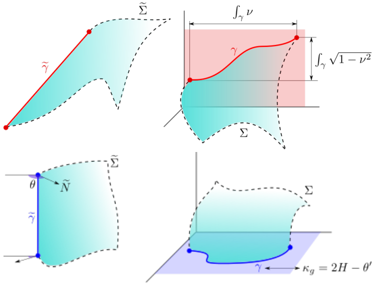
This yields the following cases:

minimal surface in	gives an H-surface in		
	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{H}^2 \times \mathbb{R}$	$\mathbb{R}^3$
$\mathbb{S}^3_b(4H^2+\epsilon, H)$	$H > 0$	$H > 1/2$	$H > 0$
Nil_3	—	$H = 1/2$	—
$\widetilde{\mathrm{SL}}_2(4H^2-1, H)$	—	$0 < H < 1/2$	—
$\mathbb{H}^2 \times \mathbb{R}$	—	$H = 0$	—
$\mathbb{S}^2 \times \mathbb{R}$	$H = 0$	—	—
$\mathbb{R}^3$	—	—	$H = 0$

The conjugate technique in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$

Let $\widetilde{\Sigma} \looparrowright \mathbb{E}(4H^2+\epsilon, H)$ and $\Sigma \looparrowright \mathbb{M}^2(\epsilon) \times \mathbb{R}$ be conjugate.

- $\widetilde{\Sigma}$ and Σ are isometric.
- Their angle function $\nu = \langle N, \xi \rangle = \langle \widetilde{N}, \widetilde{\xi} \rangle$ is the same.
- The tangent part of the Killing and the shape operator rotate $\frac{\pi}{2}$ degrees in Σ with respect to $\widetilde{\Sigma}$.



Boundary behavior (— & Torralbo, 2012), (Plehnert, 2014)

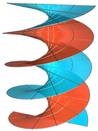
(a) A horizontal geodesic $\widetilde{\gamma} \subset \widetilde{\Sigma}$ corresponds to a planar line of symmetry $\gamma \subset \Sigma$ contained in a vertical plane $\mathbb{M}^1(\epsilon) \times \mathbb{R}$.

(b) A vertical geodesic $\widetilde{\gamma} \subset \widetilde{\Sigma}$ corresponds to a planar line of symmetry $\gamma \subset \Sigma$ contained in a horizontal slice $\mathbb{M}^2(\epsilon) \times \{t_0\}$.

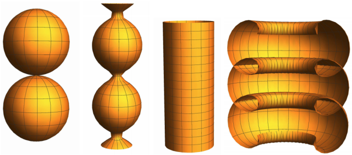
Hence Σ can be completed by succesive mirror symmetries.

- If θ is the angle between the normal to Σ and a constant reference along $\widetilde{\gamma}$, the geodesic curvature κ_g of γ in $\mathbb{M}^2 \times \{t_0\}$ verifies
$$\kappa_g = 2H - \theta'.$$

An easy but surprising example:



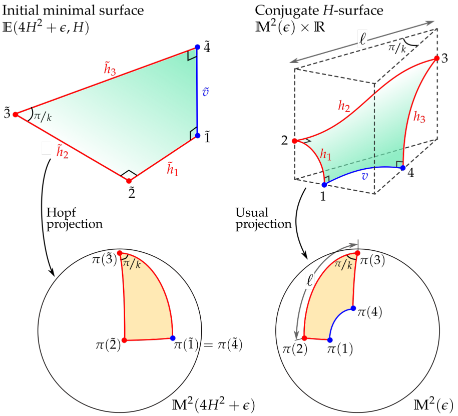
Minimal helicoids
in Berger spheres
$\mathbb{E}(4H^2+\epsilon, H)$.



Vertical Delaunay
H-surfaces in $\mathbb{R}^3$,
$\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

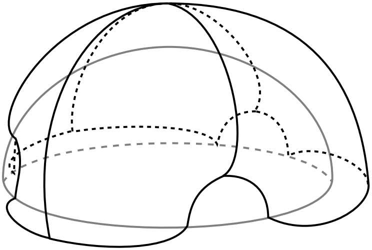
Theorem (— & Torralbo, 2019)

Given $0 < H < 1/2$ and $g \geq 0$, there is a compact embedded H-surface in $\mathbb{S}^2 \times \mathbb{R}$ with genus g. It is invariant under a dihedral group of symmetries of $\mathbb{S}^2$.



Steps in the construction

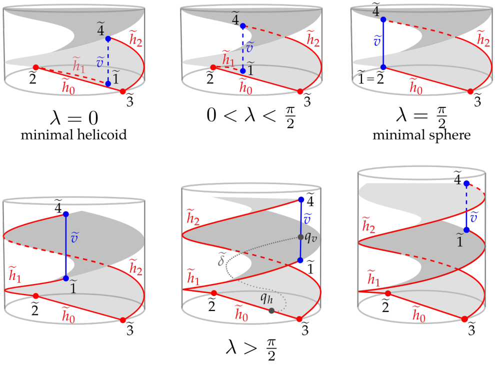
1. Find the points in which $\nu = 0$ and $\nu = -1$ by comparing with umbrellas and Hopf tori.
2. Fit the length $\ell = \frac{\pi}{2}$ via continuity.
3. Embeddedness follows from estimates of the curvature of the boundary (convex circles).



Horizontal Delaunay surfaces 1. Solution of the Plateau problem

We will consider the local model for the Berger spheres

$$\left[\mathbb{R}^3, \frac{\mathrm{d}x^2+\mathrm{d}y^2}{(1+\frac{4H^2+\epsilon}{4}(x^2+y^2))^2} + \left(\mathrm{d}z + \frac{H(x\mathrm{d}y - x\mathrm{d}y)}{1+\frac{4H^2+\epsilon}{4}(x^2+y^2)}\right)^2\right].$$



Boundary: $\widetilde{\Gamma}_\lambda = \widetilde{h}_0 \cup \widetilde{h}_1 \cup \widetilde{h}_2 \cup \widetilde{v}$.

Solution of the Plateau problem: $\widetilde{\Sigma}_\lambda$

- Mean convex body with the helicoid and the cylinder as barriers.
- $\widetilde{\Sigma}_\lambda$ is a graph if and only if $0 \leq \lambda \leq \frac{\pi}{2}$.
- $\widetilde{\Gamma}_\lambda$ is not a Nitsche graph if $\lambda > \frac{\pi}{2}$.

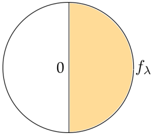
Uniqueness of solution:

- $\widetilde{\Gamma}_\lambda$ is actually a Nitsche graph in the direction of the Killing vector field

$$\widetilde{X} = -y\partial_x + x\partial_y + \frac{2H}{4H^2+\kappa}\partial_z$$

giving rise to a Killing submersion structure inside the cylinder.

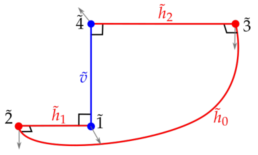
- Then $\widetilde{\Sigma}_\lambda$ is the solution to a Dirichlet problem over a half-disk:



- Understanding the model is crucial.

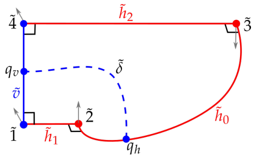
Horizontal Delaunay surfaces 2. Analysis of the angle function

The case $0 \leq \lambda \leq \frac{1}{2}$



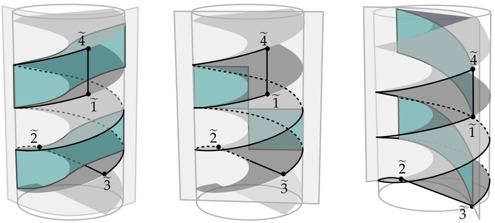
- Vertical points ($\nu = 0$): $\widetilde{v}$
- Horizontal points ($\nu = -1$): $\widetilde{2}$ and $\widetilde{3}$

The case $\lambda > \frac{1}{2}$



- Vertical points ($\nu = 0$): $\widetilde{v} \cup \widetilde{\delta}$
- Horizontal points ($\nu = \pm 1$): $\widetilde{2}$ and $\widetilde{3}$

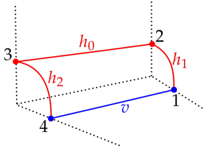
Analysis of vertical points: the zeroes of ν and $\nabla \nu$ can be captured by looking at the intersection with a tangent Clifford cylinder:



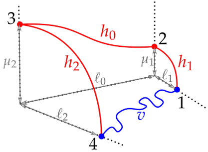
- $\nu = 0 \rightsquigarrow$ at least 2 curves in the intersection.
- $\nu = \nabla \nu = 0 \rightsquigarrow$ at least 3 curves in the intersection.

Horizontal Delaunay surfaces 3. Depiction of the conjugate surfaces

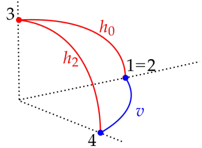
The case $\lambda = 0$: equivariant H-tori



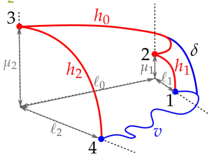
The case $0 < \lambda < \frac{\pi}{2}$: H-unduloids



The case $\lambda = \frac{\pi}{2}$: equivariant H-spheres



The case $\lambda > \frac{\pi}{2}$: H-nodoids



Monotonicity properties

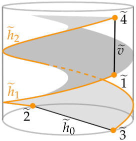
- The signed lengths ℓ_0, ℓ_1, ℓ_2 of the projections depend monotonically on λ.
- The signed heights μ_1, μ_2 of the points 2 and 3 depend monotonically on λ.
- The lengths of h_0 and v are equal and coincide with those of vertical Delaunay H-surfaces (not depending on λ).

Embeddedness

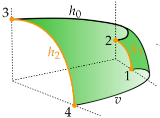
If the boundary $\widetilde{\Gamma}_\lambda$ projects one-to-one to $\mathbb{H}^2$, then the fundamental piece is embedded by maximum principle. However, it is hard to control the curve $\widetilde{v}$.

Horizontal Delaunay surfaces 4. Embeddedness

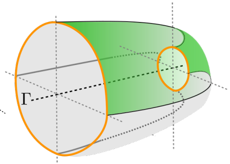
Embeddedness of the fundamental annulus



$\widetilde{X} = -y\partial_x + x\partial_y + \frac{2H}{4H^2+\kappa}\partial_z$



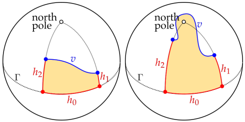
Fundamental piece



Fundamental annulus

- The initial minimal surface is a graph in the direction of $\widetilde{X}$.
- $\widetilde{u} = \langle \widetilde{X}, \widetilde{N} \rangle$ lies in the kernel of the common stability operator and extends to the fundamental annulus A_λ giving $\lambda_1(A_\lambda) = 0$.
- Let X be the Killing vector field in $\mathbb{M}^2(\epsilon) \times \mathbb{R}$ coming from translations along the axis Γ, so $u = \langle X, N \rangle$ vanishes on ∂A_λ.
- Hence, $u = a_\lambda \widetilde{u}$ has sign and A_λ is a X-multigraph.

The unduloids do not go over the north pole if $H > \frac{1}{2}$

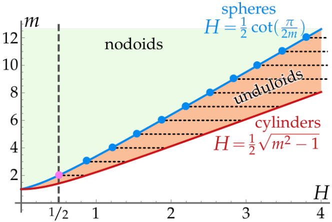


- If $H > \frac{1}{2}$, the sphere $(\lambda = \frac{\pi}{2})$ does not go over the north pole. Then neither do the curves $\widetilde{h}_1$ and $\widetilde{h}_2$ by monotonicity.
- If an interior point goes over the north pole, then $u = 0$ at that point (contradiction).

Moduli space

We obtain a family of examples in terms of 2-parameters

- $H > 0$: the value of the mean curvature.
- $m > 1$: a half of the number of fundamental pieces we need to complete the equator of $\mathbb{S}^2$.



Each point of the dotted horizontal lines represents an embedded H-torus with dihedral symmetry group D_m.

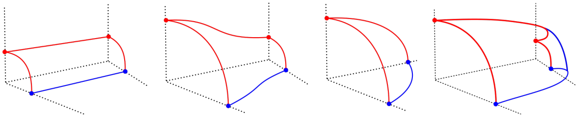
- The picture looks like that of invariant Delaunay surfaces in $\mathbb{S}^3$.
- The limit of tangent $\frac{1}{2}$-spheres shows up again.

Horizontal Delaunay surfaces 5. The case of $\mathbb{H}^2 \times \mathbb{R}$

Theorem (— & Torralbo, 2020)

For each $H > \frac{1}{2}$, there is a 1-parameter family $\overline{\Sigma}_\lambda$, $\lambda > 0$, of H-surfaces lying at bounded distance from a horizontal geodesic Γ:

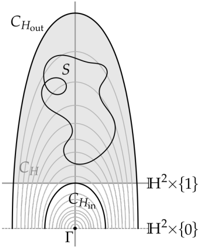
- If $\lambda = 0$, then $\overline{\Sigma}_\lambda$ is an H-cylinder invariant by hyperbolic translations.
- If $0 < \lambda < \frac{\pi}{2}$, then $\overline{\Sigma}_\lambda$ is a properly embedded H-unduloid.
- If $\lambda = \frac{\pi}{2}$, then $\overline{\Sigma}_\lambda$ is a rotationally invariant H-sphere.
- If $\lambda > \frac{\pi}{2}$, then $\overline{\Sigma}_\lambda$ is a proper (non-Alexandrov-embedded) H-nodoid.



Theorem (— & Torralbo, 2020)

There are no properly immersed H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ at bounded distance from a horizontal geodesic with $H \leq \frac{1}{2}$.

Sketch. There is a foliation $C_H = \overline{\Sigma}_0$ of $(\mathbb{H}^2 \times \mathbb{R}) - \Gamma$ by the H-cylinders with $\frac{1}{2} < H < \infty$. Apply Mazet's halfspace theorem.



Thanks for your attention...

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