The generalized Tanaka-Webster connection of real hypersurfaces in symmetric spaces

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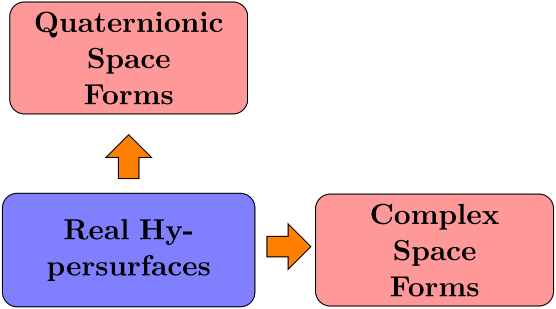
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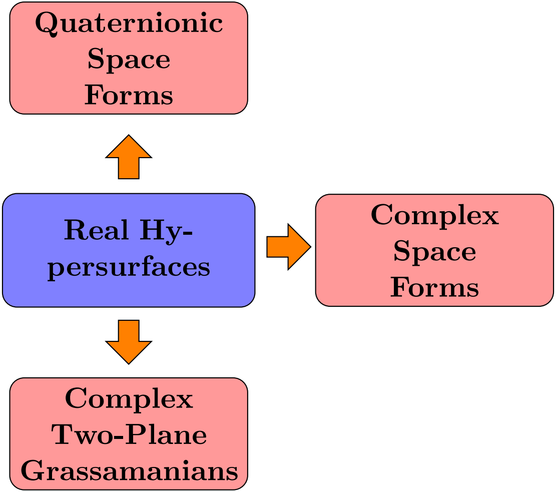
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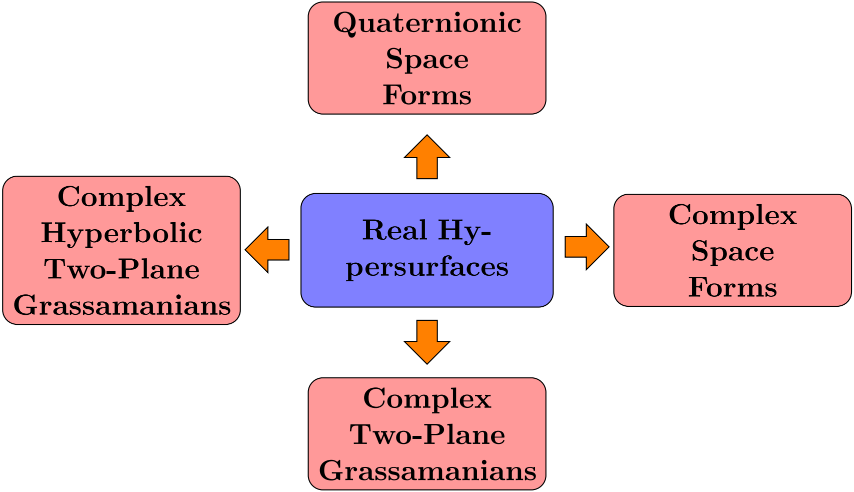
Quaternionic Space Forms
Complex Hyperbolic Two-Plane Grassamanians
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Complex Space Form *is a Kaehler manifold equipped with a complex structure* J*,* $(J^2 = -I)$*, whose holomorphic sectional curvature* c *is constant for all the* J*-invariant planes* Π *in* $T_P M$*, for all points* $P \in M$*.*

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$M_n(c), c \neq 0 \longrightarrow$ Non-Flat Complex Space Form

Definition

Almost contact structure *or* (φ, ξ, η) *-* ***structure*** *is a tensor field* φ *of type (1,1), a vector field* ξ *and a 1-form* η*, which satisfy the following relations*

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1,$$

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Compatible Metric *of an almost contact manifold is a Riemannian metric such that*

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Gauss and Weingarten equations

$$\overline{\nabla}_Y X = \nabla_Y X + g(AY, X)N,$$

$$\overline{\nabla}_X N = -AX,$$

where ∇ is the Levi-Civita connection of M, A is the shape operator of M and g the induced Riemannian metric on M.

Definition of the (φ, ξ, η, g) - structure on a real hypersurface

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$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \varphi = 0, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \;\; g(X, \varphi Y) = -g(\varphi X, Y),$$

$$\nabla_X \xi = \varphi AX, \qquad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

for any tangent vector field X, Y on M.

Gauss equation:

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\varphi Y, Z)\varphi X$$

$$-g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z] + g(AY, Z)AX - g(AX, Z)AY$$

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Codazzi equation:

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X,Y)\xi],$$

where R denotes the Riemannian curvature tensor on M.

The tangent space $T_P M$: $T_P M = \text{span}\{\xi\} \oplus \mathbb{D}$, $\mathbb{D} = \ker \eta = \{X \in T_P M : \eta(X) = 0\}$ **(maximal) holomorphic distribution (if** $n \geq 3$**).**

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A real hypersurface is a ***Hopf hypersurface****, if the structure vector field* ξ *is principal, i.e.* $A\xi = \alpha\xi$.

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If M is a Hopf hypersurface then α is constant

Complex Projective Space $\mathbb{C}P^n, n \geq 2$

Takagi (1973) [26], Cecil, Ryan (1982) [2], Wang (1983) [30], Kimura (1986) [5]

- $(A1)$ geodesic spheres of radius r, where $0 < r < \frac{\pi}{2}$,
- $(A2)$ tubes of radius r over totally geodesic complex projective space $\mathbb{C}P^k$, where $0 < r < \frac{\pi}{2}$ and $1 \leq k \leq n-2$,
- (B) tubes of radius r over complex quadrics and $\mathbb{R}P^n$, where $0 < r < \frac{\pi}{4}$,
- (C) tubes of radius r over the Serge embedding of $\mathbb{C}P^1 \times \mathbb{C}P^{\frac{(n-1)}{2}}$, where $0 < r < \frac{\pi}{4}$ and $n \geq 2\kappa + 3$, $\kappa \in \mathbb{N}^*$,

- (D) tubes of radius r over the *Plucker* embedding of the complex Grassmannian manifold $G_{2,5}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,
- (E) tubes of radius r over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.

Case of $\mathbb{C}P^2$

A three dimensional real hypersurface M is locally congruent to

- (A1) a geodesic sphere of radius r, where $0 < r < \frac{\pi}{2}$,
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Type	α	λ	ν	m_α	m_λ	m_ν
A1	$2\cot 2r$	$\cot r$	-	1	2	-
B	$2\cot 2r$	$\cot(r - \frac{\pi}{4})$	$-\tan(r - \frac{\pi}{4})$	1	1	1

Complex Hyperbolic Space $\mathbb{C}H^n, n \geq 2$

Montiel (1985) [10], Berndt (1989) [1]

- (A0) horospheres
- (A1,0) geodesic spheres of radius $r > 0$,
- (A1,1) tubes of radius $r > 0$ over totally geodesic complex hyperbolic hyperplanes $\mathbb{C}H^{n-1}$,
- (A2) tubes of radius $r > 0$ over totally geodesic submanifold $\mathbb{C}H^k$, $1 \leq k \leq n-2$,
- (B) tubes of radius $r > 0$ over totally real hyperbolic space $\mathbb{R}H^n$.

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- (A1,0) a geodesic sphere of radius $r > 0$,
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Type	α	λ	ν	m_α	m_λ	m_ν
A0	2	1	-	1	2	-
A1,0	$2\coth(2r)$	$\coth(r)$	-	1	2	-
A1,1	$2\coth(2r)$	$\tanh(r)$	-	1	2	-
B	$2\tanh(2r)$	$\tanh(r)$	$\coth(r)$	1	1	1

Ruled real hypersurfaces

Construction: consider a regular curve γ in $M_n(c)$ with tangent vector field X. Then at each point of γ there is a unique hyperplane of $M_n(c)$ cutting γ in a way to be orthogonal to both X and JX. The union of all these hyperplanes is the ruled hypersurface.

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$$A\xi = \alpha\xi + \beta U, \;\; AU = \beta\xi, \;\; AZ = 0, \; Z \text{ orthogonal to } \xi.$$

In case of $\mathbb{C}P^n$,$n \geq 2$,$\longrightarrow$ Maeda [7] and in case of $\mathbb{C}H^n$,$n \geq 2$,$\longrightarrow$ Montiel [8].

Theorem

Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then
i) If W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then $(\lambda - \frac{\alpha}{2})A\varphi W = (\frac{\lambda\alpha}{2} + \frac{c}{4})\varphi W$.
ii) If the vector field W satisfies $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ then $\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}$.

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In case of $\mathbb{C}P^n$, $n \geq 2$,$\longrightarrow$ Okumura [11] and in case of $\mathbb{C}H^n$, $n \geq 2 \longrightarrow$ Montiel-Romero [9].

Theorem

Let M be a real hypersurface in $M_n(c)$, $n \geq 2$, then $\varphi A = A\varphi$ if and only if M is an open subset of real hypersurfaces of type (A).

Definition [Tanno (1989)]

Generalized Tanaka-Webster connection *for contact metric manifolds is given by*

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\varphi Y$$

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Definition [Cho (1999)]

k-th Generalized Tanaka-Webster connection *for real hypersurfaces in complex space forms is given by*

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y,$$

where X, Y *are tangent to* M *and* k *is a real non-null number.*

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Properties of k-th Generalized Tanaka-Webster Connection

- $\nabla^{(k)}\xi = 0, \;\; \nabla^{(k)}g = 0, \;\; \nabla^{(k)}\varphi = 0, \;\; \nabla^{(k)}\eta = 0$

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- $\nabla^{(k)}\xi = 0, \;\; \nabla^{(k)}g = 0, \;\; \nabla^{(k)}\varphi = 0, \;\; \nabla^{(k)}\eta = 0$
- If the shape operator satisfies $\varphi A + A\varphi = 2k\varphi$, then the k-th generalized Tanaka-Webster connection coincides with the Tanaka-Webster connection.

Levi-Civita Connection

Parallel Shape Operator

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Cho (1999) Cho (1999) Type (A)

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Levi-Civta Connection

Parallel Structure Jacobi operator
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Result:
Ortega, Perez and Santos (2006):
Non-existence of real hypersurfaces in complex space form $M_n(c)$, $n \geq 2$, whose structure Jacobi operator is parallel.

Definition

A tensor field T of type (1,1) on M is of ***Codazzi type*** *when it satisfies*

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Perez, Santos, Suh (2007): Non-existence of real hypersurfaces in complex projective space whose structure Jacobi operator is of Codazzi type.

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Perez, Santos, Suh (2007): Non-existence of real hypersurfaces in complex projective space whose structure Jacobi operator is of Codazzi type.

Theofanidis, Xenos (2010), (2014):

1) Non-existence of three dimensional real hypersurfaces in non-flat complex space forms, whose structure Jacobi operator is of Codazzi type.

2) Non-existence of real hypersurfaces in complex hyperbolic space, whose structure Jacobi operator is of Codazzi type.

Definition

*A tensor field T of type (1,1) is of **Codazzi type with respect to the generalized Tanaka-Webster connection** when it satisfies*

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QUESTION

Are there real hypersurfaces in $M_n(c)$ whose structure Jacobi operator is of Codazzi type with respect to the generalized Tanaka-Webster connection?

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where X, Y are tangent to M.

QUESTION

Are there real hypersurfaces in $M_n(c)$ whose structure Jacobi operator is of Codazzi type with respect to the generalized Tanaka-Webster connection?

$$(\hat{\nabla}^{(k)}_X l)Y = (\hat{\nabla}^{(k)}_Y l)X.$$

Theorem [Kaimakamis, - , Pérez, Kodai J. Math. (2016)]

Every real hypersurface M in $M_2(c)$, whose structure Jacobi operator is of Codazzi type with respect to the generalized Tanaka-Webster connection and also commutes with the shape operator, is a Hopf hypersurface. Furthermore, if $\alpha \neq 2k$ then M is locally congruent to a real hypersurface of type (A) or to a Hopf hypersurface with $A\xi = 0$.

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Corollary [Kaimakamis, - , Pérez, Kodai J. Math. (2016)]

Let M be a real hypersurface in $M_2(c)$ with $\alpha \neq 2k$, whose structure Jacobi operator is parallel with respect to the generalized Tanaka-Webster connection and also commutes with the shape operator. Then M is locally congruent to a real hypersurface of type (A) or to a Hopf hypersurface with $A\xi = 0$.

Proposition[Kaimakamis, - , Pérez, Kodai J. Math. (2016)]

Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator is of Codazzi type with respect to the generalized Tanaka-Webster connection and $\alpha \neq 2k$. Then M is locally congruent

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Proposition [Kaimakamis, - , Pérez, Kodai J. Math. (2016)]

There are no ruled real hypersurfaces in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator is of Codazzi type with respect to the generalized Tanaka-Webster connection and $\alpha \neq 2k$.

k-th Cho Operator

Definition [Pérez (2014)]

*The **k-th Cho Operator** with respect to X is defined by*

$$F_X^{(k)}Y = g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y.$$

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The generalized Tanaka-Webster connection becomes

$$(\hat{\nabla}_X^{(k)})Y = \nabla_X Y + F_X^{(k)}Y.$$

Definition[- , Pérez (2015)]

The Lie derivative of a tensor field T of type (1,1) with respect to the generalized Tanaka-Webster connection is given by

$$(\hat{\mathcal{L}}_X T)Y = \hat{\nabla}_X(TY) - \hat{\nabla}_{TY}(X) - T\hat{\nabla}_X(Y) + T\hat{\nabla}_Y(X),$$

and it will be called ***generalized Tanaka-Webster Lie derivative of T****.*

Derivatives

Let T be a (1,1) tensor field on a real hypersurface.

- covariant derivative $(\nabla_X T)Y$

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PROBLEM

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PROBLEM

Classify real hypersurfaces when the derivatives of them coincides

CASE I: $\hat{\nabla}_X^{(k)} T = \nabla_X T$

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Let T be a tensor field of type $(1,1)$ on M and X a vector field tangent to M.

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The eigenspaces of T are preserved by $F_X^{(k)}$

PROBLEM

Classify real hypersurfaces in $M_n(c)$ in terms of their tensor fields satisfying commuting conditions with k-th Cho operator or Cho operator.

Shape Operator

Theorem [Pérez, Suh Monatsh. Math. (2015)]

Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Then, $F_\xi^{(k)} A = A F_\xi^{(k)}$ for any non-null constant k if and only if M is locally congruent to a real hypersurface of type (A).

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Theorem [Pérez, Suh, Monatsh. Math. (2015)]

Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Then, $F_X A = A F_X$ for any $X \in \mathbb{D}$ and any non-null constant k if and only if M is locally congruent to a ruled real hypersurface.

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Theorem [Pérez, Suh, Monatsh. Math. (2015)]

There do not exist real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, such that for any nonnull k $F_X A = AF_X$ for any X tangent to M.

Theorem [- , Pérez (work in progress)]

Let M be a Hopf hypersurface in $M_2(c)$ whose shape operator satisfies relation $F_\xi^{(k)} A = A F_\xi^{(k)}$. Then M is locally congruent to a real hypersurface of type (A).

Theorem [- , Pérez (work in progress)]

Let M be a Hopf hypersurface in $M_2(c)$ whose shape operator satisfies relation $F_\xi^{(k)} A = A F_\xi^{(k)}$. Then M is locally congruent to a real hypersurface of type (A).

Theorem [- , Pérez (work in progress)]

Let M be a real hypersurface in $M_2(c)$ whose shape operator satisfies relation $F_X A = A F_X$ any $X \in \mathbb{D}$. Then M is locally congruent to a ruled real hypersurface.

Structure Jacobi operator

Theorem [Pérez, Ann. Mat. Pura Appl. (2015)]

Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Let k be a non-null constant. Then, $F_\xi^{(k)} l = l F_\xi^{(k)}$ if and only if M is locally congruent to either a tube of radius $\frac{\pi}{4}$ over a complex submanifold of $\mathbb{C}P^n$ or to a type (A) real hypersurface with radius $r \neq \frac{\pi}{4}$.

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Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Then, $F_X l = l F_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.

Corollary

There do not exist real hypersurfaces M in $\mathbb{C}P^n$, $n \geq 3$, such that for a nonnull constant k $F_X^{(k)} l = l F_X^{(k)}$ for any $X \in TM$.

Theorem [- , Pérez (2015), Open Math. (2015)]

Every real hypersurface M in $M_2(c)$, whose structure Jacobi operator satisfies relation $F_{\xi}^{(k)} l = l F_{\xi}^{(k)}$, is Hopf and is locally congruent

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Let M be a real hypersurface in $M_2(c)$, whose structure Jacobi operator satisfies relation $F_X l = l F_X$ for any $X \in \mathbb{D}$. Then, M is locally congruent to a ruled real hypersurface.

Ricci tensor

The **Ricci tensor S** with respect to the Levi-Civita connection is given by

$$SX = \sum_{i=1}^{2n-1} R_{E_i}(X) = \sum_{i=1}^{2n-1} R(X, E_i)E_i,$$

where $\{E_i\}_{i=1,\ldots,2n-1}$ is an orthonormal basis of TM, for any X tangent to M.

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where $\{E_i\}_{i=1,\dots,2n-1}$ is an orthonormal basis of TM, for any X tangent to M.
More analytically,

$$SX = \frac{c}{4}[(2n+1)X - 3\eta(X)\xi] + hAX - A^2X.$$

Theorem [Kaimakamis, - , Pérez (work in progress)]

There do not exist Hopf hypersurfaces in $M_n(c)$, $n \geq 2$, whose Ricci tensor satisfies relation $F_X S = S F_X$, for any $X \in \mathbb{D}$.

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Overview

Condition	$M_2(c)$	$\mathbb{C}P^n, n \geq 3$	$\mathbb{C}H^n, n \geq 3$
$F_\xi^{(k)} l = l F_\xi^{(k)}$	✔	✔	✔
$F_X l = l F_X$	✔	✔	✔
$F_\xi^{(k)} A = A F_\xi^{(k)}$	✔	✔	?
$F_X A = A F_X$	✔	✔	?
$F_X S = S F_X$	✔	$h = \alpha$	$h = \alpha$
$F_\xi S = S F_\xi$	?	?	?

CASE II: $\hat{\mathcal{L}}_X^{(k)} T = \mathcal{L}_X T$ Shape Operator

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Theorem [Pérez, Medit. J. Math. (2015)]

Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Then, $\hat{\mathcal{L}}_\xi^{(k)} A = \mathcal{L}_\xi A$ for some nonnull k if and only if M is locally congruent to a real hypersurface of type (A).

CASE II: $\hat{\mathcal{L}}^{(k)}_X T = \mathcal{L}_X T$ Shape Operator

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Theorem [Pérez, Medit. J. Math. (2015)]

There do not exist real hypersurfaces in in $\mathbb{C}P^n$, $n \geq 3$, such that $\hat{\mathcal{L}}^{(k)}_X A = \mathcal{L}_X A$ for any $X \in \mathbb{D}$ and some non-null constant k.

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Let M be a real hypersurface in $M_2(c)$, whose shape operator satisfies relation $\hat{\mathcal{L}}_\xi^{(k)} A = \mathcal{L}_\xi A$ for some nonnull k. Then, M is a Hopf hypersurface and is locally congruent to a real hypersurface of type (A).

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Theorem [Pérez (submitted)]

Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Then, $\hat{\mathcal{L}}_{\xi}^{(k)} l = \mathcal{L}_{\xi} l$ for some nonnull k if and only if M is locally congruent to a real hypersurface of type (A).

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Every real hypersurface in $M_2(c)$, whose structure Jacobi operator satisfies relation $\hat{\mathcal{L}}_\xi^{(k)} l = \mathcal{L}_\xi l$ is a Hopf hypersurface. Moreover, M is locally congruent either to a real hypersurface of type (A), or to a Hopf hypersurface with $A\xi = 0$.

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Corollary

There do not exist real hypersurfaces in $M_2(c)$ such that $\hat{\mathcal{L}}_X^{(k)} l = \mathcal{L}_X l$, for any nonnull constant k $X \in TM$.

Overview

Condition	$M_2(c)$	$\mathbb{C}P^n, n \geq 3$	$\mathbb{C}H^n, n \geq 3$
$\mathcal{L}_\xi^{\hat{(k)}} A = \mathcal{L}_\xi A$	✔	✔	?
$\mathcal{L}_X^{\hat{(k)}} A = \mathcal{L}_X A$	✔	✔	?
$\mathcal{L}_\xi^{\hat{(k)}} l = \mathcal{L}_\xi l$	✔	✔	?
$\mathcal{L}_X^{\hat{(k)}} A = \mathcal{L}_X A$	✔	✔	?

CASE III: $\hat{\nabla}_X^{(k)} T = \mathcal{L}_X T$

Perez and Santos (2009): Classification of real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose structure Jacobi operator satisfies the relation $\mathcal{L}_\xi l = \nabla_\xi l$.

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Panagiotidou (2013):

1) Non-existence of real hypersurfaces in $M_2(c)$, whose structure Jacobi operator satisfies relation $\mathcal{L}_X l = \nabla_X l$ for any $X \in \mathbb{D}$.

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1) Non-existence of real hypersurfaces in $M_2(c)$, whose structure Jacobi operator satisfies relation $\mathcal{L}_X l = \nabla_X l$ for any $X \in \mathbb{D}$.
2) Non-existence of real hypersurfaces in $M_2(c)$, whose shape operator satisfies relation $\mathcal{L}_X A = \nabla_X A$ for any $X \in \mathbb{D}$.

Shape Operator

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Theorem [Pérez, Int. J. Math. (2015)]

Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Then, $\hat{\nabla}_\xi^{(k)} A = \mathcal{L}_\xi A$ for some nonnull k if and only if M is locally congruent to a real hypersurface of type (A).

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Theorem [Kaimakamis, - , Pérez (work in progress)]

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Theorem [Kaimakamis, - , Pérez (work in progress)]

Let M be a real hypersurface in $M_2(c)$, whose shape operator satisfies relation $\hat{\nabla}^{(k)}_X A = \mathcal{L}_X A$ for any $X \in \mathbb{D}$ and any nonnull constant k. Then, M is locally congruent to a a ruled real hypersurface.

Structure Jacobi Operator

Theorem [Kaimakamis, - , Pérez (work in progress)]

Let M be a real hypersurface in $M_2(c)$, whose structure Jacobi operator satisfies relation $\hat{\nabla}^{(k)}_{\xi} l = \mathcal{L}_{\xi} l$. Then, M is locally congruent to a real hypersurface of type (A).

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Theorem [Kaimakamis, - , Pérez (work in progress)]

There do not exist Hopf hypersurfaces in $M_2(c)$, whose structure Jacobi operator satisfies relation $\hat{\nabla}^{(k)}_{X} l = \mathcal{L}_{X} l$ for any $X \in \mathbb{D}$.

Overview

Condition	$M_2(c)$	$\mathbb{C}P^n, n \geq 3$	$\mathbb{C}H^n, n \geq 3$
$\hat{\nabla}^{(k)}_{\xi} A = \mathcal{L}_{\xi} A$	✔	✔	?
$\hat{\nabla}^{(k)}_{X} A = \mathcal{L}_{X} A$	✔	?	?
$\hat{\nabla}^{(k)}_{\xi} l = \mathcal{L}_{\xi} l$	✔	?	?
$\hat{\nabla}^{(k)}_{X} l = \mathcal{L}_{X} l$	✔	?	?

The Ricci tensor with respect to the k- th generalized Tanaka-Webster connection is denoted by $\hat{S}$ and is called **generalized Tanaka-Webster Ricci tensor**.

$$\hat{S}(X,Y) = Trace\ of\ \{X \mapsto \hat{R}(Z,X)Y.\}$$

More analytically

$$\hat{S}(X,Y) = \frac{cn}{2}g(X,Y) + [tr(A) - \eta(A\xi) + k]g(AX,Y)$$

$$-g(A^2X,Y) - g(\phi A\phi AX,Y) - kg(\phi A\phi X,Y) + \eta(AX)g(A\xi,Y)$$

$$+\eta(Y)\{-\frac{cn}{2}\eta(X) - \eta(AX)[tr(A) - \eta(A\xi) + k\eta(AX)]\}.$$

Theorem [Kon, Publ. Debrecen (2011)]

Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, whose generalized Tanaka-Webster Ricci tensor satisfies the relation $\hat{S}(X, \phi Y) = \mu g(X, \phi Y)$, for any vector field X, Y and μ being a function. Then M is locally congruent to a real hypersurface of type (A).

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Theorem [Kaimakamis, - , Pérez (work in progress)]

Let M be a Hopf hypersurfaces in $M_2(c)$, whose generalized Tanaka-Webster Ricci tensor satisfies the relation $\hat{S}X = 0$, for any vector fields X tangent to M. Then M is locally congruent to a real hypersurface of type (A) or type (B).

Generalized Tanaka-Webster Ricci Soliton

The **generalized Tanaka-Webster Ricci soliton** is a Riemannian metric g on a real hypersurface M satisfying relation

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and **generalized expanding** if $\lambda > 0$.

Theorem [Kaimakamis, - , Pérez (work in progress)]

Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, admitting a generalized Tanaka-Webster Ricci soliton with potential vector field ξ. Then M is locally congruent to a real hypersurface of type (A) and the generalized Tanaka-Webster Ricci soliton is generalized steady.

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Theorem [Kaimakamis, - , Pérez (work in progress)]

Let M be a Hopf hypersurface in $M_2(c)$ admitting a generalized Tanaka-Webster Ricci soliton with potential vector field ξ. Then M is locally congruent to a real hypersurface of type (A) or (B) and the generalized Tanaka-Webster Ricci soliton is generalized steady.

1^{st} Step

Proposition

Every real hypersurface in $M_2(c)$, whose structure Jacobi operator satisfies relation $\hat{\mathcal{L}}_\xi^{(k)} l = \mathcal{L}_\xi l$ is a Hopf hypersurface.

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The shape operator has the form

$$AU = \gamma U + \delta\varphi U + \beta\xi, \;\; A\varphi U = \delta U + \mu\varphi U, \;\; A\xi = \alpha\xi + \beta U.$$

Sketch of Proof

Lemma 1

Let M be a non-Hopf real hypersurface in $M_2(c)$. Then the following relations hold on M

$$AU = \gamma U + \delta\varphi U + \beta\xi, \;\; A\varphi U = \delta U + \mu\varphi U, \;\; A\xi = \alpha\xi + \beta U,$$

$$\nabla_U\xi = -\delta U + \gamma\varphi U, \;\; \nabla_{\varphi U}\xi = -\mu U + \delta\varphi U, \;\; \nabla_\xi\xi = \beta\varphi U,$$

$$\nabla_U U = \kappa_1\varphi U + \delta\xi, \;\; \nabla_{\varphi U}U = \kappa_2\varphi U + \mu\xi, \nabla_\xi U = \kappa_3\varphi U,$$

$$\nabla_U\varphi U = -\kappa_1 U - \gamma\xi, \nabla_{\varphi U}\varphi U = -\kappa_2 U - \delta\xi, \nabla_\xi\varphi U = -\kappa_3 U - \beta\xi,$$

where $\gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M.

Lemma 2 [15]

Let M be a non-Hopf real hypersurface in $M_2(c)$. Then the following relations hold on M

$$\begin{aligned}
&U\beta - \xi\gamma = \alpha\delta - 2\delta\kappa_3\\
&\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2\\
&U\alpha - \xi\beta = -3\beta\delta\\
&\xi\mu = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3\\
&(\varphi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu\\
&(\varphi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu\\
&U\delta - (\varphi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu\\
&U\mu - (\varphi U)\delta = \gamma\kappa_2 + \beta\delta - \kappa_2\mu - 2\delta\kappa_1
\end{aligned}$$

2^{nd} Step: We consider a point $P \in M$ and we choose principal vector field $W \in \ker(\eta)$ at P such that

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Case of ruled real hypersurfaces:
Examine if Hopf hypersurfaces satisfies the condition.
Define the shape operator with respect to the local orthonormal basis.

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