Quaternionic Holomorphic Geometry: transformations of minimal surfaces

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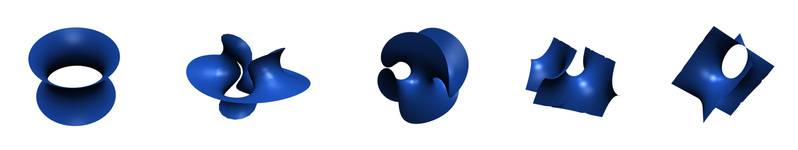

Overview

1. Minimal surfaces and holomorphic data

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4. Transformations
    - The left and right associated family
    - Simple factor dressing
    - López-Ros deformation

Minimal surfaces

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$$H = 0 \iff \quad f : M \to \mathbb{R}^3 \text{ is harmonic, i.e. } \Delta f = 0\,.$$

Weierstrass representation

Given $g : M \to \mathbb{C}$ meromorphic, $\omega \in \Omega^1(M, \mathbb{C})$ meromorphic on simply connected M, then

$$\Phi = \int \left(\frac{1}{2}(1 - g^2)\omega, \frac{\mathbf{i}}{2}(1 + g^2)\omega, g\omega \right)$$

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is a holomorphic null curve, and $f = \mathrm{Re}\,\Phi$ is a minimal surface in $\mathbb{R}^3$ with Gauss map given by the stereographic projection of g:

$$N = \frac{1}{1 + |g|^2}(2\mathrm{Re}\,g, 2\mathrm{Im}\,g, |g|^2 - 1)$$

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$$\omega = d\Phi_1 - \mathbf{i} d\Phi_2, \quad g = \frac{d\Phi_3}{d\Phi_1 - \mathbf{i} d\Phi_2}$$

The associated family

Let $f : M \to \mathbb{R}^3$ be minimal with conjugate minimal surface f^*. Then

$$f_{\cos t, \sin t} = f \cos t + f^* \sin t, \quad t \in \mathbb{R},$$

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is called the associated family of f. The minimal surfaces f_t are isometric to f.

The López-Ros deformation

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Let $f : M \to \mathbb{R}^3$ be minimal on simply connected M with Weierstrass data (g, ω). Then for $z \in \mathbb{C}_*$ the pair

$$(zg, \frac{\omega}{z})$$

is the Weierstrass data of a minimal immersion $f_z : M \to \mathbb{R}^3$.

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Note that from $d\Phi_3 = g\omega$ we see that this deformation does not change the third coordinate

$$(f_z)_3 = f_3$$

for all $z \in \mathbb{C}_*$.

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$\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ with inner product given by

$$ab = - < a, b > + a \times b$$

for $a, b \in \mathbb{R}^3 = \operatorname{Im}\mathbb{H}$.

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\begin{aligned}
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f \text{ is minimal} \quad &\iff \quad *dN = -NdN \\
f^* \text{ is conjugate of } f \quad &\iff \quad *df = -df^*
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$$S = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & -N \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}^{-1}$$

on $M \times \mathbb{H}^2$.

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on $M \times \mathbb{H}^2$. The Hopf field A of the conformal Gauss map is defined by

$$-2 * A = \frac{1}{2}(dS - S * dS)\,.$$

If f is minimal then

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Thus: the Gauss map of N is harmonic.

The associated family of the Gauss map

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Put $Q = -\frac{1}{2} * dN$. Then

$$d_\lambda = d + (\lambda - 1)Q^{1,0} + (\lambda^{-1} - 1)Q^{0,1}$$

is a flat connection on $M \times \mathbb{C}^2$ for all $\lambda \in \mathbb{C}_*$.

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Let $\mu \in \mathbb{C}_*$ and $a = \frac{\mu+\mu^{-1}}{2}, b = i\frac{\mu^{-1}-\mu}{2}$. Then

$$d_\mu \beta = 0 \iff \beta = Nm + m\frac{b}{a-1}\,, m \in \mathbb{H}\,.$$

The conformal Gauss map and its associated family

Let $f : M \to \mathbb{R}^3$ be minimal then its conformal Gauss map $N : M \to S^2$ is harmonic since

$$d(S * dS) = 4d * A = 0\,.$$

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$$d_\mu \varphi = 0 \quad \Longleftrightarrow \quad \varphi = \begin{pmatrix} n \\ 0 \end{pmatrix}, n \in \mathbb{H}, \text{ or}$$

$$\varphi = \begin{pmatrix} \alpha + f\beta \\ \beta \end{pmatrix} \text{ with } \beta = Nm + m\frac{b}{a-1}\,, \text{ and}$$

$$\alpha = -f^* m - fm\frac{b}{a-1}\,, m \in \mathbb{H}\,.$$

The left and right associated family

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A surface $f_{p,q}$ (or $f^{p,q}$) is isometric to f if and only if it is an element of the classical associated family, up to an isometry of $\mathbb{R}^4$.

Simple factor dressing (Terng–Uhlenbeck)

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Put

$$r_\lambda = \begin{cases} \frac{\lambda-\mu}{\lambda-\bar{\mu}^{-1}} \frac{1-\bar{\mu}^{-1}}{1-\mu} & \text{on } \beta\mathbb{C} \\ 1 & \text{on } (\beta\mathbb{C})^\perp \end{cases}$$

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Then the gauge

$$\hat{d}_\lambda = r_\lambda \cdot d_\lambda$$

of d_λ by r_λ is the associated family of a harmonic map $\hat{N} : M \to S^2$, the simple factor dressing of N.

Simple factor dressing of a minimal surface

Using an explicit formula for the simple factor dressing given by [BDLQ] we obtain

Lemma (L-Moriya)

In the case of the harmonic Gauss map $N : M \to S^2$ of a minimal surface $f : M \to \mathbb{R}^3$, the simple factor dressing of N is given by

$$\hat{N} = (N + \rho)N(N + \rho)^{-1}$$

with $\rho = m\frac{b}{a-1}m^{-1}$, $a = \frac{\mu+\mu^{-1}}{2}$, $b = i\frac{\mu^{-1}-\mu}{2}$, $m \in \mathbb{H}$.

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Let $f : M \to \mathbb{R}^4$ be Willmore with harmonic conformal Gauss map S. Let d_λ be the associated family of S and assume that $M = (\varphi_1, \varphi_2)$ is invertible where φ_i are d_μ–parallel sections.

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Then

$$\hat{S} = T^{-1} S T$$

is the conformal Gauss map of a Willmore surface $\hat{f} : M \to \mathbb{R}^4$ where

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We call $\hat{S}$ the simple factor dressing of S.

Simple factor dressing of a minimal surface

Theorem (L–Moriya)

Let $f : M \to \mathbb{R}^3$ be minimal and $\hat{f}$ the Willmore surface given by the simple factor dressing of the conformal Gauss map S of f given by

$$\varphi_1 = \begin{pmatrix} n \\ 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} \alpha + f\beta \\ \beta \end{pmatrix},$$

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Then $\hat{f} : M \to \mathbb{R}^4$ is minimal.

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Then $\hat{f} : M \to \mathbb{R}^4$ is minimal. We call $\hat{f}$ a simple factor dressing of f with parameters (μ, m, n).
Moreover,

- *f is complete if and only if $\hat{f}$ is complete.*

Simple factor dressing of a minimal surface

Theorem (L–Moriya)

Let $f : M \to \mathbb{R}^3$ be minimal and $\hat{f}$ the Willmore surface given by the simple factor dressing of the conformal Gauss map S of f given by

$$\varphi_1 = \begin{pmatrix} n \\ 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} \alpha + f\beta \\ \beta \end{pmatrix},$$

where $\alpha = -f^*m - fm\frac{b}{a-1}$ and $\beta = Nm + m\frac{b}{a-1}$, $n, m \in \mathbb{H}_*$.

Then $\hat{f} : M \to \mathbb{R}^4$ is minimal. We call $\hat{f}$ a simple factor dressing of f with parameters (μ, m, n).
Moreover,

- *f is complete if and only if $\hat{f}$ is complete.*
- *f has finite total curvature if and only if $\hat{f}$ has finite total curvature.*

Simple factor dressing

Theorem (L, Moriya)

Let $f : M \to \mathbb{R}^3$ be minimal. Then the simple factor dressing $\hat{f}$ of f with parameters (μ, m, m) is a minimal surface in $\mathbb{R}^3$.

Simple factor dressing

Theorem (L, Moriya)

Let $f : M \to \mathbb{R}^3$ be minimal. Then the simple factor dressing $\hat{f}$ of f with parameters (μ, m, m) is a minimal surface in $\mathbb{R}^3$.

The Gauss map of $\hat{f}$ is the simple factor dressing

$$\hat{N} = (N + \rho)N(N + \rho)^{-1}$$

of the Gauss map N of f where $\rho = m\frac{b}{a-1}m^{-1}$, $a = \frac{\mu+\mu^{-1}}{2}$, $b = i\frac{\mu^{-1}-\mu}{2}$.

The López-Ros deformation

Theorem

Let $f = (f_1, f_2, f_3) : M \to \mathbb{R}^3$ *be a minimal surface in* $\mathbb{R}^3$ *with conjugate surface* $f^* = (f_1^*, f_2^*, f_3^*)$.

The López-Ros deformation

Theorem

Let $f = (f_1, f_2, f_3) : M \to \mathbb{R}^3$ *be a minimal surface in* $\mathbb{R}^3$ *with conjugate surface* $f^* = (f_1^*, f_2^*, f_3^*)$*. The López-Ros deformation* f_z *with complex parameter* $z = e^{x+iy} \in \mathbb{C}_*$ *is given by*

$$f_z = \begin{pmatrix} -(f_1 \cosh x - f_2^* \sinh x)\cos y - (f_2 \cosh x + f_1^* \sinh x)\sin y \\ (f_1 \cosh x - f_2^* \sinh x)\sin y - (f_2 \cosh x + f_1^* \sinh x)\cos y \\ f_3 \end{pmatrix}.$$

The López-Ros deformation

Theorem

Let $f = (f_1, f_2, f_3) : M \to \mathbb{R}^3$ *be a minimal surface in* $\mathbb{R}^3$ *with conjugate surface* $f^* = (f_1^*, f_2^*, f_3^*)$. *The López-Ros deformation* f_z *with complex parameter* $z = e^{x+iy} \in \mathbb{C}_*$ *is given by*

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Moreover, the López-Ros deformation with complex parameter z *is the simple factor dressing of* f *with parameters* (μ, m) *for* $\mu = \mu(z) \in \mathbb{C}_*$ *and* $m = 1 - i - j - k$.

The López-Ros deformation

Theorem

Let $f = (f_1, f_2, f_3) : M \to \mathbb{R}^3$ be a minimal surface in $\mathbb{R}^3$ with conjugate surface $f^* = (f_1^*, f_2^*, f_3^*)$. The López-Ros deformation f_z with complex parameter $z = e^{x+iy} \in \mathbb{C}_*$ is given by

$$f_z = \begin{pmatrix} -(f_1 \cosh x - f_2^* \sinh x) \cos y - (f_2 \cosh x + f_1^* \sinh x) \sin y \\ (f_1 \cosh x - f_2^* \sinh x) \sin y - (f_2 \cosh x + f_1^* \sinh x) \cos y \\ f_3 \end{pmatrix} .$$

Moreover, the López-Ros deformation with complex parameter z is the simple factor dressing of f with parameters (μ, m) for $\mu = \mu(z) \in \mathbb{C}_*$ and $m = 1 - i - j - k$.

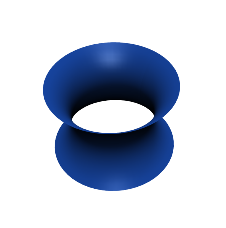
All simple factor dressings $\hat{f}$ with parameter (μ, m, m) are given by a rotation R_m in $\mathbb{R}^3$ as

$$\hat{f} = R_m^{-1} f_z R_m$$

where z depends on μ.

Examples

Catenoid

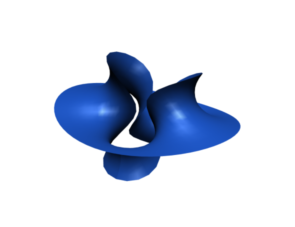


$f = \mathrm{Re}\,\Phi$ where

$$\Phi(z) = \begin{pmatrix} z \\ \cosh z \\ -\mathbf{i}\sinh z \end{pmatrix}, \quad z \in \mathbb{C}\,.$$

Examples

Examples with one planar end



$f = \mathrm{Re}\,\Phi$ where

$$\Phi = \begin{pmatrix} \frac{1}{2}\left(-\frac{1}{z} - \frac{z^{2n+1}}{2n+1}\right) \\ \frac{i}{2}\left(-\frac{1}{z} + \frac{z^{2n+1}}{2n+1}\right) \\ \frac{z^n}{n} \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{0\} = \mathbb{C}_* \,.$$

Scherk's first surface

$f = \mathrm{Re}\,\Phi$ where

$$\Phi(z) = \begin{pmatrix} \mathbf{i}\log(\frac{z+\mathbf{i}}{z-\mathbf{i}}) \\ \mathbf{i}\log(\frac{z+1}{z-1}) \\ \log(\frac{z^2+1}{z^2-1}) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{\pm 1, \pm \mathbf{i}\}.$$

Thanks!