An inclusive immersion into a quaternion manifold and its invariants

Kazuyuki HASEGAWA

Kanazawa University (Japan)

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Motivation.

Surfaces in 4-spaces :

$(M, \mathrm{or}, [g])$: 4-dim. oriented mfd. with conformal str. $[g]$
Σ : oriented surface
$f : \Sigma \to M$: immersion

are investigated very well.

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are investigated very well.
In particular,

"conformal properties and invariant"

are focused on, e.g.,

Willmore functional, twistor holomorphic immersion, etc.

Since $(\mathrm{or}, [g]) =$ quaternion structure, it is interesting to study

(M, Q) : quaternion mfd. with quaternion str. Q,
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• extrinsic invariants and properties w.r.t. quaternion str. Q are our interest.

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• extrinsic invariants and properties w.r.t. quaternion str. Q are our interest.
• If $\dim M = 4$, then $Q = (\mathrm{or}, [g])$, and hence, such invariants are *conformal* ones.

In this talk, we consider

- one candidate for quaternion object of the Willmore func. is introduced.
- relation to twistorical object,
- lower bound,
- critical points (corresponding to Willmore immersion)

Quaternion manifolds and twistor spaces.

Definition 2.1

(M, Q) : *quaternion mfd.*

$:\Longleftrightarrow$

(i) $Q \subset \mathrm{End}(TM)$ *with* $\mathrm{rank}\, Q = 3$,

(ii) Q *is locally spaned by sections* I_1, I_2, I_3 *with* $\mathbb{H}$*-relations,*

(iii) $\exists \nabla$: *a torsion free affine connection s.t.* $\nabla \Gamma(Q) \subset \Gamma(Q)$.

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- quaternion connection is *not* unique.
- (I_1, I_2, I_3) is called a *admissible frame*.

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We set

$$\dim M = 4n$$

$$\mathcal{Z}_x := \{J \in Q_x \mid J^2 = -id\}$$
$$\mathcal{Z} := \bigcup_{x \in M} \mathcal{Z}_x$$
$$\pi_{tw} : \mathcal{Z} \to M : \text{ bundle projection}$$

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On $\mathcal{Z}$, we can define an almost complex str. $I^{\mathcal{Z}}_{\nabla}$ as follows:
(i) decompose $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$
(ii) define on each space by

$$(I^{\mathcal{Z}}_{\nabla})_J(X) = (J(p_*(X)))^h_J$$

for all $X \in \mathcal{H}_J$ at $J \in \mathcal{Z}(\tilde{M})$ and

$$(I^{\mathcal{Z}}_{\nabla})_J(Y) = \mathcal{J}(Y)$$

for all $Y \in \mathcal{V}_J$ at $J \in \mathcal{Z}(\tilde{M})$.
- $(\,\cdot\,)^h$ stands for the horizontal lift
- $\mathcal{J}$ is the standard complex structure on each fiber ($\cong S^2$)

Lemma 2.2

If connections ∇^1 and ∇^2 are q-connections, then $I^{\mathcal{Z}}_{\nabla^1} = I^{\mathcal{Z}}_{\nabla^2}$.

Then we are allowed to write $I^{\mathcal{Z}}$ for $I^{\mathcal{Z}}_{\nabla}$ with no confusions in *quaternion geometry*.

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Lemma 2.3

$I^{\mathcal{Z}}(= I^{\mathcal{Z}}_{\nabla})$ is always integrable if $n \geq 2$. When $n = 1$, $I^{\mathcal{Z}}$ is integrable iff $Q = (\mathrm{or}, [g])$ is anti-self-dual (ADS).

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Definition 2.4

We call $(\mathcal{Z}(M), I^{\mathcal{Z}})$ the twistor space of M.

Inclusive immersions.

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Definition 3.1

f is inclusive

$:\Longleftrightarrow$

$f_{*x}(T_x\Sigma)$ is contained in real 4-dim. quaternion subspace of $T_{f(x)}M$ for each $x \in \Sigma$.

- If f is inclusive, then there exists unique $I_1 : \Sigma \to \mathcal{Z}$ s.t.

(i) $I_1(f_*(T\Sigma)) \subset f_*(T\Sigma)$ (so the complex str. I on Σ is induced)
(ii) the induced cpx. str. I is compatible with the orientation of Σ.

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Remark 3.2

When $n = 1$, that is $Q = (\mathrm{or}, [g])$, then any immersions are inclusive and f^*g is compatible with I. Therefore, when $n = 1$,

$$\text{inclusive immersion} = \text{conformal immersion}$$

Here we summarize our setting :

$\dim M = 4n$	$n = 1$	$n \geq 2$
structure	conformal	quaternion
integrability of $I^{\mathcal{Z}}$	anti-self-dual	always
immersion from a surface	conformal	inclusive
invariant	conformal	quaternion

Table: setting

An invariant.

Hereafter assume that Σ is compact

$f : \Sigma \to M$: inclusive immersion
∇ : q-connection
$f^{\#}\nabla$: the pull-back conn. of ∇ on $f^{\#}TM$.

To introduce an quaternion invariant for an inclusive immersion, we need some notations/definitions :

(1) Operators on $\mathrm{End}(f^{\#}TM)$

$$A_X^{f^{\#}\nabla\prime} := \frac{1}{4}\left(I_1(f^{\#}\nabla)_X I_1) + (f^{\#}\nabla)_{IX} I_1\right)$$
$$A_X^{f^{\#}\nabla\prime\prime} := \frac{1}{4}\left(I_1(f^{\#}\nabla)_X I_1) - (f^{\#}\nabla)_{IX} I_1\right)$$

for $X \in TM$.

(2) 2-form $a_\Omega(s)\Omega$:

On a complex manifold Σ of $\dim_{\mathbf{R}} \Sigma = 2$, we can choose an area form Ω on Σ, which satisfies $\Omega(X, IX) \neq 0$ for all nonzero $X \in T\Sigma$.

Definition 4.1

For a symmetric $(0, 2)$-tensor s on Σ, we define

$$a_\Omega(s) = \frac{s(X, X) + s(IX, IX)}{\Omega(X, IX)}$$

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for a nonzero $X \in T\Sigma$.

- $a_\Omega(s)\Omega = a_{\Omega'}(s)\Omega'$ if $\Omega' = c\Omega$ for $c \neq 0$.

(3) hermitrazation :
θ : $(0,2)$-tensor on M

$$\Pi_h(\theta)(X, Y) := \frac{1}{4}\left(\theta(X, Y) + \sum_{i=1}^{3} \theta(I_i X, I_2 Y)\right)$$

for X, $Y \in TM$.

(4) θ^s : symmetrization of θ

(5) Ric^{∇} : Ricci tensor of ∇

Definition 4.2

For an inclusive immersion $f:\Sigma\to M$, we define

$$\mathcal{W}_Q(f) := \frac{1}{2}\int_\Sigma a_\Omega\{f^*(Ric^\nabla)^s - \frac{2}{n+2}f^*(\Pi_h(Ric^\nabla)^s) \\ -(\mathrm{Tr}\, A^{f^\#\nabla\prime}_{(\ \cdot\ )} A^{f^\#\nabla\prime}_{(\ \cdot\ )})\}\Omega.$$

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Theorem 4.3

For an inclusive immersion $f : \Sigma \to M$,

$$\mathcal{W}_Q(f) \text{ is a quaternion invariant,}$$

that is, it is independent of the choice of q-connections.

Remark 4.4

Assume that M is quaternion Kähler with $n \geq 2$ or Q =ASD and g =Einstein $(n = 1)$. If f is inclusive, then

$$\mathcal{W}_Q(f) = \frac{Sc}{4(n+2)} Area(\Sigma, f^*g) + \frac{1}{n} \int_\Sigma \|H\|^2 \Omega,$$

and f^*g is compatible with I. In particular, if M =(4-dim. space form of constant curvature),

$$\mathcal{W}_Q = \text{(Willmore functional)}.$$

Definition 4.5

If the natural twistor lift I_1 is holomorphic, then f is called a twistor holomorphic (t-hol.).

- The property that f is t-hol. is independent of the choice of q-connections.
- f is t-hol. $\iff A^{f^{\#}\nabla''} = 0$.
- When M is QK, f is minimal $\iff A^{f^{\#}\nabla'} = 0$.

Theorem 4.6

Assume that $n \geq 2$ or $Q =$ASD $(n = 1)$. Then we have

$$\mathcal{W}_Q(f) \geq 2\pi \int_\Sigma c_1(f^\# TM, I_1).$$

The equality holds if and only if f is twistor holomorphic.

Theorem 4.6

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The equality holds if and only if f is twistor holomorphic.

In general, when we study
  (M, S) : mfd. with geometric str. S
  Σ : submanifold in M,
one of elementary approach to study Σ is
  (i) find an (extrinsic) invariant of Σ w.r.t S,
  (ii) give a lower or upper bound for it,
  (iii) characterize the equality case of (ii)

One of the advantage of considering “the twistor space” for the study of quaternion structures is that “complex geometry” can be applied.

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Hence, for inclusive surfaces in quaternion manifolds, it is important to give a relation among the quaternion invariants and complex geometric objects.

In particular, if the twistor lift of an inclusive immersion from a compact surface into the quaternion projective space $\mathbb{H}P^n$ is holomorphic, its image is an algebraic curve in the complex projective space $\mathbb{C}P^{2n+1}$.

Theorem 4.7

If

$f : \Sigma \to \mathbb{H}P^n$ *is a t-hol. inclusive immersion*

$d =$*the degree of the image* $I_1(\Sigma) \subset \mathbb{C}P^{2n+1}$

then we have

$$\mathcal{W}_Q(f) = 4\pi n d.$$

Example 4.8

Consider the Veronese map

$$\mathbb{C}P^1 \ni [W_0, W_1] \mapsto [W_0^{2n+1}, W_0 W_1^{2n}, \cdots, W_1^{2n+1}] \in \mathbb{C}P^{2n+1}.$$

Its image is a nondegenerate curve of degree $2n+1$, which is called the rational normal curve. Then the twistor projection of this curve is a twistor holomorphic (nondegenerate) surface with

$$\mathcal{W}_Q(f) = 4\pi n(2n+1).$$

We give an applications of Theorem 4.7.

Consider the quotient bundle $N := f^{\#}TM/T\Sigma$ and define the complex structure I^N on N by $I^N([\xi]) = [I_1\xi]$ for $\xi \in \Gamma(f^{\#}TM)$.

Corollary 4.9

If

$f : \Sigma \to \mathbb{H}P^n$ *is a t-hol inclusive immersion*

$d =$*the degree of the image* $I_1(\Sigma) \subset \mathbb{C}P^{2n+1}$

$q =$*genus of* Σ

then we have

$$\int_\Sigma c_1(N) = 2(nd + q - 1).$$

Friedrich proves the following [Ann. Global Anal. Geom. 1984] :

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Fact 4.10

If $f : \Sigma \to S^4 (\cong \mathbb{H}P^1)$ is a t-hol. conformal immersion, then the Euler class of the normal bundle is non negative. Moreover, its Euler class vanishes if and only if f is totally umblic.

Therefore Corollary 4.9 is a generalized and an improved result of his result as above.

Quaternion Willmore immersions

We consider critical points of $\mathcal{W}_Q$.

Definition 5.1

We say that f is quaternion Willmore (resp. constrained quaternion Willmore) if

$$\frac{d}{dt}\mathcal{W}_Q(f_t)\Big|_{t=0} = 0$$

for any variation $\{f_t\}_{t\in J}$ of $f = f_0$ such that f_t is inclusive for each $t \in J$ (resp. f_t is inclusive for each $t \in J$ and the induced complex structure on Σ does not vary).

- An explicit expression of the first variation formula has been obtained.

Example 5.2

Any twistor holomorphic immersions are quaternion Willmore.

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If f is an immersion into 4-dim space form of constant curvature with holomorphic mean curvature vector field, then f is constrained Willmore.

• f is constrained Willmore if f is a stationary point of the Willmore functional under any variations of f such that the induced conformal structure do not vary.

In [H-, J. Geom. Phys. 57 (2007)],

Fact 5.5

Let f be an immersion into 4-dim space form with the natural twistor lift l_1. Then the mean curvature vector field is holomorphic if and only if l_1 is a harmonic section.

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Fact 5.5

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• ξ is a harmonic section (or vertically harmonic) of a Riemanian vector bundle if a stationary point the restricted energy functional to the space of sections with unit length.

Theorem 5.6

If

(M, Q, g) *: QK mfd. with* $n \geq 2$ *or* $Q =$*ASD,* $g =$*Einstein*

I_1 *: a harmonic section*

then

f *is constrained quaternion Willmore.*

This theorem is a generalization and a quaternion version of [Burstall and Calderbank,2010] above.

Remark.

- $\mathcal{W}_Q > 0$ when $M = \mathbb{H}P^n$. Find a positive constant C s.t. $\mathcal{W}_Q \geq C$ ($C = 4\pi n$? When $n = 1$, it is true).

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Thank you for your attention.