Geometric relative Hardy inequality

and

the discrete spectrum of Schrödinger operators

on manifolds

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Contents

▨1 Two-body Problem in QM On $\mathbb{R}^3$

Hamiltonian Mechanics (Classical Mechanics)

- $H(q_j, p_j) := \frac{1}{2m}\Sigma_{j=1}^{3}(p_j)^2 + V(q_j)$: **Hamiltonian fct**

  = **Energy of isolated system**

- **Hamilton's canonical equations** $\begin{cases} q_j'(t) = \frac{\partial H}{\partial p_j} \\ p_j'(t) = -\frac{\partial H}{\partial q_j} \end{cases}$

Quantization $\Downarrow$ $p_j \rightsquigarrow -\sqrt{-1}\hbar\nabla_j$, $V \rightsquigarrow V\times$

Quantum Mechanics (Set $m := \frac{1}{2}$, $\hbar := 1$)

- $H := -\Delta + V(x)$: **Schrödinger operator**

- $\sqrt{-1}\frac{\partial\psi}{\partial t} = H\psi$: **Schrödinger equation**

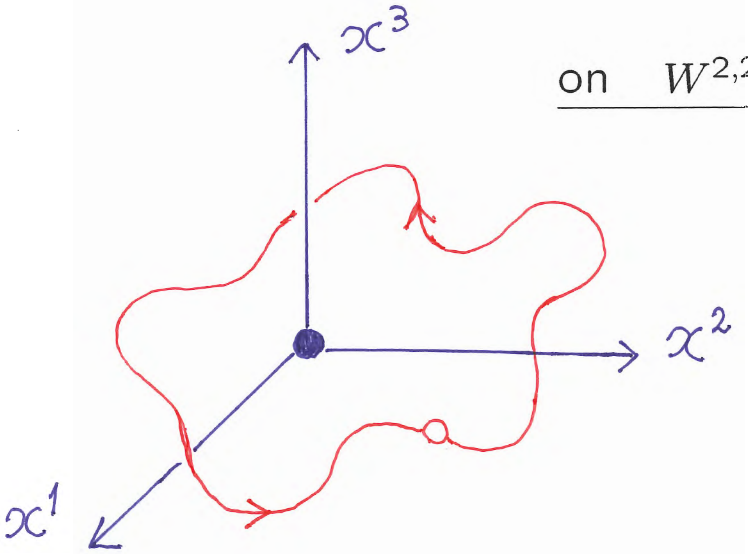
$\rightsquigarrow$ $\boxed{\psi(x,t) := \phi(x)\cdot e^{-\sqrt{-1}\lambda t}}$: **Sol.** of the above eq.

& λ : **Discrete specrtum** for H ($H\phi = \lambda\phi$)

= **Bound state energies** of particles for H

Two-body Problem *for* $H := -\Delta + V(x)$

on $W^{2,2}(\mathbb{R}^3) \subset {}^{\exists}\mathcal{D} \subset L^2(\mathbb{R}^3)$



H : **Closed** operator

- $\sigma(H) := \{\lambda \in \mathbb{R} \mid \lambda\, I - H \ :\ \nexists \text{bounded inverse}\}$
  Spectrum of H

- $\sigma_{\text{disc}}(H) := \{\lambda \in \sigma(H) \mid \lambda \ :\ $ isolated eigenvalues
  & finite multiplicities
  Discrete spectrum of H

- $\sigma_{\text{ess}}(H) := \sigma(H) - \sigma_{\text{disc}}(H)$
  Essential spectrum of H
  = **Scattering state energies** of particles
  = "Energies of free particles"

$\rightsquigarrow$ $$\sigma(H) = \sigma_{\text{disc}}(H) \bigsqcup \sigma_{\text{ess}}(H)$$

Example **Hydrogen Atom**

$V(x) = -\ e^2/|x|$ ($e > 0$: electric charge)

Coulomb potential

$\rightsquigarrow\ \sigma_{\text{disc}}(H) = \{ -\ e^4/4n^2 \ \mid n = 1, 2, \cdots \}$

$\rightsquigarrow\ \left|\sigma_{\text{disc}}(H)\right| = \infty$

& $\sigma_{\text{ess}}(H) = [0, \infty)$

Energies of free particles

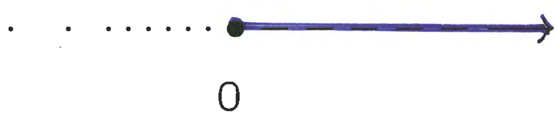
$\Updownarrow$

For $\forall \lambda \in [0, \infty)$, $\exists \{\psi_n\} \subset L^2(\mathbb{R}^3)$ s.t.

(i) $||\psi_n||_{L^2} = 1$

(ii) $\forall$bounded set $B \subset \mathbb{R}^n$, $spt(\psi_n) \cap B = \emptyset$ for $n \gg 1$

(iii) $||(H - \lambda)\psi_n||_{L^2} \longrightarrow 0$ as $n \rightarrow \infty$



$\oslash_2$ Fund. Question and Known Results

- $-\Delta_g + V(x)$: **Schrödinger operator**

  with $\underline{V \in C^0(M)}$

  on **noncompact** complete n-manifold (M, g)

Fact If $V(x) \to 0$ as $x \to \infty$, then

- $\sigma_{\text{ess}}(-\Delta_g + V) = \sigma_{\text{ess}}(-\Delta_g)$: **unchanged !**
- $(-\Delta_g + V)|_{C_c^\infty(M)}$: **essentially self-adjoint**

Note

- $\underline{M = \mathbb{R}^n} \rightsquigarrow \sigma_{\text{ess}}(-\Delta) = [0, \infty)$,

  $\sigma_{\text{disc}}(-\Delta) = \emptyset$

- $\underline{M = (\mathbb{H}^n, h)} \rightsquigarrow \sigma_{\text{ess}}(-\Delta_h) = [\frac{(n-1)^2}{4}, \infty)$,

  $\sigma_{\text{disc}}(-\Delta_h) = \emptyset$

Aim of this talk

Fundamental Question

- $\left|\sigma_{\text{disc}}(-\Delta_g + V)\right|$ $\underline{< \infty}$ or $\underline{= \infty}$?

in terms of the **asymptotic behavior** of V

Assumption for the potential $V(x)$

- $V \in C^0(M)$
- $(-\Delta_g + V)|_{C_c^\infty(M)}$: **essentially self-adjoint**
- $\sigma_{\text{ess}}(-\Delta_g + V) = \sigma_{\text{ess}}(-\Delta_g)$: **unchanged !**

Note $V(x) = -\, e^2/|x|$

- $n \leq 3, \quad V \in L^2 + L^{\infty,0} \rightsquigarrow (1) + (2)$
- $n \geq 4, \quad V \in L^p + L^{\infty,0} \ (p > \frac{n}{2}) \rightsquigarrow (1) + (2)$

Fundamental Criterion $\longleftarrow$ Known Result

Thm RSK (Reed–Simon, Kirsch–Simon)

- $(M, g) = \mathbb{R}^n, \quad n \geq 3$

(i) $$V(x) \geq -\frac{(n-2)^2}{4r^2} \quad \text{for} \quad r := |x| \geq \exists R_0 > 0$$

$$\Rightarrow \quad \left|\sigma_{\text{disc}}(-\Delta_g + V)\right| < \infty$$

(ii) $$V(x) \leq -(1 + \exists\delta)\frac{(n-2)^2}{4r^2} \quad \text{for} \quad r \geq \exists R_1 > 0$$

$$\Rightarrow \quad \left|\sigma_{\text{disc}}(-\Delta_g + V)\right| = \infty$$

$\Uparrow$ Key Lemma

Classical Hardy Inequality (CHI)

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \frac{(n-2)^2}{4r^2} u^2 dx \quad \text{for} \quad u \in C_c^\infty(\mathbb{R}^n)$$

Note CHI $\leftrightsquigarrow$ **Uncertainty Principle Lemma**

Related Known Result

Carron 1997

(M^n, g) : complete Riem. n-mfd ($n \geq 3$) satisfying

$$\boxed{\Delta_g r \geq \frac{{}^{\exists}C}{r}} \qquad \text{for} \quad r(\bullet) := \mathrm{dist}_g(\bullet, {}^{\exists}p_0)$$

$$\Downarrow$$

$$\boxed{\int_M |\nabla u|^2 dv_g \geq \int_M \frac{(C-1)^2}{4r^2}\, u^2 dv_g} \qquad \text{for} \quad u \in C_c^\infty(M)$$

$$\Downarrow$$

$$\Downarrow$$

Note $\quad (M^n, g) = \mathbb{R}^n \ \rightsquigarrow\ \Delta_g r = \frac{n-1}{r} \ \rightsquigarrow\ \frac{(C-1)^2}{4} = \frac{(n-2)^2}{4}$

$$\Downarrow$$

$$\boxed{\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \frac{(n-2)^2}{4r^2}\, u^2 dx} \qquad \text{for} \quad u \in C_c^\infty(\mathbb{R}^n)$$

Weak Point   Not done relativization!

Classical Hardy Inequality (CHI)

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \frac{(n-2)^2}{4r^2} u^2 dx \qquad \text{for} \quad u \in C_c^\infty(\mathbb{R}^n)$$

Proof I (Physical side)

(1) $\sum_j \sqrt{-1}\left[|x|^{-1}p_j|x|^{-1}, x_j\right] = n|x|^{-2}, \quad p_j := -\sqrt{-1}\nabla_j$

$\Downarrow$

(2) $n\left\||x|^{-1}u\right\|_{L^2}^2 = \sum_j \int_{\mathbb{R}^n} \overline{u}\left(\sqrt{-1}\left[|x|^{-1}p_j|x|^{-1}, x_j\right]u\right) dx$

$$= -2\sum_j \mathrm{Im}\left\langle p_j u, \frac{x_j}{|x_j|^2}u\right\rangle_{L^2} + 2\left\||x|^{-1}u\right\|_{L^2}^2$$

$\Downarrow$ • • •

• $\frac{(n-2)^2}{4}\left\||x|^{-1}u\right\|_{L^2}^2 \leq \left\langle u, -\Delta u\right\rangle_{L^2} = \left\||\nabla u|\right\|_{L^2}^2$ **QED**

Proof II (Mathematical side)

(1) $\mathbb{R}^n \cong [0,\infty) \times S^{n-1}(1), \quad g_{\mathbb{E}} = dr^2 + r^2 \cdot g_{S^{n-1}(1)}$

(2) **Original Hardy ineq.** For $f \in C_c^1([0,\infty)), \ f(0) = 0$

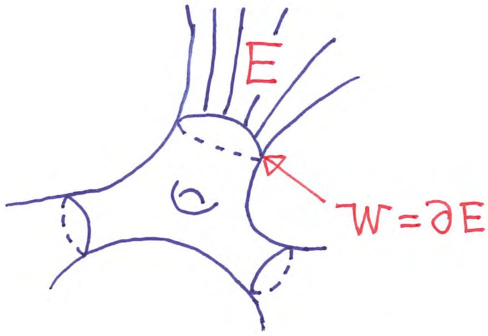
$$\int_0^\infty (f')^2 dt \geq \int_0^\infty \frac{f^2}{4t^2} dt$$

(1) + (2) $\rightsquigarrow$ **CHI** **QED**

Note **Proof II** $\rightsquigarrow$ **Hardy inequality** on manifolds **with a pole**

Generalization!

Relative version!



⊘3 Main Results

Geometric Relative Hardy Ineq. (GRHI) $(n \geq 2)$

E : one of **ends** of (M^n, g) satisfies the following :

- $H_W \geq \frac{1}{R}$ $(^\exists R > 0)$ on W, $W := \partial E$

  ↖ **mean curv.** w. r. t. **inward** unit normal vector

- $\exp_W : \{v \in TM|_W \,|\, v \perp TW, \text{ outward}\} \to E$: **diffeo.**

$\rho(\bullet) := \mathrm{dist}_g(\bullet, W)$, $r(\bullet) := \rho(\bullet) + R$ on E

Important!

$$\Rightarrow \boxed{\int_E |\nabla u|^2 dv_g} \cdots \boxed{\text{GRHI}} \quad \text{for } u \in C_c^\infty(M)$$

$$\geq \int_E \left\{ \frac{1}{4r^2} + \frac{1}{4}(\Delta_g r)^2 - \frac{1}{2}|\nabla dr|^2 - \frac{1}{2}\mathrm{Ric}_g(\nabla r, \nabla r) \right\} u^2 dv_g$$

If (M, g) has a pole $p_0 \in M$, $r(\bullet) := \mathrm{dist}_g(\bullet, p_0)$ on M

$$\Rightarrow \boxed{\int_M |\nabla u|^2 dv_g \geq \int_M \{\cdots\} u^2 dv_g} \Leftrightarrow$$

$$-\Delta_g \geq \frac{1}{4r^2} + \frac{1}{4}(\Delta_g r)^2 - \frac{1}{2}|\nabla dr|^2 - \frac{1}{2}\mathrm{Ric}_g(\nabla r, \nabla r)$$

Note $\Delta r = \frac{n-1}{r}$, $|\nabla dr|^2 = \frac{n-1}{r^2}$, $\mathrm{Ric} = 0$ on $\mathbb{R}^n$

$$\boxed{\text{GRHI}} \rightsquigarrow \boxed{\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n} \frac{(n-2)^2}{4r^2} u^2 dx}$$

Proof of GRHI **Step 1**

- $$g(r,y) = dr^2 + g_{\alpha\beta}(r,y)dy^\alpha dy^\beta$$ on $E \cong [R,\infty) \times W$ ∋ (r, y)

$$dv_g = \sqrt{g}\,dr\, dy^2 \cdots dy^n, \quad \sqrt{g} := \sqrt{\det(g_{\alpha\beta})}$$

⇓ by **some formulae**

$$\int_R^\infty |\nabla u|^2 \sqrt{g} dr \geq \int_R^\infty |\partial_r u|^2 \sqrt{g} dr$$

$$= \int_R^\infty |\partial_r(\sqrt{g}^{1/2} u)|^2 dr$$

$$+ \int_R^\infty \left\{ \frac{1}{4}(\Delta_g r)^2 - \frac{1}{2}|\nabla dr|^2 - \frac{1}{2}\mathrm{Ric}_g(\nabla r, \nabla r) \right\} u^2 \sqrt{g} dr$$

$$+ \frac{1}{2}\left(\Delta_g r \cdot u^2 \sqrt{g}\right)\Big|_{r=R}$$

Key point

Step 2 **Relative version of OHI** For $u \in C_c^\infty(M)$

$$\int_R^\infty |\partial_r(\sqrt{g}^{1/2} u)|^2 dr \geq \int_R^\infty \frac{(\sqrt{g}^{1/2} u)^2}{4r^2} dr - \frac{1}{2}\left(\frac{1}{R} u^2 \sqrt{g}\right)\Big|_{r=R}$$

($f = \sqrt{g}^{1/2} u$, $f = u^2\sqrt{g}$; extra bad term)

⇓

$$\int_R^\infty |\nabla u|^2 \sqrt{g} dr$$

$$\geq \int_R^\infty \left\{ \frac{1}{4r^2} + \frac{1}{4}(\Delta_g r)^2 - \frac{1}{2}|\nabla dr|^2 - \frac{1}{2}\mathrm{Ric}_g(\nabla r, \nabla r) \right\} u^2 \sqrt{g} dr$$

$$+ \frac{1}{2}\left(\Delta_g r - \frac{1}{R}\right)(u^2 \sqrt{g})\Big|_{r=R}$$

QED

$\Delta_g r = H_W$, $\quad H_W \geq \dfrac{1}{R}$

$\S 4$ **Applications**

GRHI on $(\mathbb{H}^n, h)$ $(n \geq 2)$

- $r(\bullet) := \mathrm{dist}_h(\bullet, \mathbb{O})$, For $u \in C_c^\infty(\mathbb{H}^n)$

$$\Rightarrow \quad \int_{\mathbb{H}^n - B_R(\mathbb{O})} |\nabla u|^2 dv_h \qquad \text{for} \quad \forall R > 0$$

$$\geq \int_{\mathbb{H}^n - B_R(\mathbb{O})} \left\{ \frac{(n-1)^2}{4} + \frac{1}{4r^2} + \frac{(n-1)(n-3)}{4\sinh^2 r} \right\} u^2 dv_h$$

- In particular, $\int_{\mathbb{H}^n} |\nabla u|^2 dv_h \geq \int_{\mathbb{H}^n} \{\cdots\} u^2 dv_h$

Thm H • $(M, g) = (\mathbb{H}^n, h)$, $n \geq 2$

Assume : $\sigma_{\text{ess}}(-\Delta_h + V) = [\frac{(n-1)^2}{4}, \infty)$

(i) $V(x) \geq -\left(\frac{1}{4r^2} + \frac{(n-1)(n-3)}{4\sinh^2 r} \right)$ for $r \geq \exists R_0 > 0$

$$\Rightarrow \quad \left| \sigma_{\text{disc}}(-\Delta_h + V) \right| < \infty$$

(ii) $V(x) \leq -(1 + \exists \delta) \frac{1}{4r^2}$ for $r \geq \exists R_1 > 0$

$$\Rightarrow \quad \left| \sigma_{\text{disc}}(-\Delta_h + V) \right| = \infty$$

GRHI on $(\mathbb{H}^n(-c), h_c)$ $(n \geq 2)$

- $r(\bullet) := \mathrm{dist}_{h_c}(\bullet, \mathbb{O})$, For $u \in C_c^\infty(\mathbb{H}^n(-c))$

$$\Rightarrow \quad \int_{\mathbb{H}^n(-c) - B_R(\mathbb{O})} |\nabla u|^2 dv_{h_c} \qquad \text{for} \quad \forall R > 0$$

$$\geq \int_{\mathbb{H}^n(-c) - B_R(\mathbb{O})} \left\{ \frac{(n-1)^2 c}{4} + \frac{1}{4r^2} + \frac{(n-1)(n-3)c}{4\sinh^2(\sqrt{c}r)} \right\} u^2 dv_{h_c}$$

$$\Downarrow \qquad c \searrow 0$$

CRHI on $\mathbb{R}^n$ For $u \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n - B_R(\mathbb{O})} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n - B_R(\mathbb{O})} \left\{ \frac{1}{4r^2} + \frac{(n-1)(n-3)}{4r^2} \right\} u^2 dx$$

$$= \int_{\mathbb{R}^n - B_R(\mathbb{O})} \frac{(n-2)^2}{4r^2} u^2 dx$$

Use GRHI as an approximation!

Thm ALE $(n \geq 3)$ (M^n, g) satisfies the following :

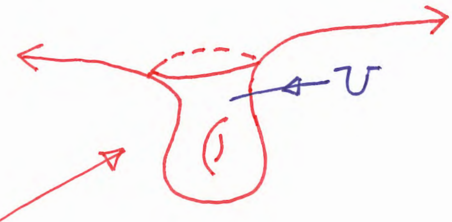
- $|\mathcal{R}_g| \leq L r^{-(2+\tau)}$, $|\nabla \mathrm{Ric}_g| \leq L' r^{-(3+\tau)}$ $(0 < \tau < 1)$ for large $r > 0$
- $\mathrm{Vol}(B_t(p_0)) \geq K t^n$ for $t > 0$
- $\sigma_{\mathrm{ess}}(-\Delta_g + V) = [0, \infty)$

(i) $\boxed{V(x) \geq -(1 - \exists\delta_0)\dfrac{(n-2)^2}{4r^2}}$ for $r \geq \exists R_0 > 0$ ($\delta_0 \to 0$)

$\Rightarrow$ $\Big|\sigma_{\mathrm{disc}}(-\Delta_g + V)\Big| < \infty$

(ii) $\boxed{V(x) \leq -(1 + \exists\delta_1)\dfrac{(n-2)^2}{4r^2}}$ for $r \geq \exists R_1 > 0$ ($\delta_1 \to 0$)

$\Rightarrow$ $\Big|\sigma_{\mathrm{disc}}(-\Delta_g + V)\Big| = \infty$



Note (M, g) : **ALE** manifold **of order** $\tau > 0$, i.e.,

Each end of $\boxed{M - U \cong \left(\mathbb{R}^n - B_R(\mathbb{O})\right)\big/\Gamma} \ni x = (x^1, \cdots, x^n)$

$g_{ij} = \delta_{ij} + O(|x|^{-\tau})$, $\partial_k g_{ij} = O(|x|^{-1-\tau})$, $\partial_k \partial_\ell g_{ij} = O(|x|^{-2-\tau})$

$\rightsquigarrow$ $\sigma_{\mathrm{ess}}(-\Delta_g) = [0, \infty)$

Use GRHI as an approximation!

Thm AH $(n \geq 2)$

(M^n, g) : **asymptotically hyperbolic** of class C^2 with

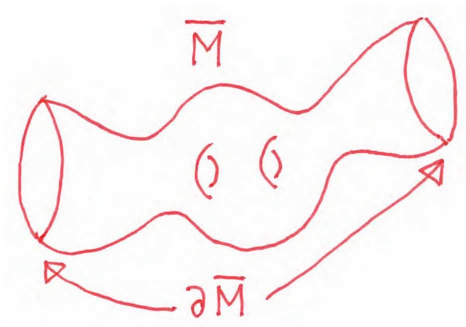
- $\sigma_{\text{ess}}(-\Delta_g + V) = [\frac{(n-1)^2}{4}, \infty)$

(i) $\boxed{V(x) \geq -(1 - \exists\delta_0)\frac{1}{4r^2}}$ for $r \geq \exists R_0 > 0$ ($\delta_0 \to 0$)

$\Rightarrow \quad \left|\sigma_{\text{disc}}(-\Delta_g + V)\right| < \infty$

(ii) $\boxed{V(x) \leq -(1 + \exists\delta_1)\frac{1}{4r^2}}$ for $r \geq \exists R_1 > 0$ ($\delta_1 \to 0$)

$\Rightarrow \quad \left|\sigma_{\text{disc}}(-\Delta_g + V)\right| = \infty$



Note (M, g) : **AH of class** C^2

$\Updownarrow$ **def.**

- $^\exists\overline{M}$: compact C^∞ manifold s.t. $\text{Int}(\overline{M}) = M$
- $^\exists$defining fct $\lambda \in C^\infty(\overline{M})$ of $\partial\overline{M}$ s.t.

  $\overline{g} := \lambda^2 \cdot g$: C^2 metric on $\overline{M}$ & $|d\lambda|^2_{\overline{g}} = 1$ on $\partial\overline{M}$

$\rightsquigarrow$ • $K_g(x) \to -1$ $(x \to \infty)$ • $\sigma_{\text{ess}}(-\Delta_g) = [\frac{(n-1)^2}{4}, \infty)$

by **R. Mazzeo** $\qquad x \to \partial\overline{M}$

Recall the below

Thm RSK (Reed–Simon, Kirsch–Simon)

- $(M, g) = \mathbb{R}^n, \quad n \geq 3$

(i) $$\boxed{V(x) \geq -\frac{(n-2)^2}{4r^2}} \quad \text{for} \quad r := |x| \geq \exists R_0 > 0$$

$$\Rightarrow \quad \left|\sigma_{\text{disc}}(-\Delta_g + V)\right| < \infty$$

(ii) $$\boxed{V(x) \leq -(1 + \exists\delta)\frac{(n-2)^2}{4r^2}} \quad \text{for} \quad r \geq \exists R_1 > 0$$

$$\Rightarrow \quad \left|\sigma_{\text{disc}}(-\Delta_g + V)\right| = \infty$$

$\Uparrow$ Key Lemma

Classical Relative Hardy Inequality (CRHI)

$$\boxed{\int_{\mathbb{R}^n - B_R(\mathbb{O})} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n - B_R(\mathbb{O})} \frac{(n-2)^2}{4r^2} u^2 dx}$$

for $R > 0$ & $u \in C_c^\infty(\mathbb{R}^n)$

Easy Proof of Thm RSK : Finiteness assertion (i)

Step 1 $(n \geq 3)$ For $u \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n - B_{R_0}(\mathbb{O})} |\nabla u|^2 dx \geq \int_{\mathbb{R}^n - B_{R_0}(\mathbb{O})} \frac{(n-2)^2 u^2}{4r^2} dx$$

$$\Downarrow \quad V(x) \geq -\frac{(n-2)^2}{4r^2} \quad \text{for} \quad r := |x| \geq R_0 > 0$$

$$\int_{\mathbb{R}^n - B_{R_0}(\mathbb{O})} (|\nabla u|^2 + V u^2) dx \geq 0 \quad \cdots \quad (*)$$

Step 2 Consider **Neumann eigenvalue problem**

for $-\Delta + V$ on $A := \overline{B_{R_0}(\mathbb{O})}$ & $B := \mathbb{R}^n - B_{R_0}(\mathbb{O})$

$\mu_1 < \mu_2 \leq \cdots$: union of all eigenvalues on both A and B

$\lambda_1 < \lambda_2 \leq \cdots$: **negative** eigenvalues on $\mathbb{R}^n$

$\Downarrow$ **Domain monotonicity of eigenvalues**

$$\mu_j \leq \lambda_j < 0 \quad \text{for} \quad \forall j \geq 1$$

By $(*)$ and A : compact

$$\rightsquigarrow \left|\{\mu_j < 0\}\right| < \infty \rightsquigarrow \left|\{\lambda_j < 0\}\right| < \infty \quad \boxed{\text{QED}}$$