Results on Real Hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$

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Contents I

Definition

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$M_n(c), c \neq 0 \longrightarrow$ Non-Flat Complex Space Form

Definition

Almost contact structure *or* (φ, ξ, η) *-* ***structure*** *is a tensor field* φ *of type (1,1), a vector field* ξ *and a 1-form* η*, which satisfy the following relations*

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Gauss and Weingarten equations

$$\overline{\nabla}_Y X = \nabla_Y X + g(AY, X)N,$$

$$\overline{\nabla}_X N = -AX,$$

where ∇ is the Levi-Civita connection of M, A is the shape operator of M and g the induced Riemannian metric on M.

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$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \varphi = 0, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \;\; g(X, \varphi Y) = -g(\varphi X, Y),$$

$$\nabla_X \xi = \varphi AX, \qquad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

for any tangent vector field X, Y on M.

Gauss equation:

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X$$

$$-g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z] + g(AY,Z)AX - g(AX,Z)AY$$

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Codazzi equation:

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X,Y)\xi],$$

where R denotes the Riemannian curvature tensor on M.

The tangent space $T_P M$: $T_P M = \text{span}\{\xi\} \oplus \mathbb{D}$, $\mathbb{D} = \ker(\eta) = \{X \in T_P M : \eta(X) = 0\}$ (**holomorphic distribution**).

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Definition

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If M is a Hopf hypersurface then α is constant

Complex Projective Space $\mathbb{C}P^n, n \geq 2$

Takagi (1973) [23], Cecil, Ryan (1982) [2], Wang (1983) [25], Kimura (1986) [6]

- $(A1)$ geodesic spheres of radius r, where $0 < r < \frac{\pi}{2}$,
- $(A2)$ tubes of radius r over totally geodesic complex projective space $\mathbb{C}P^k$, where $0 < r < \frac{\pi}{2}$ and $1 \leq k \leq n-2$,
- (B) tubes of radius r over complex quadrics and $\mathbb{R}P^n$, where $0 < r < \frac{\pi}{4}$,
- (C) tubes of radius r over the Serge embedding of $\mathbb{C}P^1 \times \mathbb{C}P^{\frac{(n-1)}{2}}$, where $0 < r < \frac{\pi}{4}$ and $n \geq 2\kappa + 3$, $\kappa \;\epsilon\; \mathbb{N}^*$,

- (D) tubes of radius r over the *Plucker* embedding of the complex Grassmannian manifold $G_{2,5}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,
- (E) tubes of radius r over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.

Complex Hyperbolic Space $\mathbb{C}H^n, n \geq 2$

Montiel (1985) [7], Berndt (1989) [1]

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- (A0) horospheres
- (A1,0) geodesic spheres of radius $r > 0$,
- (A1,1) tubes of radius $r > 0$ over totally geodesic complex hyperbolic hyperplanes $\mathbb{C}H^{n-1}$,
- (A2) tubes of radius $r > 0$ over totally geodesic submanifold $\mathbb{C}H^k$, $1 \leq k \leq n-2$,
- (B) tubes of radius $r > 0$ over totally real hyperbolic space $\mathbb{R}H^n$.

Case of $\mathbb{C}P^2$

A three dimensional real hypersurface M is locally congruent to

- (A1) a geodesic sphere of radius r, where $0 < r < \frac{\pi}{2}$,
- (B) a tube of radius r over complex quadrics and $\mathbb{R}P^n$, where $0 < r < \frac{\pi}{4}$.

Type	α	λ	ν	m_α	m_λ	m_ν
A1	$2\cot 2r$	$\cot r$	-	1	2	-
B	$2\cot 2r$	$\cot(r - \frac{\pi}{4})$	$-\tan(r - \frac{\pi}{4})$	1	1	1

Case of $\mathbb{C}H^2$

A three dimensional real hypersurface M is locally congruent to

- (A0) a horosphere
- (A1,0) a geodesic sphere of radius $r > 0$,
- (A1,1) a tube of radius $r > 0$ over totally geodesic complex hyperbolic hyperplanes $\mathbb{C}H^1$,
- (B) a tube of radius $r > 0$ over totally real hyperbolic space $\mathbb{R}H^2$.

Type	α	λ	ν	m_α	m_λ	m_ν
A0	2	1	-	1	2	-
A1,0	$2\coth(2r)$	$\coth(r)$	-	1	2	-
A1,1	$2\coth(2r)$	$\tanh(r)$	-	1	2	-
B	$2\tanh(2r)$	$\tanh(r)$	$\coth(r)$	1	1	1

Case of $\mathbb{C}P^n$, $n \geq 2, \longrightarrow$ Okumura, [9]

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Theorem

Let M be a real hypersurface in $M_n(c)$, $n \geq 2$, then $\varphi A = A\varphi$ if and only if M is an open subset of real hypersurfaces of type A.

Structure Jacobi Operator

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$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi$$

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Covariant Derivative of Structure Jacobi Operator

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Lie Derivative of Structure Jacobi Operator

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ξ-Parallel Structure Jacobi Operator

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QUESTION

Do there exist real hypersurfaces equipped with ξ-parallel structure Jacobi operator?

Theorem [14]

Let M be a real hypersurface in $M_2(c)$, whose structure Jacobi operator is ξ-parallel. Then

- in case of $\mathbb{C}P^2$, M is locally congruent to either a geodesic sphere of radius r, where $0 < r < \frac{\pi}{2}$, or a non-homogeneous real hypersurface, which is considered as a tube of radius $r = \frac{\pi}{4}$ over a holomorphic curve in $\mathbb{C}P^2$
- in case of $\mathbb{C}H^2$, M is locally congruent to a horosphere, or to a geodesic sphere, or to a tube over $\mathbb{C}H^1$, or to a Hopf hypersurface with $A\xi = 0$.

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Lie $\mathbb{D}$-Parallel Structure Jacobi Operator

Perez and Santos, (2005) [16]: Non-existence of real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, equipped with Lie-parallel structure Jacobi operator, i.e. $(\mathcal{L}_X l)Y = 0$, with $X, Y \in TM$.

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Theorem [-, Xenos 2012] [12]

There are no real hypersurfaces in $M_2(c)$ equipped with Lie $\mathbb{D}$-parallel structure Jacobi operator.

QUESTION

Do there exist real hypersurfaces whose structure Jacobi operator is Lie $\mathbb{D}$-parallel?

Theorem [-, Xenos 2012] [12]

There are no real hypersurfaces in $M_2(c)$ equipped with Lie $\mathbb{D}$-parallel structure Jacobi operator.

Theorem [Perez and Suh] [22]

There do not exist Hopf real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, equipped with Lie $\mathbb{D}$-parallel structure Jacobi operator.

Pseudo-parallel Structure Jacobi Operator

Definition

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$$(R(X,Y) \cdot P)(X_1, ..., X_s) = R(X,Y)(P(X_1, ..., X_s) - \sum_{j=1}^{s} P(X_1, ..., R(X,Y)X_j, ..., X_s).$$

Perez and Santos (2009)[20]: Non-existence of real hypersurfaces in $\mathbb{C}P^n$, whose structure Jacobi operator is semi-parallel, i.e. $R \cdot l = 0$.

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The structure Jacobi operator is called pseudo-parallel, i.e. $R(X,Y) \cdot l = L\{(X \wedge Y) \cdot l\}$, *where* L *is a function.*

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$$\begin{aligned} R(X,Y)lZ - l(R(X,Y)Z)) &= L\{g(lZ,Y)X - g(lZ,X)Y \\ &\quad -l(g(Z,Y)X) + l(g(Z,X)Y)\} \end{aligned}$$

Theorem [-, Xenos 2012] [13]

Let M be a real hypersurface in $M_2(c)$, whose structure Jacobi operator is pseudo-parallel. Then

- in case of $\mathbb{C}P^2$, M is locally congruent to either a geodesic sphere of radius r, where $0 < r < \frac{\pi}{2}$, or a non-homogeneous real hypersurface, which is considered as a tube of radius $r = \frac{\pi}{4}$ over a holomorphic curve in $\mathbb{C}P^2$
- in case of $\mathbb{C}H^2$, M is locally congruent to a horosphere, or to a geodesic sphere, or to a tube over $\mathbb{C}H^1$, or to a Hopf hypersurface with $A\xi = 0$.

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Relation $\mathcal{L}_\xi l = \nabla_\xi l$

Perez and Santos (2009) [21]: Classified real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose structure Jacobi operator satisfies the relation $\mathcal{L}_\xi l = \nabla_\xi l$.

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Theorem

Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$, whose structure Jacobi operator satisfies $\mathcal{L}_\xi l = \nabla_\xi l$. Then M is locally congruent:

- either to a tube of radius $\frac{\pi}{4}$ over a complex submanifold of $\mathbb{C}P^n$,
- or a tube of radius $r \neq \frac{\pi}{4}$ over $\mathbb{C}P^k$, where $0 \leq k \leq n-1$.

Theorem [-,Xenos 2012] [11]

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Relation $\mathcal{L}_X l = \nabla_X l,\ X \in \mathbb{D}$

Relation

$$(\mathcal{L}_X l)Y = (\nabla_X l)Y, X \in \mathbb{D}$$

$$\Downarrow$$

Relation $\mathcal{L}_X l = \nabla_X l$, $X \in \mathbb{D}$

Relation

$$(\mathcal{L}_X l)Y = (\nabla_X l)Y, X \in \mathbb{D}$$

⇩

Theorem [15]

There do not exist real hypersurfaces in $M_2(c)$, whose structure Jacobi operator satisfies the above relation.

Lie recurrence

Definition

A tensor field P of type (1,1) is called ***recurrent****, if there exists a 1-form ω such that $(\nabla_X P) = \omega(X)P(Y)$, where X, Y are tangent to M.*

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Theofanidis and Xenos (2012) [24]: Non-existence of real hypersurfaces in $M_n(c)$, $n \geq 3$, whose structure Jacobi operator is recurrent.

Relation

$$(\mathcal{L}_X l)Y = \omega(X) lY, X,\ Y\ \in\ TM$$

⇓

Relation

$$(\mathcal{L}_X l)Y = \omega(X)lY, X,\ Y\ \in\ TM$$

Theorem [5]

There do not exist real hypersurfaces in $M_2(c)$, whose structure Jacobi operator is Lie recurrent.

1^{st} Step: Real Hypersurface is Hopf

We consider a local orthonormal basis $\{U, \varphi U, \xi\}$.

$$A\xi = \alpha\xi + \beta U,$$

α, β functions.

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$\mathcal{V} \cup \Omega$ is open and dense in the closure of $\mathcal{N}$.

Lemma 1

Let M be a non-Hopf real hypersurface in $M_2(c)$. Then the following relations hold on M

$$AU = \gamma U + \delta\varphi U + \beta\xi, \;\; A\varphi U = \delta U + \mu\varphi U, \;\; A\xi = \alpha\xi + \beta U,$$

$$\nabla_U \xi = -\delta U + \gamma\varphi U, \;\; \nabla_{\varphi U}\xi = -\mu U + \delta\varphi U, \;\; \nabla_\xi \xi = \beta\varphi U,$$

$$\nabla_U U = \kappa_1\varphi U + \delta\xi, \;\; \nabla_{\varphi U} U = \kappa_2\varphi U + \mu\xi, \nabla_\xi U = \kappa_3\varphi U,$$

$$\nabla_U \varphi U = -\kappa_1 U - \gamma\xi, \nabla_{\varphi U}\varphi U = -\kappa_2 U - \delta\xi, \nabla_\xi \varphi U = -\kappa_3 U - \beta\xi,$$

where $\gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M.

Lemma 2

Let M be a non-Hopf real hypersurface in $M_2(c)$. Then the following relations hold on M

$$
\begin{aligned}
&U\beta - \xi\gamma = \alpha\delta - 2\delta\kappa_3 \\
&\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2 \\
&U\alpha - \xi\beta = -3\beta\delta \\
&\xi\mu = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3 \\
&(\varphi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu \\
&(\varphi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu \\
&U\delta - (\varphi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu \\
&U\mu - (\varphi U)\delta = \gamma\kappa_2 + \beta\delta - \kappa_2\mu - 2\delta\kappa_1
\end{aligned}
$$

2^{nd} Step

1) Hopf Hypersurface $\longrightarrow A\xi = \alpha\xi \longrightarrow \alpha$=constant.
2) We consider a point $P \in M$ and we choose principal vector field $Z \in \ker(\eta)$ at P such that m

$$AZ = \lambda Z \quad \text{and} \quad A\varphi Z = \nu\varphi Z,$$

λ, ν functions

$$\lambda\nu = \frac{\alpha}{2}(\lambda+\nu) + \frac{c}{4}$$

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- λ, ν distinct at $P \longrightarrow \lambda, \nu$ constant
- $\lambda = \nu \longrightarrow A\varphi = \varphi A.$

Types of Parallelness of the Structure Jacobi Operator

Condition	$\mathbb{C}P^2$ **and** $\mathbb{C}H^2$	$\mathbb{C}P^n, n \geq 3$	$\mathbb{C}H^n, n \geq 3$
ξ-Parallel	✔	?	?
Lie $\mathbb{D}$ - parallel	No	No	?
Pseudo - parallel	✔	?	?

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ξ**-Parallel**	✔	?	?
Lie $\mathbb{D}$ - parallel	**No**	**No**	?
Pseudo - parallel	✔	?	?

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ξ**-Parallel**	✔	?	?
Lie $\mathbb{D}$ - parallel	No	No	?
Pseudo - parallel	✔	?	?

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ξ-Parallel	✔	?	?
Lie $\mathbb{D}$ - parallel	No	No	?
Pseudo - parallel	✔	?	?

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ξ-Parallel	✔	?	?
Lie $\mathbb{D}$ - parallel	No	No	?
Pseudo - parallel	✔	?	?

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ξ-Parallel	✔	?	?
Lie $\mathbb{D}$ - parallel	No	No	?
Pseudo - parallel	✔	?	?

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ξ**-Parallel**	✔	?	?
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Lie $\mathbb{D}$ **- parallel**	**No**	**No**	?
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Lie $\mathbb{D}$ - parallel	No	No	?
Pseudo - parallel	✔	?	?

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ξ**-Parallel**	✔	?	?
Lie $\mathbb{D}$ **- parallel**	**No**	**No**	?
Pseudo - parallel	✔	?	?

Lie derivative of the structure Jacobi operator

Condition	$\mathbb{C}P^2$ **and** $\mathbb{C}H^2$	$\mathbb{C}P^n, n \geq 3$	$\mathbb{C}H^n, n \geq 3$
$\mathcal{L}_\xi l = \nabla_\xi l$	✔	✔	?
$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	**No**	?	?
Lie recurrence	**No**	**No**	**No**

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Condition	$\mathbb{C}P^2$ **and** $\mathbb{C}H^2$	$\mathbb{C}P^n, n \geq 3$	$\mathbb{C}H^n, n \geq 3$
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$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	**No**	?	?
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$\mathcal{L}_\xi l = \nabla_\xi l$	✔	✔	?
$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	No	?	?
Lie recurrence	No	No	No

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$\mathcal{L}_\xi l = \nabla_\xi l$	✔	✔	**?**
$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	**No**	**?**	**?**
Lie recurrence	**No**	**No**	**No**

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$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	No	?	?
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$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	**No**	**?**	**?**
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$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	**No**	**?**	**?**
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$\mathcal{L}_\xi l = \nabla_\xi l$	✔	✔	?
$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	**No**	?	?
Lie recurrence	No	No	No

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$\mathcal{L}_\xi l = \nabla_\xi l$	✔	✔	?
$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	**No**	?	?
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$\mathcal{L}_\xi l = \nabla_\xi l$	✔	✔	?
$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	No	?	?
Lie recurrence	No	No	No

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$\mathcal{L}_\xi l = \nabla_\xi l$	✔	✔	?
$\mathcal{L}_X l = \nabla_X l, X \in \mathbb{D}$	**No**	?	?
Lie recurrence	**No**	**No**	**No**

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THANK YOU