The symmetry group of Lamé's system and the associated Guichard nets for conformally flat hypersurfaces

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1. Conformally flat hypersurfaces, Guichard nets in $\mathbb{R}^3$ and Lamé's system of equations;
2. The symmetry group of Lamé's system and group invariant solutions;
3. The geometry of the Guichard nets in $\mathbb{R}^3$ associated to invariant solutions;
4. Conformally flat hypersurfaces associated to invariant solutions;
5. Final remarks and future work;

Conformally flat hypersurfaces

A hypersurface in the euclidean space $\mathbb{R}^n$ is called conformally flat if every point has a neighbourhood where the induced metric is conformal to a flat metric, i.e, to a metric with zero curvature.

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A hypersurface in the euclidean space $\mathbb{R}^n$ is called conformally flat if every point has a neighbourhood where the induced metric is conformal to a flat metric, i.e, to a metric with zero curvature.

- The investigation of conformally flat hypersurfaces has been of interest for quite some time. The answers to the problem are strongly related with the dimension of the space:

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- For higher dimensional hypersurfaces, E. Cartan in 1917 gave a complete classification for the conformally flat hypersurfaces of $\mathbb{R}^n$, when $n \geq 5$. He proved that such hypersurfaces are quasi-umbilic, i.e., one of the principal curvatures has multiplicity at least $n-2$.

- In the same paper, Cartan investigated the case $n = 4$. He showed that the quasi-umbilic hypersurfaces are conformally flat, but the converse does not hold.

- In the same paper, Cartan investigated the case $n = 4$. He showed that the quasi-umbilic hypersurfaces are conformally flat, but the converse does not hold.
- Since then, there has been an effort to obtain a complete classification of conformally hypersurfaces in $\mathbb{R}^4$, with three distinct principal curvatures. The problem is still an open question.

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a) M^3 is a cylinder over a surface, $M^2 \subset \mathbb{R}^3$, with constant curvature in $\mathbb{R}^3$;

b) M^3 is a cone over a surface in the sphere, $M^2 \subset \mathbb{S}^3$, with constant curvature;

c) M^3 is obtained by rotating a constant curvature surface of the hyperbolic space, $M^2 \subset \mathbb{H}^3 \subset \mathbb{R}^4$, where $\mathbb{H}^3$ is the half space model.

Hertrich-Jeromin in 1994, established a correspondence between conformally flat hypersufaces, with three distinct principal curvatures, and Guichard nets in $\mathbb{R}^3$. These are open sets of $\mathbb{R}^3$, with an orthogonal flat metric $g = \sum_{i=1}^{3} l_i^2 dx_i^2$, where the functions l_i satisfy the Guichard condition, namely,

$$l_1^2 - l_2^2 + l_3^2 = 0,$$

and a system of second order partial differential equations, which is called Lamé's system

$$\begin{aligned} \frac{\partial^2 l_i}{\partial x_j \partial x_k} - \frac{1}{l_j}\frac{\partial l_i}{\partial x_j}\frac{\partial l_j}{\partial x_k} - \frac{1}{l_k}\frac{\partial l_i}{\partial x_k}\frac{\partial l_k}{\partial x_j} &= 0, \\ \frac{\partial}{\partial x_j}\left(\frac{1}{l_j}\frac{\partial l_i}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{l_i}\frac{\partial l_j}{\partial x_i}\right) + \frac{1}{l_k^2}\frac{\partial l_i}{\partial x_k}\frac{\partial l_j}{\partial x_k} &= 0. \end{aligned} \tag{1}$$

For each solution $(l_1,\ l_2,\ l_3)$ for the Lamé's system that satisfy Guichard condition, Hertrich-Jeromin proved that there exists a parametrization for a conformally flat hypersurface in $\mathbb{R}^4$, with three distinct principal curvatures, whose induced metric is given by

$$g = e^{2P(x)} \left\{ l_1(x)^2 dx_1^2 + l_2(x)^2 dx_2^2 + l_3(x)^2 dx_3^2 \right\}, \tag{2}$$

where $P(x)$ is a function that depends on $x = (x_1, x_2, x_3)$.

Our objective is to find solutions of the Lamé's system, which satisfy the Guichard condition, in order to obtain an associated class of conformally flat hypersurface in $\mathbb{R}^4$;

The symmetry group of Lamé's system

A system S of n-th order differential equations in p independent and q dependent variables is given as a system of m equations

$$\Delta_r(x, u^{(n)}) = 0, \; r = 1, \ldots, m, \tag{3}$$

involving $x = (x_1, \ldots, x_p) \in X, \; u = (u_1, \ldots, u_q) \in U$ and the derivatives $u^{(n)}$ of u with respect to x up to order n.

A symmetry group of the system S is a Lie group of transformations G acting on $X \times U$ of the space of independent and dependent variables for the system, with the property that whenever $u = f(x)$ is a solution of S, and whenever $g \cdot (x, f(x)) = (\tilde{x}, \tilde{f}(\tilde{x}))$ is defined for $g \in G$, then $u = \tilde{f}(\tilde{x})$ is also a solution of the system.

A vector field V in the Lie algebra $\mathfrak{g}$ of the group G is called an infinitesimal generator.

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Consider V as a vector field on $X \times U$, with corresponding one-parameter group $\exp(\varepsilon \mathrm{V})$, i.e.,

$$\exp(\varepsilon \mathrm{V}) \equiv \Psi(\varepsilon, x), \tag{4}$$

where Ψ is the flow generated by V. In this case, V will be the infinitesimal generator of the action induced by the flow.

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$$\mathrm{V} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha},$$

The symmetry group of a given system of differential equation is obtained by using the prolongation formula and the infinitesimal criterion that are describe as follows: Given a vector field on $X \times U$,

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the n-th prolongation of V is the vector field on the corresponding jet space $X \times U^{(n)}$

$$\mathrm{pr}^{(n)}\mathrm{V} = V + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha}.$$

The second summation is taken over all (unordered) multi-indices $J = (j_1, \ldots, j_k)$, with $1 \leq j_k \leq p$, $1 \leq k \leq n$. The coefficient functions ϕ_α^J of $\mathrm{pr}^{(n)} V$ are given by the following formula:

$$\phi_\alpha^J(x, u^{(n)}) = D_J \left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha, \right),$$

where $u_i^\alpha = \dfrac{\partial u^\alpha}{\partial x_i}$, $u_{J,i}^\alpha = \dfrac{\partial u_J^\alpha}{\partial x_i}$ and D_J is given by total derivatives

$$D_J = D_{j_1} D_{j_2} \ldots D_{j_k},$$

with $D_i f(x, u^{(n)}) = \dfrac{\partial f}{\partial x_i} + \displaystyle\sum_{\alpha_1}^{p} \sum_{J} u_{J,i}^\alpha \frac{\partial f}{\partial u_J^\alpha}.$

Consider a system $\Delta_r(x, u^{(n)}) = 0$, $r = 1, \ldots, l$. Then the set of all vectors fields V on M such that

$$\mathrm{pr}^{(n)}\mathrm{V}[\Delta_r(x, u^{(n)})] = 0, \quad \text{whenever} \;\; \Delta_r(x, u^{(n)}) = 0, \qquad (5)$$

is a Lie algebra of infinitesimal generators of a symmetry group for the system. Conversely, all the connected symmetry groups can be determined by considering this criterion.

- Since the prolongation formula is given in terms of ξ^i and ϕ_α and the partial derivatives with respect to both x and u, the infinitesimal criterion provides a system of partial differential equations for the coefficients ξ^i and ϕ_α of V, called the determining equations.

- Since the prolongation formula is given in terms of ξ^i and ϕ_α and the partial derivatives with respect to both x and u, the infinitesimal criterion provides a system of partial differential equations for the coefficients ξ^i and ϕ_α of V, called the determining equations.
- By solving these equations, we obtain the vector field V that determines a Lie algebra $\mathfrak{g}$. The symmetry group G is obtained by exponentiating the Lie algebra.

From now on, we consider the following notation for the derivatives of a function $f = f(x_1, \ldots, x_n)$

$$f_{,x_i} := \frac{\partial f}{\partial x_i} \text{ and } f_{,x_i x_j} := \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

With this notation, Lamé's system (1) is given by

$$l_{i,x_j x_k} - \frac{l_{i,x_j} l_{j,x_k}}{l_j} - \frac{l_{i,x_k} l_{k,x_j}}{l_k} = 0, \tag{6}$$

$$\left(\frac{l_{i,x_j}}{l_j}\right)_{,x_j} + \left(\frac{l_{j,x_i}}{l_i}\right)_{,x_i} + \frac{l_{i,x_k} l_{j,x_k}}{l_k^2} = 0, \tag{7}$$

where i, j and k are distinct indices in the set $\{1, 2, 3\}$.

We will also consider the following notation,

$$\varepsilon_s = \begin{cases} \;\;\,1 & \text{if} \quad s = 1 \text{ or } s = 3, \\ -1 & \text{if} \quad s = 2. \end{cases} \tag{8}$$

We can now rewrite Guichard condition as

$$\varepsilon_i l_i^2 + \varepsilon_j l_j^2 + \varepsilon_k l_k^2 = 0.$$

Next, we introduce auxiliary functions in order to reduce the sytem of second order differential equations (6) and (7), into a first order one. Consider the functions h_{ij}, with $i \neq j$, given by

$$l_{i,x_j} - h_{ij} l_j = 0.$$

With these functions, we rewrite (6) and (7) as

$$\begin{aligned} h_{ij,x_k} - h_{ik} h_{kj} &= 0, \\ h_{ij,x_j} + h_{ji,x_i} + h_{ik} h_{jk} &= 0. \end{aligned}$$

Using Guichard condition, there are other relations involving the derivatives of l_i and h_{ij}. Taking the derivative of Guichard condition with respect to x_i, we have

$$\varepsilon_i l_{i,x_i} + \varepsilon_j h_{ji} l_j + \varepsilon_k h_{ki} l_k = 0,$$

for i, j, k distinct. The derivatives of the above equation with respect to x_j leads to

$$\varepsilon_i h_{ij,x_i} + \varepsilon_j h_{ji,x_j} + \varepsilon_k h_{ki} h_{kj} = 0.$$

Therefore, we summarize the last six equations in the following system of first order partial differential equations, equivalent to Lamé's system, that we call Lamé's system of first order

$$\varepsilon_i l_i^2 + \varepsilon_j l_j^2 + \varepsilon_k l_k^2 = 0, \tag{9}$$

$$l_{i,x_j} - h_{ij} l_j = 0, \tag{10}$$

$$\varepsilon_i l_{i,x_i} + \varepsilon_j h_{ji} l_j + \varepsilon_k h_{ki} l_k = 0, \tag{11}$$

$$h_{ij,x_k} - h_{ik} h_{kj} = 0, \tag{12}$$

$$h_{ij,x_j} + h_{ji,x_i} + h_{ik} h_{jk} = 0, \tag{13}$$

$$\varepsilon_i h_{ij,x_i} + \varepsilon_j h_{ji,x_j} + \varepsilon_k h_{ki} h_{kj} = 0. \tag{14}$$

Theorem 1

Let V be the infinitesimal generator of the symmetry group of Lamé's system of first order (9)-(14), given by

$$V = \sum_{i=1}^{3} \xi^i(x, l, h) \frac{\partial}{\partial x_i} + \sum_{i=1}^{3} \eta^i(x, l, h) \frac{\partial}{\partial l_i} + \sum_{i,j=1,\, i \neq j}^{3} \phi^{ij}(x, l, h) \frac{\partial}{\partial h_{ij}}. \tag{15}$$

Then the functions ξ^i, η^i and ϕ^{ij} are given by

$$\begin{aligned} \xi^i &= a x_i + a_i, \\ \eta^i &= c l_i, \\ \phi^{ij} &= -a h_{ij}, \end{aligned} \tag{16}$$

where a, c, $a_i \in \mathbb{R}$, $x = (x_1, x_2, x_3)$, $l = (l_1, l_2, l_3)$ and h the off-diagonal 3×3 matrix given by h_{ij}.

As a result of exponentiating V, we obtain the symmetry group of Lamé's system.

Corollary 1

The symmetry group of Lamé's system (9)-(14) is given by the following transformations:

1. *translation in the independent variables:* $\tilde{x}_i = x_i + v_i$;
2. *dilation in the independent variables:* $\tilde{x}_i = \lambda x_i$;
3. *dilation in the dependent variables:* $\tilde{l_i} = \rho l_i$ *and* $\tilde{h_{ij}} = \lambda^{-1} h_{ij}$;

where $v_i \in \mathbb{R}$ *and* $\lambda,\, \rho \in \mathbb{R} \setminus \{0\}$.

Group invariant solutions

- The knowledge of all the infinitesimal generators V of the symmetry group of a system of differential equations, allows one to reduce the system to another one with a reduced number of variables.
- Specifically, if the system has p independent variables and a symmetry subgroup is considered, where the orbits are s-dimensional, then the reduced system for the solutions invariant under this subgroup will depend on $p - s$ variables.

We start with the subgroup of translations. The basic invariant of this group is given by

$$\xi = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \tag{17}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is a non zero vector. We will consider solutions I_i such that

$$I_i(x_1, x_2, x_3) = I_i(\xi), \; 1 \leq i \leq 3, \tag{18}$$

where ξ is given by (17).

Theorem 2

Let $l_s(\xi)$, $s = 1, 2, 3$, *where* $\xi = \sum_{s=1}^{3} \alpha_s x_s$, *be a solution of Lamé's system (9)-(14), such that* l_s *is not constant for all* s. *Then there exist* $c_s \in \mathbb{R} \setminus \{0\}$, *such that,*

$$l_{i,\xi} = c_i l_k l_j, \quad i, j, k \;\; distinct, \tag{19}$$

$$c_1 - c_2 + c_3 = 0, \tag{20}$$

$$\alpha_1^2 c_2 c_3 + \alpha_2^2 c_1 c_3 + \alpha_3^2 c_1 c_2 = 0. \tag{21}$$

Moreover, the functions $l_i(\xi)$ are given by

$$l^2_{1,\xi} = c_2(c_2 - c_1)\left(l^2_1 - \frac{\lambda}{c_2}\right)\left(l^2_1 - \frac{\lambda}{c_2 - c_1}\right), \tag{22}$$

$$l^2_2 = \frac{c_2}{c_1}\left(l^2_1 - \frac{\lambda}{c_2}\right), \tag{23}$$

$$l^2_3 = \frac{c_2 - c_1}{c_1}\left(l^2_1 - \frac{\lambda}{c_2 - c_1}\right), \tag{24}$$

where $\lambda \in \mathbb{R}$.

Theorem 3

Let $l_s(\xi)$, $s = 1, 2, 3$, *where* $\xi = \sum_{s=1}^{3} \alpha_s x_s$, *be a solution of Lamé's system (9)-(14). Suppose that only one of the functions* l_s *is constant. Then one of the following occur:*

a) $l_1 = \lambda_1,\ l_2 = \lambda_1 \cosh(b\xi + \xi_0),\ l_3 = \lambda_1 \sinh(b\xi + \xi_0)$, *where* $\xi = \alpha_2 x_2 + \alpha_3 x_3$, $\alpha_2^2 + \alpha_3^2 \neq 0$ *and* b, $\xi_0 \in \mathbb{R}$;

b) $l_2 = \lambda_2,\ l_1 = \lambda_2 \cos\varphi(\xi),\ l_3 = \lambda_2 \sin\varphi(\xi)$, *where* $\xi = \alpha_1 x_1 + \alpha_3 x_3$, $\alpha_1^2 + \alpha_3^2 \neq 0$ *and* φ *is one of the following:*
   - b.1) $\varphi(\xi) = b\xi + \xi_0$, *if* $\alpha_1^2 \neq \alpha_3^2$, *where* ξ_0, $b \in \mathbb{R}$;
   - b.2) φ *is any function of* ξ, *if* $\alpha_1^2 = \alpha_3^2$;

c) $l_3 = \lambda_3,\ l_2 = \lambda_3 \cosh(b\xi + \xi_0),\ l_1 = \lambda_3 \sinh(b\xi + \xi_0)$, *where* $\xi = \alpha_1 x_1 + \alpha_2 x_2$, $\alpha_1^2 + \alpha_2^2 \neq 0$ *and* b, $\xi_0 \in \mathbb{R}$.

Next, we consider the solutions invariant under the subgroup involving translations and dilations. In this case, the basic invariant is given by

$$\eta = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{b_1 x_1 + b_2 x_2 + b_3 x_3}, \tag{25}$$

where the vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent.

Theorem 4

Let $l_i(\eta)$*, with* η *given by (25), be a solution of Lamé's system invariant under the subgroup involving translations and dilations. Suppose that* $(a_s, b_s) \neq (0,0)$, $\forall s$*, then the solutions* $l_i(\eta)$ *are constant.*

Theorem 5

Let $l_i(\eta)$, with η given by (25), be a solution of Lamé's system invariant under the subgroup involving translations and dilations. Suppose that one of the pairs $(a_s, b_s) = (0, 0)$. Then one of the following occur:

a) *If* $(a_1,\ b_1) = (0,\ 0)$ *then*
$l_1 = \lambda_1,\ l_2 = \lambda_1 \cosh\varphi(\eta),\ l_3 = \lambda_1 \sinh\varphi(\eta)$, *where*
$\eta = \dfrac{a_2x_2 + a_3x_3}{b_2x_2 + b_3x_3}$ *and* φ *is given by*

$$\varphi(\eta) = \frac{C_0}{a_2b_3 - a_3b_2} \arctan\left[\frac{b_2^2 + b_3^2}{a_3b_2 - a_2b_3}\left(\eta - \frac{a_2b_2 + a_3b_3}{b_2^2 + b_3^2}\right)\right] + C_1, \tag{26}$$

where $C_0,\ C_1 \in \mathbb{R}$.

b) *If* $(a_2,\ b_2) = (0,\ 0)$ *then*
$l_2 = \lambda_2,\ l_1 = \lambda_2 \cos\varphi(\eta),\ l_3 = \lambda_2 \sin\varphi(\eta)$, *where*
$\eta = \dfrac{a_1 x_1 + a_3 x_3}{b_1 x_1 + b_3 x_3}$ *and* φ *is given as follows:*

b.1) *if* $b_1 = b_3 = b$, *then*

$$\varphi(\eta) = \frac{D_0}{2b(a_3 - a_1)} \log(2b\eta - a_1 - a_3) + D_1, \qquad (27)$$

where $D_0,\ D_1 \in \mathbb{R}$;

b.2) *if* $b_1 \neq b_3$, *then*

$$\varphi(\eta) = \frac{D_2}{2(a_1 b_3 - a_3 b_1)} \log\left[\frac{(b_3 + b_1)\eta - (a_3 + a_1)}{(b_3 - b_1)\eta - (a_3 - a_1)}\right] + D_3, \qquad (28)$$

where $D_2,\ D_3 \in \mathbb{R}$.

c) *If* $(a_3,\ b_3) = (0,\ 0)$, *then*

$l_3 = \lambda_3,\ l_2 = \lambda_3 \cosh \varphi(\eta),\ l_1 = \lambda_3 \sinh \varphi(\eta)$, *with*
$\eta = \dfrac{a_1 x_1 + a_2 x_2}{b_1 x_1 + b_2 x_2}$ *and* φ *is given by*

$$\varphi(\eta) = \frac{E_0}{a_2 b_1 - a_1 b_2} \arctan\left[\frac{b_2^2 + b_1^2}{a_2 b_1 - a_1 b_2}\left(\eta - \frac{a_2 b_2 + a_1 b_1}{b_2^2 + b_1^2}\right)\right] + E_1, \tag{29}$$

where $E_0,\ E_1 \in \mathbb{R}$.

Definition 1

Let M^n be a Riemannian manifold and let $f : M \to \mathbb{R}$ be a differentiable function. The level submanifolds of f are said to be *geodesically parallel* if $|\mathrm{grad} f|$ is a non zero constant, along each level submanifold.

Theorem 6

Let (U, g), $U \subset \mathbb{R}^3$, be a Riemannian manifold with coordinates (x_1, x_2, x_3) and metric $g = \sum_{s=1}^{3} l_s^2(\xi) dx_i^2$, where $\xi = \sum_{s=1}^{3} \alpha_s x_s$. Then the level surfaces

$$P_{\xi_0} = \left\{ (x_1, x_2, x_3) \in U; \sum_{s=1}^{3} \alpha_s x_s = \xi_0 \right\}, \text{ where } \xi_1 < \xi_0 < \xi_2,$$

endowed with the induced metric are geodesically parallel. Moreover, each level surface has zero Gaussian curvature and constant mean curvature (depending on ξ_0).

Theorem 7

Let (U, g), $U \subset \mathbb{R}^3$, be a Riemannian manifold, with coordinates (x_1, x_2, x_3) and metric $g = \sum_{s=1}^{3} l_s^2(\xi) dx_i^2$, with $\xi = \sum_{s=1}^{3} \alpha_s x_s$. Then each coordinate surface of $U \subset \mathbb{R}^3$, $x_i =$constant, endowed with the induced metric, has constant Gaussian curvature K_i. Moreover,

$$K_1 + K_2 + K_3 = 0. \tag{30}$$

Conformally flat hypersurfaces

By Guichard condition, any conformally flat hypersurface in $\mathbb{R}^4$ has a local parametrization where the induced metric is given by

$$g = e^{2P(x)} \left\{ \sin^2 \varphi(x) dx_1^2 + dx_2^2 + \cos^2 \varphi(x) dx_3^2 \right\}, \tag{31}$$

where $x = (x_1, x_2, x_3)$, or

$$g = e^{2\tilde{P}(x)} \left\{ \sinh^2 \tilde{\varphi}(x) dx_1^2 + \cosh^2 \tilde{\varphi}(x) dx_2^2 + dx_3^2 \right\}. \tag{32}$$

- Suyama proved in 2005 that φ is a conformal invariant and he classified the hypersurfaces conformal to the products given by Lafontaine as the hypersurfaces where φ depends only on two variables;
- Hertrich-Jeromin and Suyama classified in 2007 the hypersurfaces where φ has two vanishing mixed derivatives, namely:

- These conformally flat hypersurfaces are associated to the so called *cyclic Guichard nets*, which are characterized by $\varphi_{,x_1x_2} = \varphi_{,x_2x_3} = 0$, when g is of the form (31) and by $\varphi_{,x_1x_3} = \varphi_{,x_2x_3} = 0$, when g is given by (32).
- Conformally flat hypersurfaces are given by means of linear Weingarten surfaces in space forms.
- Moreover, the authors showed that all the known cases of conformally flat hypersurfaces, at that time, are associated to cyclic Guichard nets.

Theorem 8

Let M^3 be a conformally flat hypersurface in $\mathbb{R}^4$, associated to a solution of Lamé's system $l_i(x_1, x_2, x_3) = l_i(\xi)$, with $\xi = \sum_{s=1}^{3} \alpha_s x_s$ and $\alpha_s \neq 0$, for all s, given in terms of elliptic functions by (22)-(24). Then its first fundamental form g is given by

$$g = e^{2P(x)} \left\{ \cos^2 \varphi(\xi)(dx_1)^2 + (dx_2)^2 + \sin^2 \varphi(\xi)(dx_3)^2 \right\}, \quad (33)$$

where φ satisfies,

$$\varphi_{,\xi}^2 = c(a \cos^2 \varphi - b), \quad (34)$$

or g is given by

$$g = e^{2\tilde{P}(x)}\left\{\sinh^2\tilde{\varphi}(\xi)(dx_1)^2 + \cosh^2\tilde{\varphi}(\xi)(dx_2)^2 + (dx_3)^2\right\} \quad (35)$$

where $\tilde{\varphi}$ satisfies

$$\tilde{\varphi}_{,\xi}^2 = c(b\cosh^2\tilde{\varphi} - b). \quad (36)$$

where a, b, $c \in \mathbb{R}\setminus\{0\}$, $P(x)$ and $\tilde{P}(x)$ are differentiable functions. In both cases, $\xi \in I \subset \mathbb{R}$, where I is an open interval such that g is positive definite.

Corollary 2

Let $M^3 \subset \mathbb{R}^4$ *be a conformally flat hypersurface associated to the solutions of Lamé's system* $l_i(\xi)$ *with* $\xi = \sum_{s=1}^{3} \alpha_s x_s$ *and* $\alpha_s \neq 0$ *for all* s*, given in terms of elliptic functions by (22)-(24). Then the associated Guichard net of* M^3 *is not cyclic.*

Corollary 2

Let $M^3 \subset \mathbb{R}^4$ be a conformally flat hypersurface associated to the solutions of Lamé's system $l_i(\xi)$ with $\xi = \sum_{s=1}^{3} \alpha_s x_s$ and $\alpha_s \neq 0$ for all s, given in terms of elliptic functions by (22)-(24). Then the associated Guichard net of M^3 is not cyclic.

- When $\xi = \alpha_1 x_1 + \alpha_2 x_2$, the associated conformally flat hypersurfaces is conformal to the product $M^2 \times I$, where M^2 is a flat surface in the hyperbolic 3-space.

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- When $\xi = \alpha_1 x_1 + \alpha_3 x_3$, the associated conformally flat hypersurfaces is conformal to the product $M^2 \times I$, where M^2 is a flat surface in the standard 3-sphere.
- Explicit parametrizations are given by means of solutions of Klein-Gordon equation

$$F(x_1, x_i) - b\alpha_1\alpha_i F_{,x_1 x_i}(x_1, x_i) = 0,$$

where $i = 2, 3$

- An explicit parametrization for the general case is still unknown;
- it is related to solutions of the following partial differential equation:

$$F + F_{,x_1x_1} - F_{,x_2x_2} + F_{,x_3x_3} - \beta_{12}F_{,x_1x_2} - \beta_{13}F_{,x_1x_3} - \beta_{23}F_{,x_2x_3} = 0,$$

where

$$\begin{aligned}\beta_{12} &= \left(\frac{\alpha_1^2 - \alpha_2^2 - \alpha_3^2}{\alpha_1\alpha_2}\right),\\ \beta_{13} &= \left(\frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}{\alpha_1\alpha_3}\right),\\ \beta_{23} &= \left(\frac{\alpha_3^2 - \alpha_1^2 - \alpha_2^2}{\alpha_2\alpha_2}\right).\end{aligned}$$

Final remarks and future work

- Hertrich-Jeromin and Suyama have considered recently (2013) non-cyclic Guichard nets where the coordinates surfaces have constant Gaussian curvature. They call these nets Bianchi-type Guichard nets.

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- Hertrich-Jeromin and Suyama have considered recently (2013) non-cyclic Guichard nets where the coordinates surfaces have constant Gaussian curvature. They call these nets Bianchi-type Guichard nets.
- The geometry of the flat surfaces in $\mathbb{H}^3$ related to the case $\xi = \alpha_1 x_1 + \alpha_2 x_2$ was studied in a joint work with Martinez and Tenenblat (Pacific J. Math., V. 264, N.1, 2013);
- The geometry of the flat surfaces in $\mathbb{S}^3$ related to the case $\xi = \alpha_2 x_2 + \alpha_3 x_3$ is a work in progress.

Invariant solutions under dilations:

- a system of partial differential equations in two variables that provides a new class class of conformally flat hypersurfaces;
- possibility of hidden symmetries;

These solutions are being investigated it will appear in a future work.

A complete description of the solutions of Lamé's system

$$
\begin{aligned}
\frac{\partial^2 l_i}{\partial x_j \partial x_k} - \frac{1}{l_j}\frac{\partial l_i}{\partial x_j}\frac{\partial l_j}{\partial x_k} - \frac{1}{l_k}\frac{\partial l_i}{\partial x_k}\frac{\partial l_k}{\partial x_j} &= 0, \\
\frac{\partial}{\partial x_j}\left(\frac{1}{l_j}\frac{\partial l_i}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{l_i}\frac{\partial l_j}{\partial x_i}\right) + \frac{1}{l_k^2}\frac{\partial l_i}{\partial x_k}\frac{\partial l_j}{\partial x_k} &= 0,
\end{aligned}
$$

with the Guichard condition $l_1^2 - l_2^2 + l_3^2 = 0$, is still unknown, as well as the correspondent conformally flat hypersurfaces.