Subvariedades de codimensión dos en espaciotiempos

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Outline

1. **Parte I (Senovilla). Propiedades generales.**
   - Introducción. Notación y nomenclatura, ecuaciones básicas
   - Clasificación basada en la orientación causal de H
   - Variación del área y curvatura media. Subvariedades y simetrías
   - Estabilidad de MOTS. Operador de estabilidad AMS
   - Deformación de esferas redondas en simetría esférica $(n = 4)$
   - Núcleo de la región atrapada espacio-temporal
   - Deformación de MOTS en general. Una fórmula para el autovalor principal
2. **Parte II (Mars). Operador de estabilidad; propiedades de existencia, unicidad, y evolución de MOTS. Topología. Relación con simetrías, Desigualdades de tipo área-momento angular, MOTS en horizontes de Killing, caracterizaciones del autovalor principal, etc.**

Basic concepts and notation

- $(\mathcal{V}, g)$ is an n-dimensional, oriented and time-oriented, Lorentzian manifold with metric tensor g of signature $(-, +, \dots, +)$.
- $\forall x \in \mathcal{V}$, the isomorphism between $T_x\mathcal{V}$ and $T_x^*\mathcal{V}$ is denoted by

$$\begin{array}{rcl} \flat \,: T_x\mathcal{V} & \longrightarrow & T_x^*\mathcal{V} \\ v & \longmapsto & v^\flat \end{array}$$

and defined by $v^\flat(w) = g(v, w)$, $\forall w \in T_x\mathcal{V}$.
Its inverse map is denoted by $\sharp$. These maps extend naturally to the tangent and co-tangent bundles.

Definition (Codimension-two (imbedded) submanifold)

*A **codimension-two submanifold** —a "surface" if $n = 4$—is (S, Φ), where S is an $(n-2)$-dimensional oriented manifold and $\Phi : S \longrightarrow \mathcal{V}$ is an imbedding.*

(S will be identified with its image $\Phi(S) \subset \mathcal{V}$.)

1st fundamental form and orthogonal splitting

- The **first fundamental form**: $h \equiv \Phi^* g$.
- h is assumed to be positive definite on S, so that S is spacelike.
- Then, at any $x \in S$ one has the orthogonal decomposition

$$T_x\mathcal{V} = T_xS \oplus T_xS^{\perp}\,.$$

- $\mathfrak{X}(S)$ (respectively $\mathfrak{X}(S)^{\perp}$) will denote the set of smooth vector fields tangent (resp. orthogonal) to S.
- We will often give definitions and properties on $\mathfrak{X}(S)$, but they of course have always a previous, more fundamental, version on each T_xS.

Special bases on $\mathfrak{X}(S)^\perp$. Boost freedom

- S having co-dimension 2, there are two *independent* normal vector fields on S: two sections on the normal bundle which are linearly independent at each $x \in S$.
- A possible choice is that they constitute an orthonormal (ON) basis on $\mathfrak{X}(S)^\perp$. Notation: $u, m \in \mathfrak{X}(S)^\perp$, with

$$g(m,m) = -g(u,u) = 1, \quad g(u,m) = 0\,.$$

- u will be assumed to point to the future
- Any two such ON bases are related by a boost:

$$\begin{pmatrix} u' \\ m' \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} u \\ m \end{pmatrix}, \qquad \beta \in C^\infty(S) \tag{1}$$

The null normals

- Another possible choice, which will be fundamental in our discussion, is to take two independent normal **null** vector fields (and future-pointing say). We will denote these by $k, \ell \in \mathfrak{X}(S)^{\perp}$, so that

$$g(\ell, \ell) = g(k, k) = 0, \quad g(\ell, k) = -1$$

- The last of these is a convenient normalization condition
- Observe that, to any ON basis $\{u, m\}$ on $\mathfrak{X}(S)^{\perp}$, one can associate a null basis given by $\sqrt{2}\,\ell = u + m$ and $\sqrt{2}\,k = u - m$ (also $\ell = u + m$ and $2k = u - m$, etc.)
- The boost freedom becomes now simply

$$\ell \longrightarrow \ell' = e^{\beta}\ell, \qquad k \longrightarrow k' = e^{-\beta}k \tag{2}$$

so that the two independent null *directions* are uniquely determined on S.

Volume forms and Hodge dual operators

- The canonical volume element $(n-2)$-form associated to (S,h) is denoted by $\bar{\epsilon}$, while ϵ is the volume n-form in $(\mathcal{V},g)$.
- There is also a volume element 2-form on $\mathfrak{X}(S)^{\perp}$, induced by ϵ and $\bar{\epsilon}$ and denoted by $\epsilon^{\perp}$. The corresponding Hodge dual operator is written and defined by

$$\star^{\perp}N \equiv (i_N\epsilon^{\perp})^{\sharp}, \ \forall N \in \mathfrak{X}(S)^{\perp}$$

- The orientations of $(\mathcal{V},g)$ and of (S,h) will be chosen such that the operator $\star^{\perp}$ acts on the previous bases as follows

$$\star^{\perp}u = m, \ \star^{\perp}m = u; \qquad \star^{\perp}\ell = \ell, \ \star^{\perp}k = -k.$$

- $\star^{\perp}N$ defines the unique normal direction in $\mathfrak{X}(S)^{\perp}$ orthogonal to the normal $N \in \mathfrak{X}(S)^{\perp}$.

Covariant derivatives

- Let ∇ denote the canonical connection in $(\mathcal{V}, g)$
- The *Levi-Civita connection* $\overline{\nabla}$ on (S, h) can be defined as

$$\overline{\nabla}_X Y \equiv (\nabla_X Y)^T \qquad \forall X, Y \in \mathfrak{X}(S)$$

- The *normal connection* D acts, in turn, on $\mathfrak{X}(S)^\perp$

$$D_X : \mathfrak{X}(S)^\perp \longrightarrow \mathfrak{X}(S)^\perp$$

for $X \in \mathfrak{X}(S)$, and is given by the standard definition

$$D_X N \equiv (\nabla_X N)^\perp, \quad \forall N \in \mathfrak{X}(S)^\perp \quad \forall X \in \mathfrak{X}(S).$$

The normal connection one-form s

- For a fixed ON basis on $\mathfrak{X}(S)^{\perp}$, a one-form $s \in \Lambda^1(S)$ is defined by

$$s(X) \equiv -g(u, D_X m) = g(D_X u, m) \qquad \forall X \in \mathfrak{X}(S)$$

- For $\sqrt{2}\ell = u + m$ and $\sqrt{2}k = u - m$ one also has

$$s(X) \equiv -g(k, D_X \ell) = g(D_X k, \ell) \qquad \forall X \in \mathfrak{X}(S)$$

- Therefore, for all $X \in \mathfrak{X}(S)$

$$D_X u = s(X) m, \; D_X m = s(X) u;$$
$$D_X \ell = s(X) \ell, \; D_X k = -s(X) k$$

- Observe that s is not invariant under boost rotations. It is a "connection": $s'(X) = s(X) + X(\beta)$ or simply

$$s' = s + d\beta.$$

- ds is thus invariant and well-defined (related to the normal curvature).

Extrinsic geometry

- The basic extrinsic object is the **shape tensor** (also called **second fundamental form tensor**) of S in $(\mathcal{V}, g)$:

$$\mathrm{I\!I} : \mathfrak{X}(S) \times \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)^{\perp}$$

defined by

$$-\mathrm{I\!I}(X,Y) \equiv (\nabla_X Y)^{\perp} \qquad \forall X, Y \in \mathfrak{X}(S)$$

- Observe that

$$\nabla_X Y = \overline{\nabla}_X Y - \mathrm{I\!I}(X,Y) \qquad \forall X, Y \in \mathfrak{X}(S)$$

- $\mathrm{I\!I}$ contains the information concerning the "shape" of S within $\mathcal{V}$ along *all* directions normal to S.

Definition (Totally geodesic S)

The submanifold S is called totally geodesic if every geodesic within (S,h) is a geodesic of the space-time $(\mathcal{V}, g)$. Equivalently, if

$$\mathrm{I\!I} = 0$$

Second fundamental forms

Definition (Second fundamental form relative to $N \in \mathfrak{X}(S)^\perp$)

$\forall N \in \mathfrak{X}(S)^\perp$, the second fundamental form of S in $(\mathcal{V}, g)$ relative to N is the 2-covariant symmetric tensor field on S defined by

$$K_N(X,Y) \equiv g\left(N, \mathbb{I}(X,Y)\right), \qquad \forall X,Y \in \mathfrak{X}(S)\,.$$

The shape tensor decomposes as

$$\mathbb{I}(X,Y) = -K_k(X,Y)\ \ell - K_\ell(X,Y)\ k$$

in the null basis, or as

$$\mathbb{I}(X,Y) = -K_u(X,Y)\ u + K_m(X,Y)\ m$$

in any ON basis, $\forall X,Y \in \mathfrak{X}(S)$.

Observe that these formulae are invariant under the boost freedom.

Weingarten operators

Definition (Weingarten operator relative to $N \in \mathfrak{X}(S)^\perp$)

The Weingarten operator $A_N : \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)$ associated to $N \in \mathfrak{X}(S)^\perp$ is defined by

$$A_N(X) \equiv (\nabla_X N)^T \qquad \forall X \in \mathfrak{X}(S)$$

(Sometimes A_N is denoted by χ_N in the Physics literature)

Observe that $\nabla_X N = A_N(X) + D_X N$ and that

$$h(A_N(X), Y) = K_N(X, Y), \quad \forall X, Y \in \mathfrak{X}(S)$$

Therefore, at each $x \in S$, $A_N|_x$ is a self-adjoint (with respect to h) linear transformation on $T_x S$. As such, it is always diagonalizable over $\mathbb{R}$.

The mean curvature vector H

Definition (The mean curvature vector field H)

The mean curvature vector field $H \in \mathfrak{X}(S)^{\perp}$ is defined as the trace of the shape tensor with respect to h, or explicitly

$$H \equiv -(\mathrm{tr}\, A_k)\ \ell - (\mathrm{tr}\, A_\ell)\ k \tag{3}$$

in a null basis, or

$$H \equiv -(\mathrm{tr}\, A_u)\ u + (\mathrm{tr}\, A_m)\ m \tag{3'}$$

in ON bases.

Notice that H and

$$\star^{\perp} H = -(\mathrm{tr}\, A_k)\ \ell + (\mathrm{tr}\, A_\ell)\ k = -(\mathrm{tr}\, A_u)\ m + (\mathrm{tr}\, A_m)\ u$$

are well-defined, they are invariant under the boost gauge freedom, and actually under arbitrary changes of basis.

The null expansions

Definition (Expansion along $N \in \mathfrak{X}(S)^{\perp}$)

Each component of H along a particular normal direction

$$\theta_N \equiv g(H, N) = \mathrm{tr} A_N$$

is termed "expansion along N" of S. In particular,

$$\theta_k \equiv g(H, k) = \mathrm{tr} A_k, \qquad \theta_\ell \equiv g(H, \ell) = \mathrm{tr} A_\ell$$

are called the null expansions.

Important: Note that the expansions are not invariant under the boost freedom, e.g. $\theta_{k'} = e^{-\beta}\theta_k$ and $\theta_{\ell'} = e^{\beta}\theta_\ell$, however their signs are invariant.

Another extrinsic vector field G

- For each Weingarten operator A_N, its trace-free part

$$\tilde{A}_N \equiv A_N - \frac{\theta_N}{n-2}\mathbf{1} \qquad (\mathrm{tr}\tilde{A}_N = 0)$$

  is called the shear matrix relative to $N \in \mathfrak{X}(S)^\perp$.
- The shear scalar —or simply the shear—relative to $N \in \mathfrak{X}(S)^\perp$ is the non-negative scalar given by

$$\sigma_N^2 \equiv \mathrm{tr}\tilde{A}_N^2$$

- We define the vector field $G \in \mathfrak{X}(S)^\perp$ by

$$\boxed{G \equiv \sigma_k\,\ell + \sigma_\ell\,k} \qquad (\neq \sigma_u\,u - \sigma_m\,m)!!$$

- G, as well as

$$\star^\perp G = \sigma_k\,\ell - \sigma_\ell\,k$$

  are invariant under the boost freedom (2).

Totally umbilical submanifolds

Definition (Totally umbilical points)

A point $x \in S$ is said to be totally umbilical if

$$\tilde{A}_N|_x = 0 \qquad \forall N \in \mathfrak{X}(S)^\perp.$$

S is called totally umbilical if every $x \in S$ is totally umbilical. Equivalently,

$$\mathrm{II}(X,Y) = \frac{1}{n-2} h(X,Y) H, \quad \forall X, Y \in \mathfrak{X}(S)$$

Result

S is totally umbilical if and only if $G = 0$.

The Casorati operator and curvature

For any ON basis $\{e_i\}$ in $\mathfrak{X}(S)$, set by definition

$$\mathbb{J}(X,Y) \equiv \sum_{i=1}^{n-2} g\left(\mathrm{I\!I}(e_i,X),\mathrm{I\!I}(e_i,Y)\right) \qquad \forall X,Y \in \mathfrak{X}(S)\,.$$

$\mathbb{J}(X,Y)$ is a 2-covariant symmetric tensor field on S.

Then, the Casorati operator

$$B\ : \mathfrak{X}(S) \to \mathfrak{X}(S)$$

is simply defined by

$$g(B(X),Y) \equiv \mathbb{J}(X,Y) \qquad \forall X,Y \in \mathfrak{X}(S)\,.$$

One can check that $B = -\{A_k, A_\ell\}$.

B is invariant under the boost freedom. Observe furthermore that

$$\mathrm{tr}B = g(\mathrm{I\!I},\mathrm{I\!I}).$$

sometimes called the Casorati curvature.

Curvatures

- The intrinsic curvature for (S,h) has the usual definition

$$\overline{R}(X,Y)Z \equiv \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, \quad \forall X,Y,Z \in \mathfrak{X}(S)$$

- Similarly, the normal curvature is defined on S by

$$R^{\perp}(X,Y)N \equiv D_X D_Y N - D_Y D_X N - D_{[X,Y]}N, \quad \forall N \in \mathfrak{X}(S)^{\perp}$$

- A simple calculation provides

$$R^{\perp}(X,Y)N = ds(X,Y)\star^{\perp}N, \quad \forall X,Y \in \mathfrak{X}(S), \ \forall N \in \mathfrak{X}(S)^{\perp}$$

This justifies that s characterizes the normal connection and that ds defines its curvature.

Forms of the Gauss equation

For all $X, Y, Z, W \in \mathfrak{X}(S)$

$$R(W,Z,X,Y) = \overline{R}(W,Z,X,Y) + g\left(\mathrm{I\!I}(X,Z), \mathrm{I\!I}(Y,W)\right) - g\left(\mathrm{I\!I}(Y,Z), \mathrm{I\!I}(X,W)\right)$$

where I use the notation

$$R(W,Z,X,Y) \equiv g(W, R(X,Y)Z)$$

and analogously for $\overline{R}$.

$$\begin{aligned} \mathrm{Ric}(X,Y) + R(\ell,X,k,Y) + R(k,X,\ell,Y) &= \\ \overline{\mathrm{Ric}}(X,Y) + \mathbb{J}(X,Y) - g(\mathrm{I\!I}(X,Y),H) &= \\ \overline{\mathrm{Ric}}(X,Y) + \mathbb{J}(X,Y) - K_H(X,Y) & \end{aligned}$$

$$S(g) + 4\mathrm{Ric}(\ell,k) - 2R(\ell,k,\ell,k) = S(h) + \mathrm{tr}B - g(H,H)$$

For $n = 4$, S is 2-dimensional and $S(h)$ is uniquely determined by its Gaussian curvature $K(S) = S(h)/2$. The Gauss equation:

$$2\,K(S) = S(g) + 4\,\mathrm{Ric}(\ell,k) - 2R(\ell,k,\ell,k) + g(H,H) - \mathrm{tr}B$$

Forms of the Codazzi equation

For all $X, Y, Z \in \mathfrak{X}(S)$, for all $N \in \mathfrak{X}(S)^{\perp}$,

$$
\begin{aligned}
(R(X,Y)Z)^{\perp} &= \left(\overline{\nabla}_X + D_X\right) \mathrm{I\!I}(Y,Z) - \left(\overline{\nabla}_Y + D_Y\right) \mathrm{I\!I}(X,Z) \\
&= \big\{ -\overline{\nabla}_X K_\ell(Y,Z) - s(X) K_k(Y,Z) \\
&\quad + \overline{\nabla}_Y K_\ell(X,Z) + s(Y) K_k(X,Z) \big\}\, k \\
&\quad - \big\{ \overline{\nabla}_X K_k(Y,Z) - s(X) K_\ell(Y,Z) \\
&\quad - \overline{\nabla}_Y K_k(X,Z) + s(Y) K_\ell(X,Z) \big\}\, \ell
\end{aligned}
$$

$$
\begin{aligned}
(R(X,Y)N)^{T} = &-\overline{\nabla}_X \left(A_N(Y)\right) + \overline{\nabla}_Y \left(A_N(X)\right) \\
&- A_{D_Y N}(X) + A_{D_X N}(Y) + A_N\left([X,Y]\right)
\end{aligned}
$$

Forms of the Ricci equation

For all $X, Y \in \mathfrak{X}(S)$, for all $N, M \in \mathfrak{X}(S)^\perp$

$$\begin{aligned}(R(X,Y)N)^\perp &= \mathrm{I\!I}\left(X, A_N(Y)\right) - \mathrm{I\!I}\left(Y, A_N(X)\right) + R^\perp(X,Y)N \\ &= \mathrm{I\!I}\left(X, A_N(Y)\right) - \mathrm{I\!I}\left(Y, A_N(X)\right) + ds(X,Y) \star^\perp N\end{aligned}$$

$$\begin{aligned}R(M,N,X,Y) &= g\left(A_M\left(A_N(Y)\right), X\right) - g\left(A_M\left(A_N(X)\right), Y\right) \\ &\quad + ds(X,Y)\; g(\star^\perp N, M) \\ &= g\left(\left[A_M, A_N\right](Y), X\right) + ds(X,Y)\; g(\star^\perp N, M)\end{aligned}$$

The (null and normal) Raychaudhuri equation

Starting at each point $x \in S$ one can issue the unique geodesic with tangent vector $\ell|_x$ at x. This defines a congruence of null geodesics, orthogonal to a family of codimension two submanifolds (which include S) whose tangent vector field will still be denoted by ℓ. Then, one has **the Raychaudhuri equation**

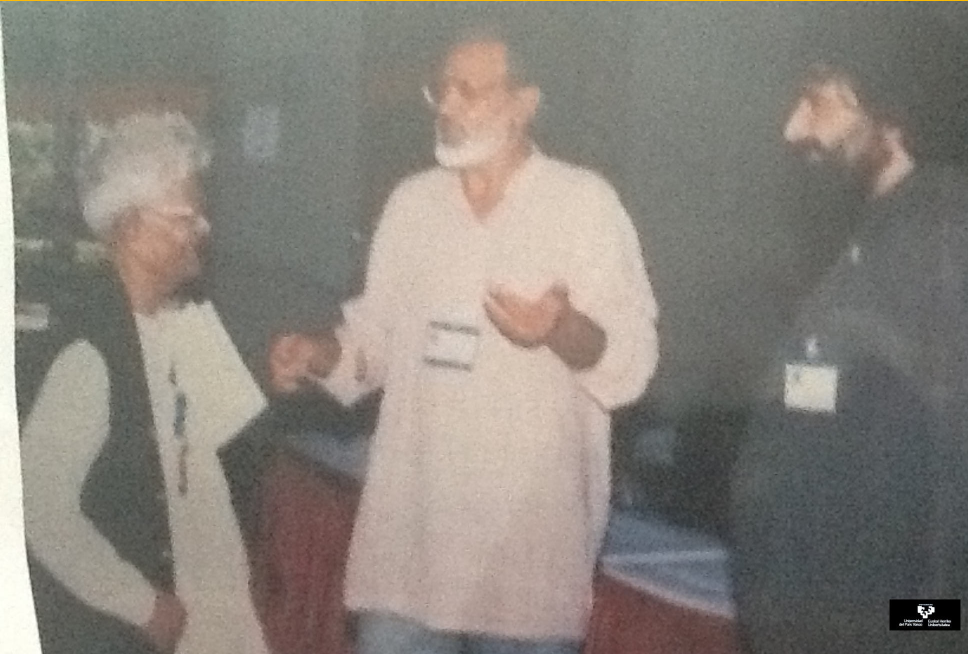
$$\boxed{\ell(\theta_\ell) = -\mathrm{tr} A_\ell^2 - \mathrm{Ric}(\ell, \ell)}$$

In the Physics literature this is usually written as

$$\ell(\theta_\ell) = -\sigma_\ell^2 - \frac{\theta_\ell^2}{n-2} - \mathrm{Ric}(\ell, \ell)$$

Notice: $\mathrm{tr} A_\ell^2 \geq 0$, so that if $\mathrm{Ric}(\ell, \ell) \geq 0$ too one knows that θ_ℓ never increases (usually decreases) along the congruence defined by ℓ. This is the basis of the geodesic focusing in General Relativity and is fundamental for the singularity theorems.

Raychaudhuri



Curvature and "energy" conditions

Definition (Convergence or energy conditions)

$(\mathcal{V}, g)$ is said to satisfy the null convergence condition (NCC) —also known as the null energy condition (NEC)— if

$$\mathit{Ric}(k,k) \geq 0 \qquad \forall k \in T\mathcal{V} \text{ such that } g(k,k) = 0.$$

And it satisfies the dominant energy condition (DEC) if

$$\mathrm{Ein}(v,w) \geq 0 \qquad \forall v,w \in T\mathcal{V}$$

such that $g(v,v) \leq 0$, $g(w,w) \leq 0$, and $g(v,w) \leq 0$.

Here

$$\mathsf{Ein} \equiv \mathsf{Ric} - \frac{1}{2} S(g) g$$

is the Einstein tensor of $(\mathcal{V}, g)$. Observe that Ein(k,k) =Ric(k,k) for all null $k \in T\mathcal{V}$.

Extrinsic classification of S

- A complete local classification of spacelike surfaces in $n = 4$ was put forward on (JMMS, CQG 24 (2007) 3091–3124), and its generalization to arbitrary dimensions was also discussed.
- The classification is algebraic and based, at each point, on the properties of the two *null* Weingarten operators A_ℓ and A_k.
- Each Weingarten operator is a self-adjoint matrix which can be readily classified algebraically according to the signs of their (real) eigenvalues. This produces 8 different types for each of A_ℓ and A_k, and therefore 64 types of points for generic spacelike surfaces.
- This number increases a lot with the dimension n
- The above was not enough and an extra parameter has to be associated to each point $x \in S$ taking into account the *relative orientation* of the two null Weingarten operators there.
- In $n = 4$ one can prove that the parameter is simply related to the commutator

$$[A_k, A_\ell]$$

Notation for the causal character of H

Fortunately, we are only interested on the so-called primary classification, which concerns the causal orientation of H exclusively (equivalently, the signs of the null expansions).
This part of the classification is independent of n.
A useful symbolic notation denotes the causal orientation of H as follows (future = upwards, null =45^o):

H	Causal orientation
$\downarrow$	past-pointing timelike
$\swarrow$ or $\searrow$	past-pointing null ($\propto \ell$ or k)
$\leftarrow$ or $\rightarrow$	spacelike
$\cdot$	vanishes
$\nearrow$ or $\nwarrow$	future-pointing null ($\propto \ell$ or k)
$\uparrow$	future-pointing timelike

The future-trapped fauna: H is future on S

Symbol	Expansions	Type of submanifold
$\cdot$	$\theta_\ell = \theta_k = 0$	stationary or minimal
$\uparrow$	$\theta_\ell < 0, \theta_k < 0$	f-trapped (TS)
$.\nearrow$	$\theta_\ell = 0, \theta_k \leq 0$	marginally f-trapped (MTS)
$\nwarrow.$	$\theta_\ell \leq 0, \theta_k = 0$	marginally f-trapped (MTS)
$\nwarrow\!\uparrow\!\nearrow$	$\theta_\ell \leq 0, \theta_k \leq 0$	weakly f-trapped (WTS)

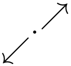
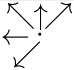
There are many other cases such as $\nwarrow\!\uparrow$, $\uparrow\!\nearrow$, $\nwarrow\!\uparrow\!\nearrow$, or $\nwarrow\!\uparrow$, etc., they deserve (and have) their own name, but are hardly considered in the literature.

(Observe also that there are many impossible cases, as H must be able to change continuously).

Other trapped-type submanifolds

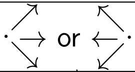
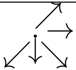
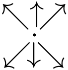
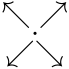
- Sometimes, only the sign of one of the expansions is relevant.
- This may happen if there is a consistent or intrinsic way of selecting a particular null normal on S.
- In the literature the preferred direction is usually selected —or declared!— to be "outer", and then the nomenclature speaks about "outer trapped" S, no matter whether or not this outer direction coincides with any particular outer or external part to the submanifold.
- Thus, (marginally) ℓ-trapped submanifolds are usually referred to as (marginally) outer trapped submanifolds ((M)OTS).
- The main possibilities are summarized in the following Table.

The "outer" trapped fauna

Symbol	Expansions	Type of submanifold
	$\theta_\ell < 0$	half converging, or ℓ-trapped (OTS)
	$\theta_\ell = 0$	null dual or marginally ℓ-trapped (MOTS)
	$\theta_\ell \leq 0$	weakly ℓ-trapped (WOTS)

Important studies concerning these submanifolds, and in particular MOTS, have been carried out recently with relevant results for Numerical Relativity, Mathematical Relativity, and black holes ($\longrightarrow$ Parte II)

Miscellaneous submanifolds

Symbol	Expansions	Type of submanifold
$\rightarrow$ or $\leftarrow$	$\theta_\ell \theta_k < 0$	untrapped
	$\theta_\ell \geq 0, \theta_k \leq 0$ or $\theta_\ell \leq 0, \theta_k \geq 0$	weakly untrapped
	$\theta_\ell > 0$	half diverging or ℓ-untrapped
	$\theta_\ell \geq 0$	weakly ℓ-untrapped
	$\theta_\ell \theta_k \geq 0$	$\ast$-submanifolds
	$\theta_\ell \theta_k = 0$	null $\ast$-submanifolds
	$\theta_\ell \theta_k = 0$ and $\theta_\ell \geq 0, \theta_k \leq 0$	null untrapped or GAH

Some comments of possible interest

- The last type of submanifold shown, Generalized Apparent Horizon, was proposed as a viable replacement for marginally trapped submanifolds in a new version of the Penrose inequality (Bray & Khuri, arXiv:0905.2622 (math.DG)).
- However, this version cannot hold as a recent counterexample has been found (Carrasco & Mars CQG 27 (2010) 062001).
- $\ast$-submanifolds are characterized by

$$H \wedge (\star^{\perp} H)(= g(H,H)\bar{\epsilon}/2) = -f^2 \bar{\epsilon}$$

- the null $\ast$-submanifolds are characterized simply by

$$H \wedge (\star^{\perp} H) = 0$$

Some examples: TSs

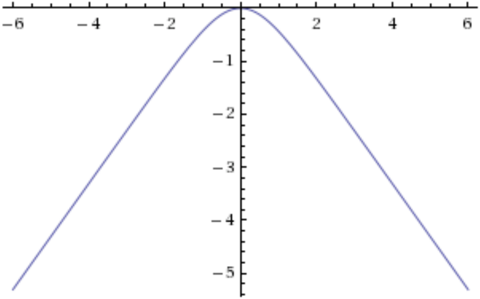
Trapped submanifolds are easily constructed in flat space-time

$$ds^2 = -dt^2 + dx_1^2 + \cdots + dx_{n-1}^2$$

One must simply let the surface "bend down" in time. An example is provided by

$$S : \{x_2 = const., \;\; e^{t_0 - t} = \cosh x_1\}.$$

These are non-compact. (They are also non-complete, but a complete example is given by $t_0 - t = \sqrt{k^2 + x_1^2}$ with $t_0 < k$.).

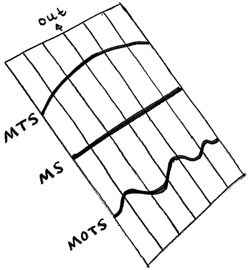


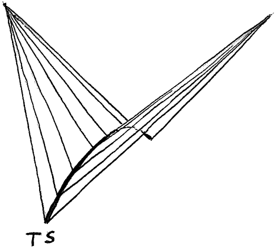
(z from −6 to 6)

Some examples: TSs, MTSs, MOTSs, MSs

Consider the simple case of 3-dimensional Minkowski space-time. Now, a codimension-two submanifold is simply a spacelike curve.

- It will be trapped if its normal vector is timelike.
- A MOTS is any spacelike curve in some null plane. It will be a MTS if the curve is concave.
- Observe that TSs and MTSs can never be closed.
- MOTS cannot be closed either. However, if the topology of space is changed to a cylinder, then there are closed MOTS in suitable null planes (but no MTS or TSs).





Spherically symmetric spacetimes

$$ds^2 = g_{ab}(x^c)dx^a dx^b + r^2(x^c)d\Omega^2$$

Here $d\Omega^2$ is the round metric on the $(n-2)$-sphere and $\det(g_{ab}) < 0$.
The mean curvature vector of any round sphere $x^b =$consts. reads

$$H^\flat = \frac{dr}{r}$$

Define the standard "mass function"

$$2m(x^a) \equiv r^{n-3}\left(1 - g(dr, dr)\right)$$

then these round $(n-2)$-spheres are (marginally) trapped if $2m/r^{n-3}$ is (equal to) less than 1.

Many examples

For many other interesting examples, do not miss I. Bengtsson's contribution (arXiv:1112.5318) to the book

"New Horizons" (S. Hayward, ed., World Scientific, in press)

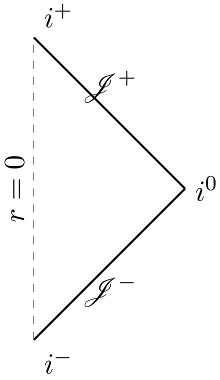
He is the author of the previous (and forthcoming) hand drawings.

One can also check my own chapter in that book.

MOTS versus MTS

- The closed trapped-type submanifolds are the only ones of relevance in Physics. They are usually formed in gravitational collapse and are the hallmark of the formation of Black Holes.
- From now on, we will concentrate on closed (=compact without boundary) submanifolds.
- The above shows a fundamental difference between (M)OTS and (M)TS, when they are closed.
- Actually, there is a general conjecture (Eardley, Phys. Rev. D 54 (1996) 4862) that the Event Horizon (EH) of black-hole spacetimes is the boundary of the region with OTSs. This was proven for some particular cases (Ben-Dov, Phys. Rev. D 75 (2007) 064007).
- Nevertheless, this cannot be the case for TS: the EH will rarely be the boundary of the region with TSs. And there is general proof of this fact (Bengtsson and JMMS, Phys. Rev. D 83 (2011) 044012).

Flat (Minkowski) spacetime conformal diagram



The event horizon

In a general situation when the asymptotic region is "Minkowskian" (called *asymptotically flat*), that is, with $\mathscr{I}^{\pm}$ and i^0, one can define the region from where $\mathscr{I}^{+}$ cannot be reached by any causal means.

The Event Horizon EH

The boundary of the past of $\mathscr{I}^{+}$.
By definition, this is always a null hypersurface.

Example: Schwarzschild's solution ($n = 4$) in "Eddington-Finkelstein" advanced coordinates

Schwarzchild (in units with $G = c = 1$)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2$$

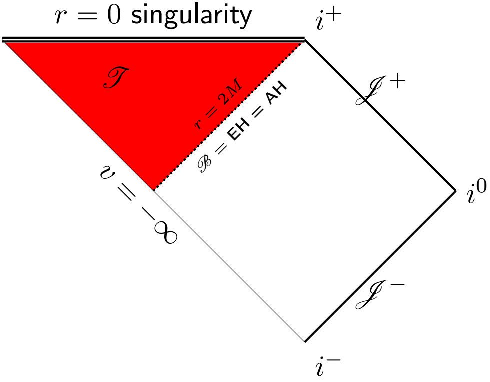
$m = M$ is constant: the total mass, and v is advanced (null) time.

From the above we know that the round spheres —defined by constant values of v and r— are trapped if and only if $r < 2M$. If $r = 2M$ they are "marginally trapped"

The Event Horizon, apparent 3-horizon AH and boundary $\mathscr{B}$

$$\text{AH} = \text{EH} = \mathscr{B} : \qquad r - 2M = 0$$

A Penrose, or conformal, diagram

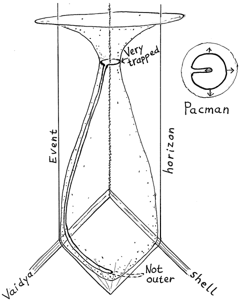


S in a spacelike hypersurface?

- In the recent literature there has been a general trend to consider the "outer" cases only, forgetting about the other null expansion.
- It is unclear whether this is just due to the difficulties in controlling the sign of the second null expansion, or it has some physical relevance. (Wait for Parte II, however).
- A possible relevant question related with these issues was raised in (Mars & JMMS, CQG 20 (2003) L293), as to whether or not there can be any closed $\star$-submanifolds in Minkowski spacetime.
- It is now known that there cannot be closed codimension-two $\star$-submanifolds imbedded in spacelike hypersurfaces of Minkowski spacetime (Khuri CQG 26 (2009) 078001).
- However, there are known examples of imbedded submanifolds in flat spacetime which cannot be imbedded in any spacelike hypersurface (Kossowski, Proc. AMS 117 (1993) 813–818).
- Thus the question remains alive...

OTS versus TS once again

- And this question, as well as the general problem of "outer" or not, has more relevance that one might think at first.
- This stems from the fact that the "outer" property is meaningful only if the submanifold is actually the boundary of some spacelike hypersurface.
- A dramatic example has been presented by Bengtsson (based on Ben-Dov's construction).



Spacetime "tubes" of closed M(O)TSs

The EH in Schwarzschild space-time is an example of a (null) hypersurface foliated by MTSs. In general, the hypersurface defined in spherical symmetry by AH $\equiv \{2m = r^{n-3}\}$ has that property (sometimes called the apparent $(n-1)$-horizon). This leads to

Definition (Marginally (outer) trapped tube, M(O)TT)

A M(O)TT is a hypersurface foliated by closed marginally (outer) f-trapped surfaces.
If the MTT is spacelike, it is called a dynamical horizon (DH).

In general EH is not an MTT.
It is known that in DHs the foliation by MTS is unique (Ashtekar & Galloway, Adv. Theor. Math. Phys. 9 (2005) 1).

However, MTTs are not unique!

Variation of volume/area and H

- The mean curvature vector is related to possible variations of volume (area) on S along normal directions.
- To be precise, let $\xi \in \mathfrak{X}(\mathcal{V})$ be defined on a neighbourhood of S, and let $\{\varphi_\tau\}_{\tau \in I \subset \mathbb{R}}$ be its flow. $S_\tau \equiv \varphi_\tau(S)$ is a family of submanifolds in $\mathcal{V}$, with corresponding imbeddings $\Phi_\tau : S \to \mathcal{V}$ given by $\Phi_\tau = \varphi_\tau \circ \Phi$. Observe that $S_0 = S$.
- Denoting by $\bar{\epsilon}_\tau$ their associated canonical volume element $(n-2)$-forms, it is a matter of simple calculation to get

$$\left.\frac{d\bar{\epsilon}_\tau}{d\tau}\right|_{\tau=0} = \frac{1}{2}\mathrm{tr}_h\left[\Phi^*(\pounds_\xi\, g)\right]\bar{\epsilon} = \Big(\delta(\Phi^*\xi^\flat) + g(\xi, H)\Big)\,\bar{\epsilon}$$

  where δ is the codifferential on S.
- $$\left.\frac{dV_\tau}{d\tau}\right|_{\tau=0} = \int_S \Big(\delta(\Phi^*\xi^\flat) + g(\xi, H)\Big)\,\bar{\epsilon}$$

  where $V_\tau = \int_{S_\tau} \bar{\epsilon}_\tau$ is the volume of S_τ.

Minimal submanifolds

Theorem (Minimal/Maximal submanifolds)

Among the set of all submanifolds without boundary (or with a fixed boundary under appropriate restrictions) those of extremal volume must have $H = 0$.

Observe that for the case of closed S the integral of $\delta(\Phi^* \xi^\flat)$ vanishes (and there is no loss of generality in assuming ξ of compact support).

It should also be noted that $\star^\perp H$ always defines a direction of no volume/area variation for arbitrary closed S.

Minimal hypersurfaces in Riemannian manifolds

Assume for a moment that S is a compact minimal hypersurface in a Riemannian manifold $(V, \tilde{g})$. Then the notion of **stability**, related to the second variation of volume/area, is fundamental.

Let m be the unit normal of S in V, and set $\psi \equiv \tilde{g}(\xi, m)$. A classical result informs us that, in this case,

$$\left.\frac{d^2 V_\tau}{d\tau^2}\right|_{\tau=0} = \oint_S \psi L_0 \psi \, \bar{\epsilon}$$

where L_0 is an elliptic operator on S given by

$$L_0 \psi = -\Delta_S \psi - \left(\mathrm{Ric}_{\tilde{g}}(m, m) + \mathrm{tr}(\kappa^2)\right)$$

Here, Δ_S is the Laplacian in S and κ is its Weingarten operator.

A similar operator will be defined for MOTS and its consequences will be of fundamental relevance.

Stability of minimal hypersurfaces

- The stability operator L_0 is self-adjoint with respect to the L^2 product $<,>$ with volume form $\bar{\epsilon}$:

$$\langle \psi, \phi \rangle = \oint_S \psi \phi \bar{\epsilon}$$

  and therefore its spectrum is real.
- Furthermore, the spectrum realizes its infimum λ_0, called **the principal eigenvalue**, from where one gets easily

$$\left. \frac{d^2 V_\tau}{d\tau^2} \right|_{\tau=0} = \oint_S \psi L_0 \psi \, \bar{\epsilon} \geq \lambda_0 \oint_S \psi^2 \bar{\epsilon}$$

- It follows that S is stable if and only if $\lambda_0 \geq 0$, and strictly stable if and only if $\lambda_0 > 0$.

An equivalent characterization of minimal stability

By noting that $L_0\psi = \frac{d\mathcal{H}_\tau}{d\tau}\big|_{\tau=0}$ where $\mathcal{H}_\tau$ denotes the mean curvature of S_τ in $(V, \tilde{g})$, one has the following equivalent characterization of (strict) stability :

Lemma

A minimal hypersurface S in a Riemannian background is (strictly) stable if and only if $\exists\psi \geq 0$ such that $\psi \not\equiv 0$ and $\frac{d\mathcal{H}_\tau}{d\tau}\big|_{\tau=0} \geq 0$ ($\frac{d\mathcal{H}_\tau}{d\tau}\big|_{\tau=0} > 0$).

Proof. Let $\phi_0 > 0$ such that $L\phi_0 = \lambda_0\phi_0$.
$\Longleftarrow$ If $\lambda_0 \geq 0$, then choose $\psi = \phi_0$.
$\Longrightarrow$ Conversely, if $\exists\psi \geq 0$ with $\frac{d\mathcal{H}_\tau}{d\tau}\big|_{\tau=0} \geq 0$, then

$$0 \leq \left\langle \phi_0, \left.\frac{d\mathcal{H}_\tau}{d\tau}\right|_{\tau=0} \right\rangle = \langle \phi_0, L_0\psi \rangle = \langle L_0\phi_0, \psi \rangle = \lambda_0 \langle \phi_0, \psi \rangle$$

and as $\psi \not\equiv 0$ implies that $\langle \phi_0, \psi \rangle > 0$ one obtains $\lambda_0 \geq 0$ (and $\lambda_0 > 0$ if $\frac{d\mathcal{H}_\tau}{d\tau}\big|_{\tau=0} > 0$).

Submanifolds and vector fields

Recall

$$\frac{1}{2}\mathrm{tr}_h\left[\Phi^*(\pounds_\xi\, g)\right] = \delta(\Phi^*\xi^\flat) + g(\xi, H)$$

1. If S is minimal, integrating the formula for closed S

$$\oint_S \mathrm{tr}_h\left[\Phi^*(\pounds_\xi\, g)\right]\bar{\epsilon} = 0\,.$$

   Observe that this relation must be satisfied for *all* $\xi \in \mathfrak{X}(\mathcal{V})$. Therefore, closed minimal submanifolds are very rare.

2. If ξ is a Killing vector, integrating again for closed S

$$\oint_S g(\xi, H)\bar{\epsilon} = 0\,.$$

   Therefore, if the Killing vector ξ is timelike on S, then S cannot be weakly trapped, unless it is minimal (M Mars and JMMS, CQG 20 (2003) L293).

Submanifolds and vector fields

Lemma

- *ξ future-pointing on $\mathscr{R} \subset \mathcal{V}$*
- *$S \subset \mathscr{R}$ a closed submanifold with $\mathrm{tr}_h\left[\Phi^*(\pounds_\xi\, g)\right] \geq 0$.*
  Then, S cannot be weakly f-trapped
  (unless $g(\xi, H) = 0$ and $\mathrm{tr}_h\left[\Phi^*(\pounds_\xi\, g)\right] = 0$.)

Proof.

Integrating for closed S

$$\oint_S g(\xi, H) = \frac{1}{2}\oint_S \mathrm{tr}_h\left[\Phi^*(\pounds_\xi\, g)\right] \geq 0$$

$\Longrightarrow$ H cannot be future pointing all over S
(unless $g(\xi, H) = \mathrm{tr}_h\left[\Phi^*(\pounds_\xi\, g)\right] = 0$.)

Integrable ξ

Assume that ξ is orthogonal to an integrable distribution

$$\xi^\flat \wedge d\xi^\flat = 0.$$

In other words, locally there exist functions F and t such that $\xi^\flat = -F dt$, hence t =const. is a family of hypersurfaces orthogonal to ξ (called the level hypersurfaces.)

Theorem (No minimum of t)

- *ξ future-pointing and hypersurface-orthogonal on $\mathscr{R} \subset \mathcal{V}$*
- *S a f-trapped surface*
- *Then, S cannot have a local minimum of t at any point $q \in \mathscr{R}$ where $\mathrm{tr}_h \left[\Phi^*(\pounds_\xi\, g)\right]|_q \geq 0$.*

Proof.

- Let $q \in S \cap \mathscr{R}$ be a point where S has a local extreme of t.
- Noting that $\Phi^* \xi^\flat = -\bar{F} d(\Phi^* t)$ with $\bar{F} \equiv \Phi^* F > 0$ (as ξ is future), we have $\Phi^* \xi^\flat |_q = 0$.
- An elementary calculation leads then to:

$$\delta \left(\Phi^* \xi^\flat \right) \Big|_q = \delta \left(-\bar{F} d(\Phi^* t) \right) \Big|_q = -\bar{F} \Delta_S (\Phi^* t) \Big|_q$$

- Introducing this in the main formula

$$\bar{F} \Delta_S (\Phi^* t) \Big|_q = -\frac{1}{2} \mathrm{tr}_h \left[\Phi^* (\pounds_\xi \, g) \right] + g(\xi, H) \Bigg|_q \leq g(\xi, H) \big|_q$$

- $\mathrm{Hess}(\Phi^* t)|_q$ cannot be positive (semi)-definite. ∎

Remarks

1. S does not need to be compact, nor contained in $\mathscr{R}$.
2. It is enough to assume $\text{tr}_h\left[\Phi^*(\pounds_\xi\, g)\right]|_q \geq 0$ only at the local extremes of t on S.
3. A positive semi-definite $\text{Hess}(\Phi^* t)|_q$ is also excluded.
4. The theorem holds true for WTSs with the only exception of

   $$\text{Hess}(\Phi^* t)\big|_q = 0 \quad \text{and} \quad \text{tr}_h\left[\Phi^*(\pounds_\xi\, g)\right]|_q = 0 \quad \text{and} \quad g(\xi, H)\big|_q = 0$$

   If $\xi|_q$ is timelike, the last of these implies that $H|_q = 0$.
5. Letting aside this exceptional possibility, t always decreases at least along one tangent direction in $T_q S$. Starting from any point $x \in S \cap \mathscr{R}$ one can always follow a connected path along $S \cap \mathscr{R}$ with decreasing t.

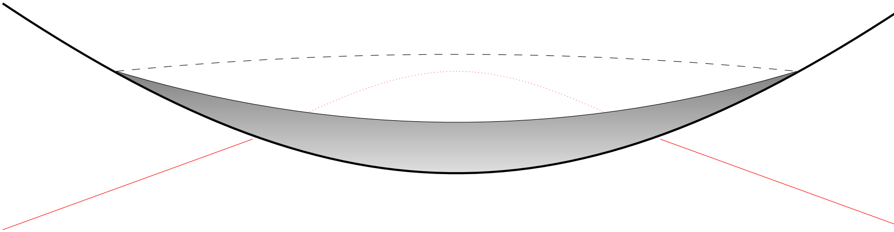
A fundamental property

Result (Bengtsson and JMMS, Phys. Rev. D 83 (2011) 044012)

No f-trapped surface (closed or not) can touch a spacelike hypersurface to its past at a single point if the latter has a positive semi-definite second fundamental form.

Proof. Exercise

The intuitive idea



MOTS and vector fields

Related results, for the case of MOTS, have been obtained (Carrasco & Mars CQG 25 (2008) 055011; *ibid.* 26 (2009) 175002)

Variation of the volume/area for MOTS

- As for the minimal case, MOTS are also critical points for the volume/area functional for variations along the the null direction with vanishing expansion. This follows from the general formula with $\xi = \ell$:

$$\left.\frac{d\bar{\epsilon}_\tau}{d\tau}\right|_{\tau=0} = \Big(\delta(\Phi^* \ell^\flat) + g(\ell, H)\Big)\,\bar{\epsilon} = 0$$

  together with the fact that $\ell \in \mathfrak{X}(S)^\perp$ and that $H \propto \ell$. Actually, this vanishes pointwise.
- Note, however, that the second variation is given by the Raychaudhuri equation

$$\left.\frac{d^2 V_\tau}{d\tau^2}\right|_{\tau=0} = -\int_S \left(\mathrm{tr} A_\ell^2 + \mathrm{Ric}(\ell, \ell)\right)\bar{\epsilon}$$

  so that if NCC holds, this is always non-positive (meaning that MOTS are unstable with respect to the volume/area functional). Observe that if one uses the boost freedom, the result is algebraic in e^β.

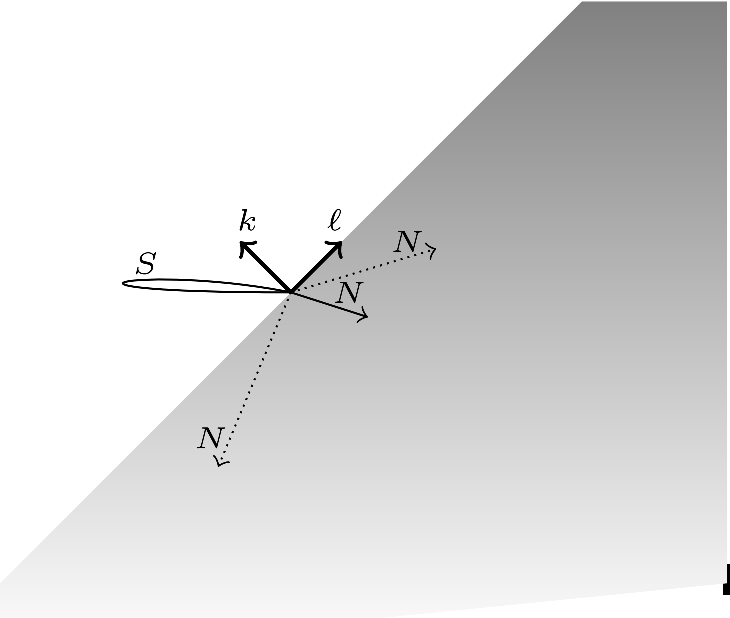
Variation of the vanishing expansion

- Nevertheless, as we saw the second variation for minimal hypersurfaces is directly related to the first variation of $\mathcal{H}$, so that one can ask whether a similar alternative can be pursued for MOTS: they are defined by $\theta_\ell = 0$, so what about the variation of θ_ℓ?
- This was first derived in (R.P.A.C. Newman, CQG 4 (1987) 277-290) , and later improved and computed in full generality in (Andersson-Mars-Simon , Adv. Theor. Math. Phys. 12 (2008) 853).
- A key point here is that S has codimension two hence, as the variations along $e^\beta \ell$ are algebraic, there will be just one differential operator ruling all variations.

The stability operator of Andersson-Mars-Simon

- Let S be a MOTS with $\theta_\ell = 0$, ergo $H = -\theta_k \ell$.
- Choose $N \in \mathfrak{X}(S)^\perp$ such that $g(N, \ell) = 1$, that is
$$N = -k + \frac{1}{2} g(N, N)\ell \,.$$
Observe that the causal orientation of N is unrestricted.
- (Warning: Mars usually uses v instead of N. This should not lead to any confusion)
- Let fN where $f \in C^\infty(S)$ be the variation vector field. Then, the variation $\delta_{fN}\theta_\ell \equiv d\theta_{\ell,\tau}/d\tau|_{\tau=0}$ is
$$\delta_{fN}\theta_\ell = -\Delta_S f + 2\overline{\nabla}_{s^\sharp} f$$
$$+ f\left(\frac{1}{2}S(h) + \delta s - h(s^\sharp, s^\sharp) - \frac{1}{2} g(N, N)\,\mathrm{tr} A_\ell^2 - \mathrm{Ein}(\ell, \star^\perp N)\right)$$
- This formula is valid for all possible normal directions *except* for ℓ itself. In that case the variation is given by the Raychaudhuri equation.

Scheme for the variation direction



Stability of MOTS

Definition (A-M-S stability of MOTS)

A MOTS S is said to be (strictly) stable along a non-timelike direction $N \in \mathfrak{X}(S)^{\perp}$ if there exists a non-identically-vanishing $f \geq 0$ such that $\delta_{fN}\theta_\ell \geq 0$ (and $\delta_{fN}\theta_\ell \not\equiv 0$).

It arises the question of whether one can characterize the stability by means of a principal eigenvalue. To that end

Definition (A-M-S stability operator for MOTS)

The stability operator L_N along a normal direction $N \in \mathfrak{X}(S)^{\perp}$ for a MOTS S is defined by $\delta_{fN}\theta_\ell \equiv L_N f$, that is

$$L_N f = -\Delta_S f + 2\overline{\nabla}_{s^\sharp} f$$
$$+f\left(\frac{1}{2}S(h) + \delta s - h(s^\sharp, s^\sharp) - \frac{1}{2}g(N,N)\,\mathrm{tr}A_\ell^2 - \mathrm{Ein}(\ell, \star^{\perp} N)\right)$$

The principal eigenvalue

- A difficulty is that L_N is not self-adjoint with respect to *any* sesquilinear product on $C^\infty(S)$ in general.
- This is due to the presence of the term $\overline{\nabla}_{s^\sharp} f$. The cases where such a product exists are given by an exact one-form s. (Recall that by using the boost freedom s is defined up to the addition of an exact one-form $d\beta$).
- Nevertheless, one can prove using the results in (Donsker & Varadhan Commun. Pure Appl. Math. 29 (1976) 591Ð621; Berestycki, Nirenberg & Varadhan, C.R. Acad. Sci. Paris 317 Série I (1993) 51Ð56; Andersson-Mars-Simon 2008)
  1. for each N, L_N has a **real principal eigenvalue** λ_N such that any other (complex) eigenvalue λ satisfies Re$(\lambda) > \lambda_N$.
  2. The principal eigenfunction ϕ_N is unique (up to a constant factor) and it does not vanish on S: thus, it can be chosen to be strictly positive.
  3. The formal adjoint operator $L_N^\dagger \equiv L_N - 4\overline{\nabla}_{s^\sharp} - 2\delta s$ with respect to the L^2 product in (S, h) has the same principal eigenvalue λ_N.

Stability in terms of λ_N

The above is enough to prove, along the same lines as in the case of minimal hypersurfaces, that

Result (Andersson-Mars-Simon 2008)

A MOTS S is (strictly) stable along a non-timelike normal direction $N \in \mathfrak{X}(S)^\perp$ if and only if the principal eigenvalue of L_N is non-negative (positive).

Result (e.g. Jaramillo, Reiris & Dain, Phys. rev. D 84 (2011) 121503)

If the NCC holds, and if a MOTS S is not stable along the null direction $-k$, then it cannot be stable along any spacelike normal direction $N \in \mathfrak{X}(S)^\perp$.

Proof. It is easily seen that $\delta_{fN}\theta_\ell = \delta_{-fk}\theta_\ell + g(N,N)\delta_{f\ell}\theta_\ell/2$ so that if $\exists f \geq 0$ such that $\delta_{fN}\theta_\ell \geq 0$ with $g(N,N) > 0$, and as $\delta_{f\ell}\theta_\ell \leq 0$, one gets $\delta_{-fk}\theta_\ell \geq 0$.

Spherically symmetric spacetimes $n = 4$

In advanced coordinates

$$ds^2 = -e^{2\alpha}\left(1 - \frac{2m(v,r)}{r}\right)dv^2 + 2e^{\alpha}dvdr + r^2 d\Omega^2$$

- For each round sphere $S \equiv \{u, v\}$ =consts., the future null normals are

$$k = -e^{-\alpha}\partial_r, \qquad \ell = \partial_v + \frac{1}{2}\left(1 - \frac{2m}{r}\right)e^{\alpha}\partial_r$$

- Their mean curvature vector H_{sph}:

$$H_{sph} = \frac{2}{r}\left(e^{-\alpha}\partial_v + \left(1 - \frac{2m}{r}\right)\partial_r\right).$$

- The null expansions:

$$\theta_\ell^{sph} = \frac{e^{\alpha}}{r}\left(1 - \frac{2m}{r}\right), \qquad \theta_k^{sph} = -\frac{2e^{-\alpha}}{r}.$$

Some properties of AH

- Recall: AH : $\quad r - 2m(r,v) = 0 \quad (\Leftrightarrow \theta_\ell^{sph} = 0)$
- One can prove (Bengtsson & JMMS 2011) that AH is actually the only spherically symmetric MTT : the only hypersurface foliated by MTSs —be they round spheres or not.
- AH can be timelike, null or spacelike depending on the sign of

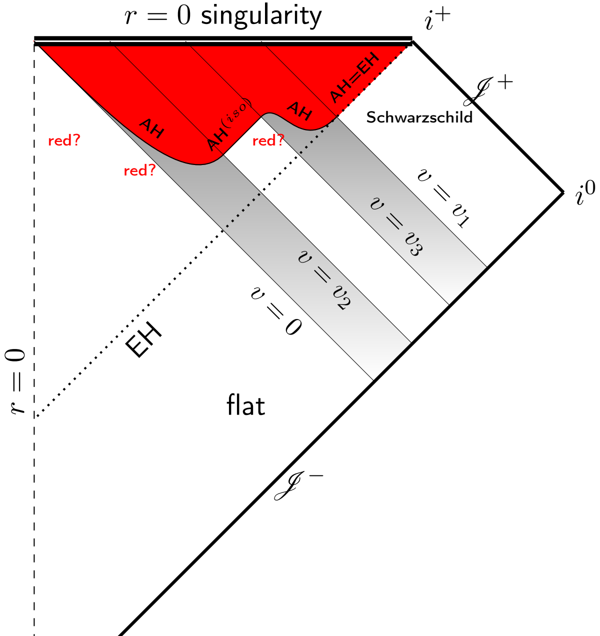
$$\frac{\partial m}{\partial v}\left(1 - 2\frac{\partial m}{\partial r}\right)\Bigg|_{AH}$$

- In particular, AH is null (in fact it is an *isolated horizon*) on any open region where $m = m(r)$. This isolated horizon portion of AH, if non-empty, is characterized also by:

$$\text{AH}^{(iso)} \equiv \text{AH} \cap \{\text{Ein}(\ell,\ell) = 0\}$$

- Recall that $\text{Ein}(\ell,\ell) = \text{Ric}(\ell,\ell)$.

A dynamical situation with non-empty AH$^{iso}\backslash$EH



The Kodama vector field (H Kodama, Prog. Theor. Phys. 63 (1980) 1217)

Recall that $\star^{\perp} H_{sph}$ is a direction of no area-variation for the round spheres. In the space-time this defines a vector field

$$\star^{\perp} H_{sph} \equiv \xi = e^{-\alpha} \partial_v$$

- ξ is hypersurface orthogonal, with the level function t defined by

$$\xi^{\flat} = -F dt = dr - e^{\alpha} \left(1 - \frac{2m(v,r)}{r}\right) dv$$

- Its norm is

$$g(\xi, \xi) = -\left(1 - \frac{2m(v,r)}{r}\right),$$

  so that ξ is future-pointing timelike on the region $\{r > 2m\}$, and future-pointing null at AH $= \{r = 2m\}$.

ξ has the necessary properties

- Let $q \in S \cap \{r \geq 2m\}$ be a point where S has a local extreme of t, or belonging to an open portion of $S \cap \{t = t_0\}$ for some constant t_0.
- it is enough to show that on any such point

$$\mathrm{tr}_h[\Phi^*(\pounds_\xi g)]\big|_q \geq 0.$$

- The Lie derivative can be easily computed

$$\pounds_\xi g = e^\alpha \frac{2}{r}\frac{\partial m}{\partial v}\ell \otimes \ell - \frac{\partial \alpha}{\partial r}\left(dr \otimes \xi^\flat + \xi^\flat \otimes dr\right)$$

- Then, given that $\Phi^* \xi^\flat|_q = 0$ we obtain

$$\mathrm{tr}_h[\Phi^*(\pounds_\xi g)]\big|_q = e^\alpha \frac{2}{r}\frac{\partial m}{\partial v}\; h(\Phi^*\ell^\flat, \Phi^*\ell^\flat)\bigg|_q \geq 0$$

provided that $\partial m/\partial v\,(q) \geq 0$.

ξ restricts the location of TSs

It follows that:

Theorem (The location of TSs in spherical symmetry)

1. *No closed (future) TS can be fully contained in the region $\{r \geq 2m\}$ (where ξ is future pointing).*
2. *All closed (future) WTSs must intersect the region $\{r \leq 2m\}$.*
3. *All closed (future) TSs must intersect the region $\{r < 2m\}$ (where ξ is spacelike).*
4. *All closed (future) WTSs other than the MTS round spheres with $r = 2m$ must intersect the region $\{r < 2m\}$.*

The stability operator at work

- Let $\varsigma \subset$ AH be any MT round sphere with $r = r_\varsigma =$const.
- The variation along a normal direction fN simplifies drastically in this case, given the marginal character of ς and its spherical symmetry. In fact it is straightforward to check that $A_\ell = 0$ (i.e., shear-free too) and that $s = 0$. In other words, most of the terms in the variation formula vanish and the variation of the zero null expansion is given by

$$\delta_{fN}\theta_\ell^{sph} = -\Delta_\varsigma f + f\left(\frac{1}{r_\varsigma^2} - \mathsf{Ein}(\ell, \star^\perp N)\right) \tag{4}$$

- Recall that $\star^\perp N$ is the following vector field orthogonal to ς and N:

$$\star^\perp N = k + \frac{1}{2}g(N,N)\ell,$$

so that $\mathsf{Ein}(\ell, \star^\perp N) = \mathsf{Ein}(\ell, k) + \frac{1}{2}g(N,N)\mathsf{Ein}(\ell,\ell)$.

- **Remark**: selecting f =constant (4) informs us that the vector N such that $1/r_\varsigma^2 - \mathsf{Ein}(\ell, \star^\perp N) = 0$ produces no variation on θ_ℓ^{sph}, meaning that N is tangent to the AH simply leading to other MT round spheres on AH.
- Let us call such a vector field M, so that

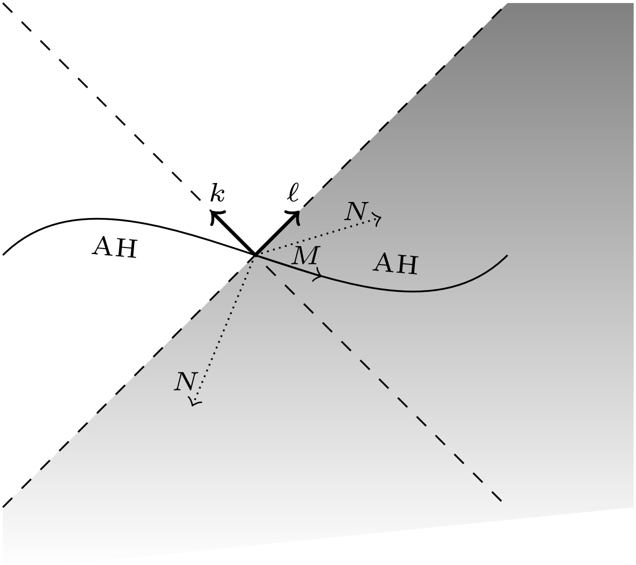
$$\frac{1}{r_\varsigma^2} - \mathsf{Ein}(\ell, k) - \frac{1}{2}\, g(M, M)\mathsf{Ein}(\ell, \ell)\bigg|_\varsigma = 0 \qquad (5)$$

together with

$$M = -\,\ell + \frac{1}{2} g(M, M) k\bigg|_\varsigma$$

characterizes AH$\setminus$AH^{iso}, since M is the unique spherically symmetric direction tangent to it.
- The exceptional isolated-horizon portion AH^{iso} has the null ℓ as its tangent vector field.

A helpful picture



Perturbations on AH^{iso}

- Assume that $\text{Ein}(\ell, \ell) = 0$ holds on a region so that we are dealing with AH^{iso}. This can be seen to be equivalent to the condition $\partial m/\partial v = 0$. From the variation formula (4) we deduce that

$$\delta_{fN}\theta_\ell^{sph} = -\Delta_\varsigma f + f\left(\frac{1}{r_\varsigma^2} - \text{Ein}(\ell, k)|_\varsigma\right)$$

  so that the perturbed expansion is independent of the direction of deformation N.

- One can check that $\text{Ein}(\ell, k)|_\varsigma = (2/r_\varsigma^2)\partial m/\partial r|_\varsigma$ and the previous relation can be rewritten as

$$\Delta_\varsigma f - f\frac{1}{r_\varsigma^2}\left(1 - 2\left.\frac{\partial m}{\partial r}\right|_\varsigma\right) = -\delta_{fN}\theta_\ell^{sph}\,. \qquad (6)$$

- Notice that the term in round brackets will generally be positive —for instance if $m =$ const. or if AH^{iso} is related to an asymptotically flat end, because then $r - 2m(r)$ changes from negative to positive values.

Perturbations on AH^{iso}

- Eq.(6) can be seen as an equation $\mathcal{L}f = -\delta_{fN}\theta^{+}$ where $\mathcal{L} = \Delta_{\varsigma} - (1/r_{\varsigma}^{2})(1 - 2\partial m/\partial r|_{\varsigma})$ is an elliptic operator on ς, and thus it is adapted for direct application of the maximum principle. In particular, if $\delta_{fN}\theta_{\ell}^{sph}$ is non-positive everywhere it follows that f must be negative everywhere on ς.
- Combining this with the known fact that arbitrary perturbations along the null generator ℓ of the isolated horizon AH^{iso} produce MTS we obtain the following theorem.

Theorem

On any isolated-horizon portion AH^{iso} of AH arbitrary deformations of its round spheres lead to MTS. Moreover, if AH^{iso} is such that $1 \geq 2\partial m/\partial r$, any other possible perturbation leading to WTS has $f < 0$, so that the deformed surfaces lie strictly outside the region $\{r > 2m\}$.

Deformations on AH$\setminus$AH^{iso}

- Consider now the parts of AH with Ein$(\ell,\ell) =$ Ric$(\ell,\ell) > 0$. From the **helpful figure** we deduce that the perturbation along fN will enter into the region with f-trapped round spheres at points with

$$f(g(N,N) - g(M,M)) > 0.$$

- For easy control of these signs we note that

$$\text{Ein}(\ell,\ell) f(g(N,N) - g(M,M)) = -2(\Delta_\varsigma f + \delta_{fN}\theta_\ell^{sph}). \quad (7)$$

- An interesting conclusion arises by integrating on ς

$$\text{Ein}(\ell,\ell) \oint_\varsigma f(g(N,N) - g(M,M)) = -2 \oint_\varsigma \delta_{fN}\theta_\ell^{sph}$$

from where we deduce the following facts:

$$\text{Ein}(\ell,\ell)\oint_{\varsigma} f(g(N,N)-g(M,M)) = -2\oint_{\varsigma}\delta_{fN}\theta_{\ell}^{sph}$$

- the deformed surface can be f-trapped only if $f(g(N,N)-g(M,M))$ is positive somewhere. Hence, a f-trapped surface (obtained in this way) must lie at least partially in the region $\{r<2m\}$. This, of course, complies with the general result shown above.
- and it can be untrapped only if $f(g(N,N)-g(M,M)$ is somewhere negative.
- if the deformed surface lies entirely within $\{r>2m\}$ —so that $f(g(N,N)-g(M,M))<0$ everywhere —, then $\delta_{fN}\theta_{\ell}^{sph}$ must be positive somewhere.
- if the deformed surface lies entirely in $\{r<2m\}$, then $\delta_{fN}\theta_{\ell}^{sph}$ must be negative somewhere.

TSs entering $\{r > 2m\}$

- In order to construct examples of TSs which lie partly in $\{r > 2m\}$, let us consider perturbations such that

$$g(N,N) - g(M,M) > 0.$$

- For this choice the deformed surface enters the region $\{r < 2m\}$ at points with $f > 0$. We introduce a constant a_0 and aim for f-trapped surfaces for which

$$\mathrm{Ein}(\ell,\ell)\, a_0(g(N,N) - g(M,M)) + 2\delta_{fN}\theta_{\ell}^{sph} = 0.$$

  By our assumptions this implies that $\delta_{fN}\theta_{\ell}^{sph} < 0$ if $a_0 > 0$, so that the deformed surface will be f-trapped.

- Now set $f \equiv a_0 + \tilde{f}$ for some as yet undetermined function $\tilde{f}$. Equation (7) becomes

$$\mathrm{Ein}(\ell,\ell)\tilde{f}(g(N,N) - g(M,M)) + 2\Delta_{\varsigma}\tilde{f} = 0. \qquad (8)$$

TSs entering $\{r > 2m\}$

- We conclude that our aim (with our assumptions) requires that

$$\frac{1}{2}\mathsf{Ein}(\ell,\ell)\left(g(N,N) - g(M,M)\right) = -\frac{\Delta_\varsigma \tilde{f}}{\tilde{f}} > 0. \qquad (9)$$

- This is a (mild) restriction on the function $\tilde{f}$. A simple solution is to choose $\tilde{f}$ to be an eigenfunction of the Laplacian Δ_ς, say

$$\tilde{f} = c_l P_l$$

for a fixed $l \in \mathbb{N}$ and constant c_l ($P_l =$ Legendre polynomials).

- Then, on using $\Delta_\varsigma P_l = -\frac{l(l+1)}{r_\varsigma^2} P_l$ the deformation direction N is determined by

$$g(N,N) - g(M,M) = \frac{2}{\mathsf{Ein}(\ell,\ell)} \frac{l(l+1)}{r_\varsigma^2} > 0$$

and the variation of the expansion then reads

$$\delta_{fN}\theta_\ell^{sph} = -a_0 \frac{l(l+1)}{r_\varsigma^2} < 0 \qquad (\text{for } a_0 > 0)$$

- As the other expansion θ_k was initially negative, by choosing a very small deformation we can always achieve that the deformed ς is f-trapped.
- It only remains to check that f realizes all signs, so that the deformed surface criss-crosses AH. Given that

$$f = a_0 + c_l P_l$$

  it is enough to adjust the constant c_l to achieve this goal. For instance, the choice $c_l < -a_0 < 0$ will do, so that f has the sign of a_0 at the region where $P_l \leq 0$, and the opposite sign around the north pole of ς where $P_l > 0$.
- Thus, we have proven the following theorem.

Theorem

In arbitrary spherically symmetric spacetimes there are closed f-trapped surfaces (topological spheres) penetrating both sides of AH *at any region where* $\mathrm{Ein}(\ell, \ell)|_{AH} > 0$*.*

MTTs are not unique

- We remark that the previous reasoning is independent of the causal character of AH, which can be spacelike, null or timelike. The only restriction is that Ein$(\ell, \ell) > 0$.
- The non-uniqueness of dynamical horizons and MTTs can be addressed now. The perturbation argument tells us that there are f-trapped surfaces penetrating into both sides of AH$\setminus$AH^{iso}.
- We also know that there are untrapped round spheres lying just outside it.
- If AH is spacelike this means that we can find a spacelike hypersurface having such an outer trapped sphere as its inner boundary and an untrapped round sphere as its outer boundary, and such that it contains a path connecting the boundaries and lying entirely outside AH (that is, inside $\{r > 2m\}$).

Many MTTs weaving each other

- There is a theorem (L. Andersson & J. Metzger, Commun. Math. Phys. 290 (2009) 941 $\longrightarrow$ Parte II) that ensures that a spacelike hypersurface with such boundaries necessarily contains a MOTS.
- By construction such a MOTS has a part lying inside $\{r > 2m\}$, and we know that it must penetrate inside $\{r < 2m\}$.
- Moreover, generically such a MOTS 'evolves' into a MOTT (L. Andersson, M. Mars & W. Simon 2005 and 2008 $\longrightarrow$ Parte II).
- As long as we stay sufficiently close to AH all the MOTS in the argument will be inner trapped as well. Thus we have obtained:

Corollary

In arbitrary spherically symmetric spacetimes there are MTTs penetrating both sides of the spherical MTT AH at any region where Ein$(\ell,\ell)|_{AH} > 0$.

How much must a TS lie inside $\{r < 2m\}$?

- As another application of the AMS stability operator, we wonder how small the fraction of any closed f-trapped surface that extends outside $\{r < 2m\}$ can be made.
- With the assumptions above this means that we must produce a C^2 function $\tilde{f}$ defined on the sphere and
  1. obeying the inequality (9),
  2. positive only in a region that we can make arbitrarily small.
- If we choose a sufficiently small constant a_0 the last requirement implies that the region where the surface extends outside $\{r > 2m\}$ can be made arbitrarily small.
- To find such a function it is convenient to introduce stereographic coordinates $\{\rho, \varphi\}$ on the sphere, so that the Laplacian takes the form

$$\Delta_\varsigma = \Omega^{-1}\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\varphi^2\right) , \qquad \Omega = \frac{4r_\varsigma^2}{(1+\rho^2)^2}.$$

An explicit solution to the problem

- A solution to the problem as stated is the axially symmetric function

$$\tilde{f}(\rho) = \begin{cases} c_1 \left(e^{\frac{1}{2a}(2a-\rho^2)} - 1 \right) & \rho^2 < 4a \\ \\ \frac{8c_1 a}{e} \frac{1}{\rho^2} - c_1(1 + e^{-1}) & \rho^2 > 4a \ . \end{cases} \tag{10}$$

- This function is C^2 (and can be further smoothed if necessary), and it is positive only if $\rho^2 < 2a$, that is on a disk surrounding the origin (the pole) whose size can be chosen at will.
- The function obeys

$$-\frac{\Delta_\varsigma \tilde{f}}{\tilde{f}} = \begin{cases} \frac{\Omega^{-1}}{a^2} \frac{2a - x^2}{1 - e^{-\frac{1}{2a}(2c-\rho^2)}} & \rho^2 < 4a \\ \\ \frac{32a\Omega^{-1}}{\rho^4} \frac{\rho^2}{(e+1)\rho^2 - 8a} \ , & \rho^2 > 4a \ . \end{cases}$$

- This is always larger than zero.

A surprising theorem

Thus we have proven the following important result.

Theorem (Bengtsson & JMMS 2001)

In spherically symmetric spacetimes, there are closed f-trapped surfaces (topological spheres) penetrating both sides of the apparent 3-horizon $AH \setminus AH^{iso}$ with arbitrarily small portions outside the region $\{r > 2m\}$.

The future-trapped region $\mathscr{T}$ and its boundary $\mathscr{B}$

The future-trapped region $\mathscr{T}$

is defined as the set of points $x \in \mathcal{V}$ such that x lies on a closed (future) TS.

This is a space-time concept, not to be confused with the trapped region within spacelike hypersurfaces, to be discussed in Parte II, which is defined as the union of the interiors of all MOTS in the given hypersurface.

The boundary $\mathscr{B}$

We denote by $\mathscr{B}$ the boundary of the future trapped region $\mathscr{T}$:

$$\mathscr{B} \equiv \partial \mathscr{T}$$

One of the mysteries concerning closed TSs is: where is $\mathscr{B}$? But this is another story....

$\mathscr{T}$ and $\mathscr{B}$ are invariant by the isometry group

Result

If G is the group of isometries of the spacetime $(\mathcal{V}, g)$, then $\mathscr{T}$ is invariant under the action of G, and the transitivity surfaces of G, relative to points of $\mathscr{B}$, remain in $\mathscr{B}$.

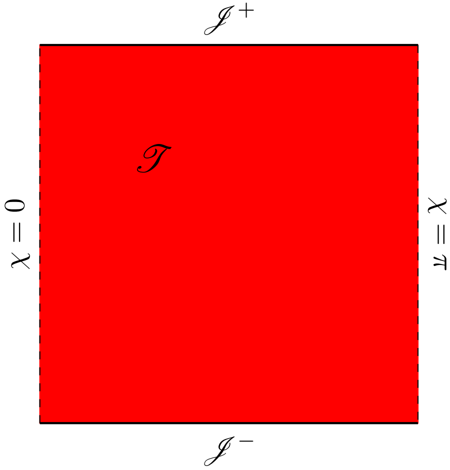
Result ($\mathscr{T}$ and $\mathscr{B}$ in spherical symmetry)

In arbitrary spherically symmetric spacetimes, $\mathscr{T}$ and $\mathscr{B}$ have spherical symmetry.

Result

In arbitrary spherically symmetric spacetimes, $\mathscr{B}$ (if not empty) is a spherically symmetric hypersurface without boundary.

Simple example: de Sitter spacetime. $\mathscr{B} = \emptyset$



Non-locality and clairvoyance of TSs

- Closed TSs are clairvoyant , highly non-local objects. They cross MTTs and even enter flat portions of the space-time.
- In conjunction with the non-uniqueness of MTTs, this poses a fundamental puzzle for the physics of black holes.
- Although several solutions can be pursued, the most natural and popular one is trying to define a preferred dynamical horizon or MTT. Hitherto, though, there has been no good definition for that.
- We have put forward a novel strategy. The idea is based on the simple question:
  what part of the spacetime is absolutely indispensable for the existence of the black hole?
- Surely enough, any flat region is certainly not essential for the existence of the black hole.
- What is?

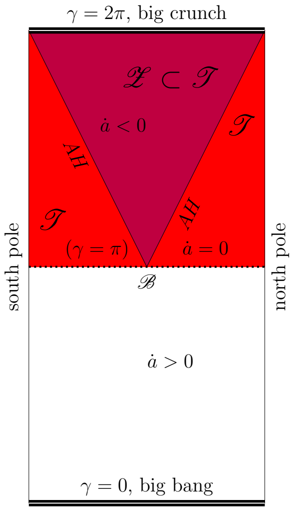
The Core of the trapped region

Definition

A region $\mathscr{Z}$ is called the core of the f-trapped region $\mathscr{T}$ if it is a minimal closed connected set that needs to be removed from the spacetime in order to get rid of all closed f-trapped surfaces in $\mathscr{T}$, and such that any point on the boundary $\partial\mathscr{Z}$ is connected to $\mathscr{B} = \partial\mathscr{T}$ in the closure of the remainder.

- Here, "minimal" means that there is no other set $\mathscr{Z}'$ with the same properties and properly contained in $\mathscr{Z}$.
- The final technical condition states that the excised space-time $(\mathcal{V}\backslash\mathscr{Z}, g)$ has the property that $\forall x \in \mathcal{V}\backslash\mathscr{Z} \cup \partial\mathscr{Z}$ there is continuous curve $\gamma \subset \mathcal{V}\backslash\mathscr{Z} \cup \partial\mathscr{Z}$ joining x and $\mathscr{B}$ (γ can have zero length if $\mathscr{B} \cap \partial\mathscr{Z} \neq \emptyset$).
  This is needed because one could identify a particular removable region, excise it, but then put back a tiny isolated portion to make it smaller. However, this is not what one wants to cover with the definition.

$\mathscr{Z} \subset \mathscr{T}$. Example: RW, $p = 0$ and $\Lambda = 0$.



Cores are not unique.

- This example also proves that $\mathscr{Z}$ is not unique: one can choose any other region $\mathscr{Z}$ equivalent to the chosen one by moving all its points by the group of symmetries on each homogeneous slice.
- Actually this kind of non-uniqueness is rather trivial, and is due to the existence of a high degree of symmetry.
- Nevertheless, even in less symmetric cases the uniqueness of the cores $\mathscr{Z}$ cannot be assumed beforehand. Actually, we have proven that it does not hold in general (see below).

Cores in spherical symmetry

Result (Bengtsson & JMMS , 2011)

The region $\mathscr{Z} \equiv \{r \leq 2m\}$ *is a core.*

Proof.

- From the result using the Kodama ξ every closed TS must intersect $\mathscr{Z}$: removing it all possible TSs are short-circuited.
- Is it minimal? Observe that $\partial\mathscr{Z} = \text{AH} = \{r = 2m\}$. Take any $\mathscr{Z}' \subset \mathscr{Z}$ such that from every $x \in \partial\mathscr{Z}'$ there is a curve to the nearest part of the boundary $\mathscr{B}$: all these curves must therefore cross AH. Thus, $\mathscr{Z} \backslash \mathscr{Z}'$ always contains an open region around AH with $\{r < 2m\}$.
- But then, the surprising theorem ensures that there are closed TSs fully contained in $\mathcal{V} \backslash \mathscr{Z}'$, so that $\mathscr{Z}'$ cannot be a core.

Corollary (AH is the boundary of a core)

The spherical MTTs given by AH $= \{r = 2m\}$ *are boundaries of a core of* $\mathscr{T}$ *.*

$\mathscr{Z} = \{r \leq 2m\}$ are unique respecting the symmetry

Result

In spherically symmetric spacetimes, $\mathscr{Z} = \{r \leq 2m\}$ are the only spherically symmetric cores of $\mathscr{T}$. Therefore, $\partial\mathscr{Z} = AH$ are the only spherically symmetric boundaries of a core.

Proof. Suppose there were another spherically symmetric core $\mathscr{Z}'$. Obviously $\mathscr{Z}'$ could not be a proper subset of $\mathscr{Z}$, nor vice versa, because both are cores. Thus $\mathscr{Z} \backslash \mathscr{Z}' \neq \emptyset$ and this set would be spherically symmetric. However, every round sphere in $\mathscr{Z}$ is f-trapped, and therefore there would be f-trapped round spheres having no intersection with $\mathscr{Z}'$, contradicting the hypothesis that $\mathscr{Z}'$ was a core of the trapped region.

Non-spherically symmetric cores

Proposition

There exist non-spherically-symmetric cores of the f-trapped region in spherically symmetric spacetimes.

Proof. For simplicity, consider the case with a spacelike AH. From previous results we know that there are non-spherically symmetric MTTs interweaving AH. Choose any of these, say T, so that T lies partly to the future of AH (and partly to its past). From a theorem in (Ashtekar & Galloway, 2005) no WTS can be fully contained in the past domain of dependence of T. Thus, removing $J^+(T)$ from $\mathcal{V}$ eliminates all closed TSs, and there must be a subset of $J^+(T)$ which is a core of $\mathscr{T}$. This new core will never include those parts of the spacetime which are to the future of AH but to the past of h. Thus, this core is not $\mathscr{Z}$, and due to the previous Result, it cannot be spherically symmetric.

Is there anything special about $\mathscr{Z}:\{r\leq 2m\}$?

- Still, the identified core $\mathscr{Z}=\{r\leq 2m\}$ might be unique in the sense that its boundary $\partial\mathscr{Z}=$ AH is an MTT.
- This would happen, for instance, if any dynamical horizon T other than AH is such that its causal future $J^+(T)$ is *not* a core —the core being a proper subset of $J^+(T)$.
- Then AH would be selected as the unique MTT which is the boundary of a core of the f-trapped region $\mathscr{T}$.
- Whether or not this happens is a very interesting open question.
- It should be observed that the concept of core is global, and requires full knowledge of the future. However, AH is local and can be defined and identified by observing just around it. How can then AH $=\partial\mathscr{Z}$?

Back to the general case

- The question arises of whether the previous results for spherically symmetric spacetimes can be translated to the general case.
- To start with, the existence of a MOTT through any given strictly stable MOTS was proven in (Andersson-Mars-Simon 2005).
- If NCC holds, then the MOTT is non-timelike.
- The MOTT is constructed in the mentioned article according to a given reference foliation by spacelike hypersurfaces, and it depends on the foliation.
- Thus, there are many MOTTs which contain any given strictly stable MOTS.
- I am going to consider the properties of these many MOTTs from another viewpoint, using the stability operators in different normal directions (and non-necessarily spacelike).

Characterization of the many MOTTs through a given MOTS

- Recall the stability operator:

$$L_N f = -\Delta_S f + 2\overline{\nabla}_{s^\sharp} f$$
$$+f\left(\frac{1}{2}S(h) + \delta s - h(s^\sharp, s^\sharp) - \frac{1}{2}g(N,N)\,\mathrm{tr}A_\ell^2 - \mathrm{Ein}(\ell, \star^\perp N)\right)$$

- Recall also: $\star^\perp N = k + \frac{1}{2}g(N,N)\ell$
- Combining them:

$$L_N f = -\Delta_S f + 2\overline{\nabla}_{s^\sharp} f$$
$$+f\left(\frac{1}{2}S(h) + \delta s - h(s^\sharp, s^\sharp) - \mathrm{Ein}(\ell, k) - \frac{1}{2}g(N,N)\,W\right)$$

with

$$W \equiv \mathrm{Ein}(\ell,\ell) + \mathrm{tr}A_\ell^2$$

- We assume that $W \neq 0$, which will happen in general unless we are in an isolated horizon (null hypersurface).

Considering the different operators

- Let $z \in C^\infty(S)$ and set

$$L_z f = -\Delta_S f + 2\overline{\nabla}_{s^\sharp} f + z f \, .$$

- L_z has a principal *real* eigenvalue λ_z —which depends on z— and the corresponding eigenfunction $\phi_z > 0$.
- The variation of $\theta_\ell = 0$ along the direction $\phi_z N$ becomes

$$\frac{L_N \phi_z}{\phi_z} = \lambda_z - z + \frac{1}{2} S(h) + \delta s - h(s^\sharp, s^\sharp) - \mathrm{Ein}(\ell, k) - \frac{1}{2} g(N, N)\, W$$

- Thus, *whenever* $W \neq 0$ on S, one can choose for any z a variation vector $M_z = -k + \ell\, g(M_z, M_z)/2$ with

$$\frac{g(M_z, M_z)}{2} = \frac{1}{W}\left(\lambda_z - z + \frac{1}{2} S(h) + \delta s - h(s^\sharp, s^\sharp) - \mathrm{Ein}(\ell, k)\right) \tag{11}$$

  such that $L_{M_z}\phi_z = \delta_{\phi_z M_z}\, \theta_\ell = 0$.
- Observe that M_z depends on the chosen function z.

Local definition of MOTTs

- The general variation along M_z reads

$$\delta_{fM_z}\theta_\ell = -\Delta_S f + 2\overline{\nabla}_{s^\sharp} f + f(z-\lambda_z) = (L_z-\lambda_z)f \quad (12)$$

  so that the stability operator L_{M_z} of S along M_z is simply $L_z-\lambda_z$ which obviously has a vanishing principal eigenvalue.
- This leads to stability of the original MOTS along M_z *if* the vector M_z is spacelike for some choice of z.
- The directions M_z define locally MOTTs including the given MOTS S (due to a result to be discussed in Parte II).
- These MOTTs will generically be different for different z. In fact, given that $\forall z_1, z_2 \in C^\infty(S)$

$$M_{z_1} - M_{z_2} = \frac{1}{W}\left(\lambda_{z_1} - z_1 - \lambda_{z_2} + z_2\right)\ell$$

  one can easily prove that

$$M_{z_1} = M_{z_2} \Longleftrightarrow z_1 - z_2 = \text{const.}$$

The stability operator along $-k$

- Notice that the stability operator L_{-k} corresponds to the choice

$$z = \hat{z} \equiv \frac{1}{2}S(h) + \delta s - h(s^\sharp, s^\sharp) - \mathrm{Ein}(\ell, k)$$

so that

$$g(M_{\hat{z}}, M_{\hat{z}}) = \frac{\lambda_{\hat{z}}}{W}$$

where $\lambda_{\hat{z}} = \lambda_{-k}$ is the principal eigenvalue.

- Strict stability along $-k$ implies that $\lambda_{\hat{z}} = \lambda_{-k} > 0$, ergo M_z is spacelike —recovering the previous result of existence of spacelike stability directions whenever strict stability along k holds.

A formula for the principal eigenvalue

- By using $\overline{\nabla}_{s^\sharp} f = \delta(fs) - f\delta s$, one deduces for any given z,

$$\oint_S L_z f = \oint_S \left(2\overline{\nabla}_{s^\sharp} f + zf\right) = \oint_S (z - 2\delta s)\, f$$

- in particular for the principal eigenfunction

$$\lambda_z \oint_S \phi_z = \oint_S (z - 2\delta s)\, \phi_z$$

- This provides
  1. a formula for the principal eigenvalue

$$\lambda_z = \frac{\oint_S (z - 2\delta s)\, \phi_z}{\oint_S \phi_z}\,. \qquad (13)$$

  2. bounds for λ_z

$$\min_S (z - 2\delta s) \leq \lambda_z \leq \max_S (z - 2\delta s)\ . \qquad (14)$$

  3. and that $\lambda_z - (z - 2\delta s)$ must vanish somewhere on S for all $z \in C^\infty(S)$.

A formula for the principal eigenvalue

- For each $z \in C^\infty(S)$ rewrite $\delta_{fN}\theta_\ell = L_N f$ using (11) so that

$$\frac{W}{2} f\,(g(N,N) - g(M_z,M_z)) = (L_z - \lambda_z)f - \delta_{fN}\theta_\ell \qquad (15)$$

- Note that

$$f(N - M_z) = (f/2)\,(g(N,N) - g(M_z,M_z))\,\ell$$

  hence fN points into the zone with OTSs —that is, the region towards which ℓ points in— if $f\,(g(N,N) - g(M_z,M_z)) > 0$. The situation is clear from the helpful figure.

- From (15) becomes evident that deformations along any N with the principal eigenfunction ϕ_z are such that the deformed submanifold becomes outer trapped (untrapped) if N points above (below) M_z.

A distinguished MOTTs

- The problem with the stability of MOTS and the local construction of MOTTs is that we have little control on the principal eigenvalue.
- This is how I have tried to get around this problem: Consider the particular function

$$z = 2\delta s$$

  This defines what I guess can lead to a preferred M(O)TT, being a natural candidate for boundary of a core.
- For such a choice let L denote the corresponding operator $L = L_{2\delta s}$, μ its principal eigenvalue, and $\phi > 0$ the corresponding eigenfunction. Observe that

$$Lf = -\Delta_S f + 2\delta(fs) = -\delta\left(df - 2fs\right).$$

- The principal eigenvalue μ vanishes. Indeed, this follows immediately from either (13) or (14). Also from

$$\oint_S Lf = 0 \quad \forall f \, , \Longrightarrow \oint_S L\phi = \mu \oint_S \phi = 0$$

A distinguished MOTTs

- For this particular choice of z, (15) reduces to

$$\frac{W}{2} f\left(g(N,N) - g(M,M)\right) = L(f) - \delta_{fN}\theta_\ell \qquad (16)$$

where now the vector $M = -k + \frac{g(M,M)}{2}\ell$ is defined by

$$\frac{g(M,M)}{2} = \frac{1}{W}\left(\frac{1}{2}S(h) - \delta s - h(s^\sharp, s^\sharp) - \mathrm{Ein}(\ell, k)\right)$$

as follows from (11).

- For any other direction M_z defining a local M(O)TT

$$\frac{W}{2}\left(g(M_z, M_z) - g(M,M)\right) = \lambda_z - (z - 2\delta s)$$

Result

The local M(O)TT defined by the direction M is such that any other nearby local M(O)TT must interweave it with non-trivial intersections to both of its sides, that is to say, the vector $M_z - M$ changes sign on any of its M(O)TSs.

- Integrating (16) on S we get

$$\frac{1}{2}\oint_S W f(g(N,N) - g(M,M)) = -\oint_S \delta_{fN}\theta_\ell$$

- from (16), deformations using $c\phi$ with constant c lead to outer untrapped surfaces if $c(n_\mu n^\mu - m_\mu m^\mu) < 0$ everywhere and to outer f-trapped surfaces if $c(n_\mu n^\mu - m_\mu m^\mu) > 0$ everywhere.
- the deformed surface can be (outer) f-trapped only if $f(g(N,N) - g(M,M)$ is positive somewhere (meaning that N points there into the region with (O)TS).
- and it can be (outer) untrapped only if $f(g(N,N) - g(M,M))$ is somewhere negative (meaning that N points there outside the region with (O)TS).
- if the deformed surface has $f(g(N,N) - g(M,M)) < 0$ everywhere then $\delta_{fN}\theta_\ell$ must be positive somewhere.
- if the deformed surface has $f(g(N,N) - g(M,M)) > 0$ everywhere then $\delta_{fN}\theta_\ell$ must be negative somewhere.

What about Cores?

- We try to follow the same steps as in the spherically symmetric case.
- Thus, the idea is to start with a function

$$f = a_0\phi + \tilde{f}$$

for a constant $a_0 > 0$ so that, as $\phi > 0$ has eigenvalue $\mu = 0$, (16) becomes

$$\frac{W}{2}(a_0\phi + \tilde{f})\left(g(N,N) - g(M,M)\right) = L\tilde{f} - \delta_{fN}\theta_\ell$$

- This can be split into two parts:

$$\frac{W}{2}a_0\phi\left(g(N,N) - g(M,M)\right) = -\delta_{fN}\theta_\ell \qquad (17)$$

$$\frac{W}{2}\tilde{f}\left(g(N,N) - g(M,M)\right) = L\tilde{f} \qquad (18)$$

- This would certainly be useful if L has more real eigenvalues, and leads to the analysis of the condition $L\tilde{f}/\tilde{f} > 0$ for some function $\tilde{f}$.

What about Cores?

- Eq.(17) tells us that $\delta_{fN}\theta_\ell < 0$ whenever N points "above" M if $a_0 > 0$ is chosen.
- Therefore, using (18) the problem one needs to solve can be reformulated as follows:

A mathematical problem

Is there a function $\tilde{f}$ on S such that

1. $L(\tilde{f})/\tilde{f} \geq \epsilon > 0$,
2. $\tilde{f}$ changes sign on S,
3. $\tilde{f}$ is positive in a region as small as desired?

- To prove that there are future-trapped surfaces penetrating both sides of the MTT it is enough to comply with points 1 and 2 only.

The case when L has real eigenvalues

Result

If the operator L has any real eigenvalue other than the principal one $\mu = 0$, then the conditions 1 and 2 do hold for the corresponding real eigenfunction. This leads to the existence of closed OTSs penetrating both sides of the local M(O)TT.

Proof. Any real eigenvalue is strictly positive (as $\mu = 0$). Hence, the corresponding eigenfunction must change sign on S, because integration of $L\psi = \lambda\psi$ on S implies $\oint \psi = 0$.

However, even if there are no other real eigenvalues the result might hold.

Some open mathematical problems

- In order to attack the mathematical problem in full, one can follow several routes.
- One possibility is to use the Hodge decomposition theorem on the compact S:
$$s = d\sigma + \delta F + \Upsilon$$
where $\sigma \in C^\infty(S)$, $F \in \Lambda^2 S$ and $\Upsilon \in \Lambda S$ with $d\Upsilon = 0$ and $\delta\Upsilon = 0$.
- For $n = 4$, one has $F = \bar{\epsilon}\Psi$ for some function $\Psi \in C^\infty(S)$, so that
$$s = d\sigma + \bar{\star} d\Psi + \Upsilon$$
- Thus, the two functions σ and Ψ encode all the information of the normal connection s for any simply connected S.

Conformal transformations in $n = 4$

- As a final, probably useful, remark, let us consider conformal transformations on S. Define

$$\tilde{h} \equiv \Omega h$$

  for some positive function Ω.

- One can prove that

$$\tilde{L}(f) = \frac{1}{\Omega} L(f)\,.$$

- It follows that the principal eigenfunction ϕ is also an eigenfunction, with vanishing eigenvalue, of $\tilde{L}$, as any solution to $L(f) = 0$ is also a solution to $\tilde{L}(f) = 0$, and viceversa.
- This result, together with the fact that any simply-connected compact surface (S, h) is globally conformal to the round sphere, allows one to reformulate the mathematical problem in the round sphere (where σ and Ψ are two given data).

Parte II

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