On the convergence of minimal graphs

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based on

• A. Del Prete, J.M. Manzano, B. Nelli. The Jenkins–Serrin problem in 3-manifolds with a Killing vector field. *In preparation*.
• A.M. Lerma, J.M. Manzano. Compact stable surfaces with constant mean curvature in Killing submersions. *Ann. Mat. Pura Appl.* **196** (2017), no. 9, 1345–1364.

Killing submersions

Definition

A Killing submersion is a Riemannian submersion $\pi : \mathbb{E} \to M$, where $\mathbb{E}$ and M are orientable and connected, such that the fibers of π are the integral curves of a nowhere vanishing Killing vector field ξ.

Examples

- Homogeneous 3-manifolds.
- Product manifolds $M \times \mathbb{R}$.
- Warped products $M \times_\mu \mathbb{R}$.

Basic features

- $\xi \rightsquigarrow \{\Phi_t\}_{t\in\mathbb{R}}$ vertical translations.
- ξ is determined up to a constant.
- The fibers need not be geodesics.
- If a fiber has finite length, then all fibers have finite length.
- π admits a global section if:
  - M is not compact, or
  - fibers have infinite length.

Classification ingredients

1. Base surface M.
2. Killing length $\mu = \|\xi\|$
   - $\mu \in C^\infty(M)$.
3. Bundle curvature $\tau = \frac{1}{\mu}\langle \nabla_{e_1}\xi, e_2\rangle$.

   $\{e_1, e_2\}$ pos. on. basis of $\ker(\mathrm{d}\pi_p)^\perp$.
   - $\tau \in C^\infty(M)$.
   - τ does not depend on ξ.
   - $\tau \equiv 0 \Leftrightarrow \ker(\mathrm{d}\pi_p)^\perp$ integrable.

Theorem (Lerma–M)

Given $\tau, \mu \in C^\infty(M)$, $\mu > 0$, and M simply connected, we can recover univocally the Killing submersion if we assume the total space is also simply connected.

Killing graphs and Killing cylinders

Let $\pi : \mathbb{E} \to M$ be a Killing submersion

- A Killing graph over $\Omega \subset M$ is a smooth section $F : \Omega \to \pi^{-1}(\Omega) \subset \mathbb{E}$.

Let $\Omega \subset M$ with a smooth zero section $F_0 : \Omega \to \mathbb{E}$, then any smooth graph over Ω can be expressed for some $u \in C^\infty(\Omega)$ as

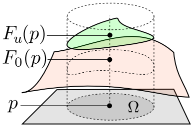
$$F_u : \Omega \to \mathbb{E}, \qquad F_u(p) = \Phi_{u(p)}(F_0(p)).$$

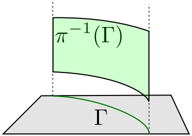
Mean curvature:

$$H = \frac{1}{2\mu}\operatorname{div}(\mu\, \pi_* N) = \frac{1}{2\mu}\operatorname{div}\left(\frac{\mu^2\, Gu}{\sqrt{1+\mu^2\|Gu\|^2}}\right)$$

Generalized gradient: $Gu = \nabla u - Z$ with $Z \in \mathfrak{X}(\Omega)$ not depending on u satisfying

$$\operatorname{div}(JZ) = \frac{-2\tau}{\mu}$$





- A Killing cylinder is the preimage by π of a curve $\Gamma \subset M$.

Mean curvature:

$$2H = \kappa_g - \langle \eta, \tfrac{1}{\mu}\nabla\mu\rangle = \mu\,\widetilde{\kappa}_g$$

with the μ-metric $\mathrm{d}s_\mu^2 = \mu^2 \mathrm{d}s_M^2$

- $H = 0 \Leftrightarrow \mu$-geodesic
- $H \geq 0 \Leftrightarrow \mu$-convex

Existence of sections (Lerma–M)

- $\pi : \mathbb{E} \to M$ admits a global section $\Leftrightarrow$ either M is not compact or $\int_M \frac{\tau}{\mu} = 0$.
- If $\Omega \subset M$ is precompact, then there is a minimal section over some open subset $\overline{\Omega} \subset U \subset M$.

Corollaries (Del Prete–M–Nelli)

- Open book decompositions by minimal vertical cylinders.
- Local classification of invariant H-surfaces.

Local convergence

Let $\Omega \subset M$ be precompact $\rightsquigarrow \pi^{-1}(\Omega(\delta))$ has bounded geometry:

$$|K_{\mathrm{sec}}^{\mathbb{E}}| \leq |K_M| + 5\tau^2 + 2\|\overline{\nabla}\tau\| + \tfrac{2\tau}{\mu}\|\overline{\nabla}\mu\| + \|\overline{\nabla}^2\mu\| \leq \Lambda.$$

Let Σ a the minimal graph given by $u \in \mathcal{C}^\infty(\Omega)$.

Rosenberg–Souam–Toubiana

- Gradient estimates:

$$\|Gu\| \leq C(C_1, C_2, \Lambda)$$

if $|u_1| \leq C_1$ and $d_M(\cdot, \partial\Sigma) \geq C_2$.

- Curvature estimates:

$$|A| \leq \frac{C(\delta^2\Lambda)}{\min\{d_M(\cdot,\partial\Sigma), \frac{\pi}{2\Lambda}, \delta\}}.$$

- Uniform graph lemma: each $q \in \Sigma$ has a ball of uniform radius that admits harmonic coordinates.
  - $\exp_q^{\mathbb{E}}(D(\rho)) \supset B_\Sigma(q, \rho_0)$ Euclidean graph of f.
  - $|f(v)| \leq M|v|^2$ for all $v \in D(\rho)$ (Pérez–Ros).

M, ρ, ρ_0 depend on $d_M(\cdot, \partial\Sigma)$ and the geometry.

Bounded gradients

Let $\{u_n\}$ be minimal graphs in Ω such that $\|Gu_n(p)\| \leq M$ for all n.

$$\{u_n - u_n(p)\} \xrightarrow[\text{sub}]{\mathcal{C}^\infty} u_\infty \text{ in } B(p, R)$$

R depends on M, $d(p, \partial\Omega)$ and the geometry.

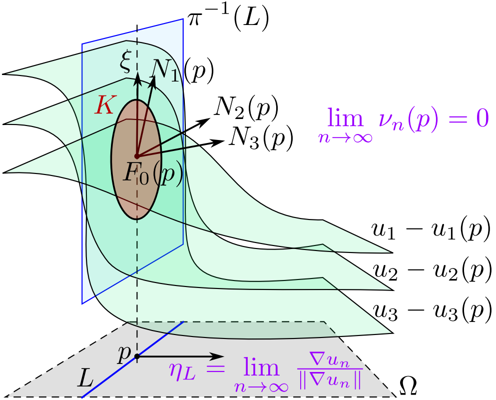
Unbounded gradients

Let $\{u_n\}$ be minimal graphs in Ω such that $\|Gu_n(p)\| \to +\infty$.

$$\{u_n - u_n(p)\} \xrightarrow[\text{sub}]{\mathcal{C}^\infty} \text{ball of a minimal cylinder}$$

(converge as surfaces, not as graphs).

Divergence lines



Let $\Sigma_n(p) = \Phi_{-u_n(p)}\Sigma_{u_n}$

Definition

A divergence line is a maximal μ-geodesic $L \subset \Omega$ such that $\Sigma_n(p)$ subconverges locally to $\pi^{-1}(L)$ around some $p \in L$.

Mazet–Rodríguez–Rosenberg

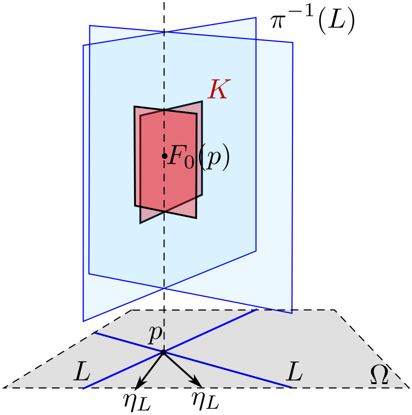
$\Sigma_n(p)$ subconverges to $\pi^{-1}(L)$ on all compact subsets $K \subset \pi^{-1}(L)$.

There are many possible configurations of μ-geodesics, but our goal is to check that divergence lines cannot behave too weirdly.

We will assume hereafter that Ω has piecewise regular boundary consisting of μ-convex arcs or closed curves, possibly μ-geodesic.

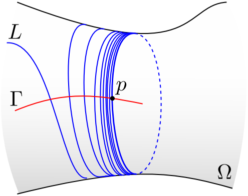
Divergence lines are properly embedded

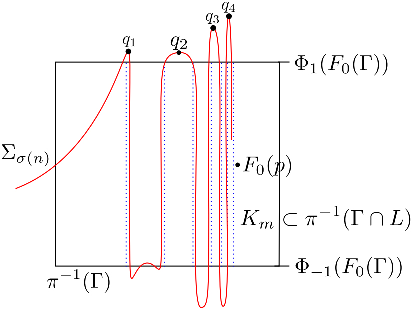
No self-intersections:



$$\eta_L = \lim_{n\to\infty} \frac{\nabla u_n}{\|\nabla u_n\|}$$

No interior accumulations:

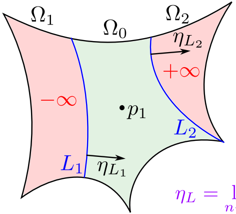




The same applies in the boundary with minor changes.

Divergence levels

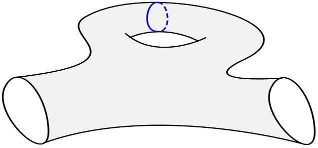
Assumption: all divergence lines are disjoint and let $\mathcal{D}$ be their union.
Ω_0: connected component of $\Omega - \mathcal{D}$
L: divergence line in $\partial\Omega_0$ such that $\pm\eta_L$ exterior to Ω_0

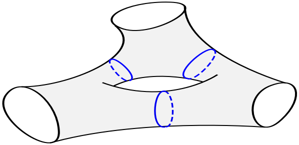


$\eta_L = \lim_{n\to\infty} \frac{\nabla u_n}{\|\nabla u_n\|}$

In the picture, we say that Ω_1 (resp. Ω_2) lies at a lower (resp. higher) divergence level than Ω_0.

These configurations are not okay:





Mazet–Rodríguez–Rosenberg

A subsequence $u_{\sigma(n)} - u_{\sigma(n)}(p_0)$, $p_0 \in \Omega_0$,

- converges in Ω_0,
- diverges to $\pm\infty$ in L with

$$\lim_{n\to\infty} \mathrm{Flux}(u_{\sigma(n)}, L) = \pm\mathrm{Length}_\mu(L),$$

- diverges to $\pm\infty$ in the adjacent component Ω_1 (if any).

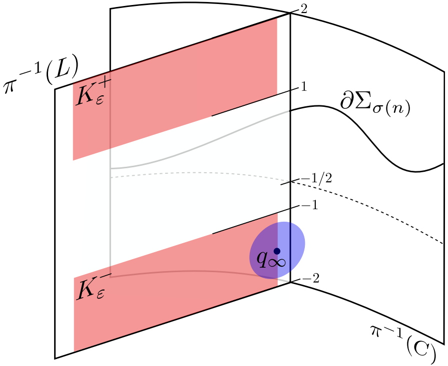
Mazet–Rodríguez–Rosenberg: the assumption is not restrictive if there are countably many divergence lines.

Preventing divergence lines from touching the interior of the arcs

Lemma (Del Prete–M–Nelli)

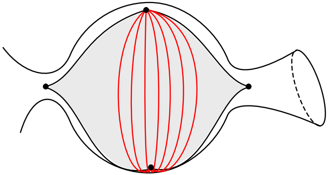
Let $C \subset \partial\Omega$ be a μ-convex arc and $p \in C$. Let $u_n \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^0(\Omega \cup C)$ be minimal graphs.

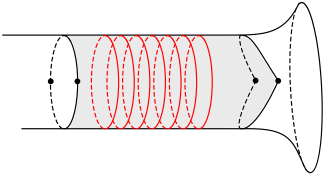
If $\{u_n|_C - u_n(p)\} \to f \in \mathcal{C}^0(C)$ uniformly, then no divergence line ends at p.



We will assume this property (locally on each component of $\partial\Omega$) hereafter, so divergence lines either connect two vertices or they are closed curves.

It doesn't solve the countability issue:





Isotopy classes of divergence lines

Lemma (Del Prete–M–Nelli)

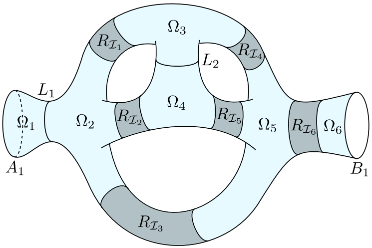
- The limit of divergence lines (as μ-geodesics) is again a divergence line.
- Isotopy classes are closed under limits.

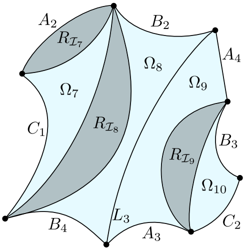
Lemma (Del Prete–M–Nelli)

We can assume all divergence lines are disjoint after considering a subsequence.

Let $\mathcal{I}$ be an isotopy class of divergence lines not isotopic to any boundary component:

- $\mathcal{I}$ admits a linear order with maximum and minimum elements $L_{\pm}$.
- The union of $\mathcal{I}$ lies in a region $R_{\mathcal{I}}$, where there is no other divergence lines.
- All lines in $\mathcal{I}$ have the same μ-length.
- The normals η_L point in the same direction.
- There are neighboring components $\Omega_{\pm}$.





Each component of $\Omega - \mathcal{D}$ is an inscribed polygon.

There are finitely many regions Ω_i and $R_{\mathcal{I}_j}$ $\rightsquigarrow$ Divergence levels apply!

The Jenkins–Serrin problem

$\Omega \subset M$ precompact with piecewise regular boundary consisting of μ-geodesic open arcs or simple closed curves $A_1, \ldots, A_r, B_1, \ldots, B_s$ and regular curves $C_1, \ldots, C_m$ which are μ-convex with respect to the inner conormal to Ω. The finite set $V \subset \partial\Omega$ of endpoints of these curves is the corner set of Ω.

We are looking for

$$u \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^0(\Omega \cup (\cup C_i))$$

with limit values:

- $f_i \in \mathcal{C}^0(C_i)$ on each C_i
- $+\infty$ on each A_i,
- $-\infty$ on each B_i.

We will assume necessarily that no two A_i or two B_i meet at a convex corner.

For an inscribed μ-polygon $\mathcal{P} \subset \Omega$:

$$\alpha(\mathcal{P}) = \mathrm{Length}_\mu((\cup A_i) \cap \mathcal{P}),$$
$$\beta(\mathcal{P}) = \mathrm{Length}_\mu((\cup B_i) \cap \mathcal{P}),$$
$$\gamma(\mathcal{P}) = \mathrm{Length}_\mu(\mathcal{P}).$$

Theorem (Del Prete–M–Nelli)

Case $\{C_i\} \neq \emptyset$. *There is solution iff*

$$2\alpha(\mathcal{P}) < \gamma(\mathcal{P}) \quad \text{and} \quad 2\beta(\mathcal{P}) < \gamma(\mathcal{P})$$

for every inscribed μ-polygon $\mathcal{P}$.

Case $\{C_i\} = \emptyset$. *There is solution iff*

$$2\alpha(\mathcal{P}) < \gamma(\mathcal{P}) \quad \text{and} \quad 2\beta(\mathcal{P}) < \gamma(\mathcal{P})$$

for every inscribed μ-polygon $\mathcal{P} \neq \partial\Omega$ and $\alpha(\partial\Omega) = \beta(\partial\Omega)$.

Existence of solutions

Consider a sequence of minimal graphs

$$u_n = \begin{cases} n & \text{on } \cup A_i, \\ -n & \text{on } \cup B_i, \\ f_i \text{ truncated at } \pm n & \text{on } C_i. \end{cases}$$

(We employ the Perron process)

Lemma (Del Prete–M–Nelli)

No divergence line of $\{u_n\}$ is isotopic to any A_i or B_i.Moreover,

$$\lim_{n\to\infty} \text{Flux}(u_n, A_i) = \text{Length}_\mu(A_i),$$
$$\lim_{n\to\infty} \text{Flux}(u_n, B_i) = -\text{Length}_\mu(B_i).$$

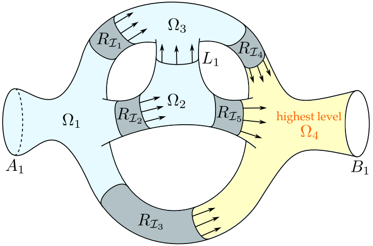
(We use the JS-conditions)

The structure of the set of the divergence lines is finally clear!

Lemma (Del Prete–M–Nelli)

$\{u_n\}$ subconverges in all Ω under the JS-conditions.

By contradiction, if there were divergence lines, then follow the arrows to a maximally high component of $\Omega - (\mathcal{D} \cup (\cup R_{\mathcal{I}}))$:



Such a component $\Omega_0 \neq \Omega$ must contain some of the A_i or B_i, so that $\mathcal{P} = \partial\Omega_0$ contradicts the JS-conditions.

We're running out of time...

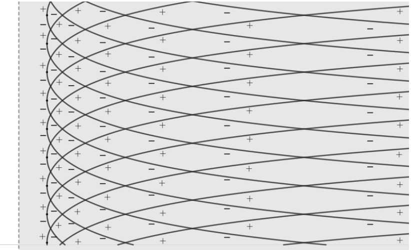
The rest of our work

1. The limit achieves the right boundary values:
   - Continuous values f_i on C_i.
   - Asymptotic values $\pm\infty$.

   Barriers: small Scherk graphs + annuli

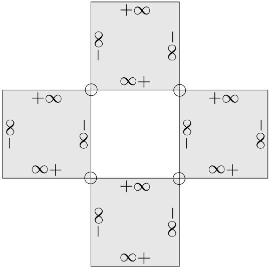
2. Uniqueness (up to vertical translations if $\cup C_i = \emptyset$).

Ex.1: New minimal surfaces in $\mathbb{R}^3$

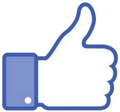


Can they be continued analytically?

Ex.2: A wild Scherk graph in Nil_3



Gracias por venir...



... y citadnos si os ha gustado.