Análisis geométrico de la distancia lorentziana en subvariedades marginalmente atrapadas

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- Obviously, $p \ll q$ implies $p < q$. As usual, $p \leq q$ means that either $p < q$ or $p = q$.
- For a subset $S \subset M$, one defines the chronological future of S as

$$I^+(S) = \{q \in M : p \ll q \text{ for some } p \in S\},$$

and the causal future of S as

$$J^+(S) = \{q \in M : p \leq q \text{ for some } p \in S\}.$$

Thus $S \cup I^+(S) \subset J^+(S)$.

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- The Lorentzian distance function $d : M \times M \to [0, +\infty]$ for an arbitrary spacetime may fail to be continuous in general, and may also fail to be finite valued.
- As a matter of fact, globally hyperbolic spacetimes[1] turn out to be the natural class of spacetimes for which the Lorentzian distance function is finite-valued and continuous.

[1] M is globally hyperbolic provided (a) $J^+(p) \cap J^-(q)$ is compact for any $p, q \in M$, and (b) M is causal, i.e, there are no closed causal curves in M.

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$$T_{-1}M|_p = \{v \in T_pM : v \text{ is a future-directed timelike unit vector}\}.$$
- Define the function $s_p : T_{-1}M|_p \to [0, +\infty]$ by
$$s_p(v) = \sup\{t \geq 0 : d_p(\gamma_v(t)) = t\},$$
where $\gamma_v : [0, a) \to M$ is the future inextendible geodesic starting at p with initial velocity v.

Lorentzian distance function from a point

- Then, one can define the subset $\tilde{\mathcal{I}}^+(p) \subset T_pM$ given by

$$\tilde{\mathcal{I}}^+(p) = \{tv : \text{ for all } v \in T_{-1}M|_p \text{ and } 0 < t < s_p(v)\}$$

and consider the subset $\mathcal{I}^+(p) \subset M$ given by

$$\mathcal{I}^+(p) = \exp_p(\mathrm{int}(\tilde{\mathcal{I}}^+(p))) \subset I^+(p).$$

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Lemma 1 (Erkekoglu, García-Río and Kupeli, 2003)

Let M be a spacetime and $p \in M$.

1. If M is strongly causal at p[a], then $s_p(v) > 0$ for all $v \in T_{-1}M|_p$ and $\mathcal{I}^+(p) \neq \emptyset$.
2. If $\mathcal{I}^+(p) \neq \emptyset$, then the Lorentzian distance function d_p is smooth on $\mathcal{I}^+(p)$ and its gradient $\overline{\nabla} d_p$ is a past-directed timelike (geodesic) unit vector field on $\mathcal{I}^+(p)$.

[a] Given any neighborhood U of p there is a neighborhood $V \subset U$ of p such that every causal curve segment with endpoints in V is entirely contained in U.

Hessian comparison results for the Lorentzian distance

- For every $c \in \mathbb{R}$, let us define

$$h_c(t) = \begin{cases} \frac{1}{\sqrt{c}} \sinh(\sqrt{c}\, t) & \text{if } c > 0 \text{ and } t > 0 \\ \quad t & \text{if } c = 0 \text{ and } t > 0 \\ \frac{1}{\sqrt{-c}} \sin(\sqrt{-c}\, t) & \text{if } c < 0 \text{ and } 0 < t < \pi/\sqrt{-c}. \end{cases}$$

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- Observe that the index of a Jacobi field along a timelike geodesic in a Lorentzian space form of constant curvature c is given by

$$I_{\gamma_c}(J_c, J_c) = -\frac{h_c'(t)}{h_c(t)} \langle x, x \rangle.$$

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- On the other hand, $\dfrac{h_c'(t)}{h_c(t)}$ is the future mean curvature of the level set

$$\Sigma_c(t) = \{q \in \mathcal{I}^+(p) : d_p(q) = t\} \subset M_c^n.$$

Hessian comparison results for the Lorentzian distance

Lemma 2 (Alías, Hurtado, Palmer, 2010)

Let M be a spacetime such that $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes in M. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$, and let $q \in \mathcal{I}^+(p)$ (with $d_p(q) < \pi/\sqrt{-c}$ when $c < 0$). Then for every spacelike vector $x \in T_qM$ orthogonal to $\overline{\nabla} d_p(q)$

$$\overline{\nabla}^2 d_p(x,x) \leq -\frac{h_c'}{h_c}(d_p(q))\langle x, x\rangle,$$

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- The proof of Lemma 2 follows from the fact that

$$\overline{\nabla}^2 d_p(x,x) = I_\gamma(J,J)$$

where γ is the radial future directed unit timelike geodesic from p to q and J is the Jacobi field along γ with $J(0) = 0$ and $J(s) = x$, and is strongly based on the maximality of the index of Jacobi fields.

- On the other hand, under the assumption that the sectional curvatures of the timelike planes of M are bounded from above by a constant c, we get the following result.

Lemma 3 (Alías, Hurtado, Palmer, 2010)

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- The proof is similar to that of Lemma 2.

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- Our first objective is to compute the Hessian of u. To do that, observe that

$$\overline{\nabla} r = \nabla u + (\overline{\nabla} r)^\perp$$

along Σ, where $\nabla u = (\overline{\nabla} r)^\top$ stands for the gradient of u on Σ and $(\overline{\nabla} r)^\perp$ denotes the normal component of $\overline{\nabla} r$.

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along Σ, where $\nabla u = (\overline{\nabla} r)^\top$ stands for the gradient of u on Σ and $(\overline{\nabla} r)^\perp$ denotes the normal component of $\overline{\nabla} r$.
- By Gauss and Weingarten formulae we get
$$\overline{\nabla}_X \overline{\nabla} r = \nabla_X \nabla u - A_{(\overline{\nabla} r)^\perp} X + \mathrm{II}(X, \nabla u) + \nabla^\perp_X (\overline{\nabla} r)^\perp,$$
for every tangent vector $X \in T\Sigma$, where II denotes the second fundamental form of the submanifold and, for every normal vector η, A_η denotes the Weingarten endomorphism with respect to η.

- It follows from here that

$$\nabla^2 u(X, Y) = \overline{\nabla}^2 r(X, Y) + \langle \mathrm{II}(X, Y), \overline{\nabla} r \rangle$$

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- Tracing this expression, one gets that the Laplacian of u is given by

$$\Delta u = \sum_{i=1}^{m} \overline{\nabla}^2 r(E_i, E_i) + m\langle \mathbf{H}, \overline{\nabla} r\rangle,$$

where $\{E_1, \ldots, E_m\}$ is a local orthonormal frame on Σ, and

$$\mathbf{H} := \frac{1}{m}\mathrm{tr}(\mathrm{II}) = \frac{1}{m}\sum_{i=1}^{m} \mathrm{II}(E_i, E_i)$$

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- Consider the function $v = \phi_c(u)$, where $\phi_c(t)$ is a primitive of $h_c(t)$:

$$\phi_c(t) = \begin{cases} \frac{1}{c}\cosh(\sqrt{c}\, t) & \text{if } c > 0 \text{ and } t > 0 \\ \frac{t^2}{2} & \text{if } c = 0 \text{ and } t > 0 \\ \frac{1}{c}\cos(\sqrt{-c}\, t) & \text{if } c < 0 \text{ and } 0 < t < \pi/\sqrt{-c}. \end{cases}$$

- Then, the Laplacian of v is given by

$$\begin{aligned}\Delta v &= \phi_c'(u)\Delta u + \phi_c''(u)|\nabla u|^2 \\ &= h_c(u)\sum_{i=1}^{m}\overline{\nabla}^2 r(E_i,E_i) + m h_c(u)\langle \mathbf{H}, \overline{\nabla} r\rangle + h_c'(u)|\nabla u|^2.\end{aligned}$$

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- Assume now that $K_M(\Pi) \geq c$ (resp. $K_M(\Pi) \leq c$) for all timelike planes in M. Then by the Hessian comparison results for r given in Lemma 2 (resp. Lemma 3), one gets that

$$\overline{\nabla}^2 r(X,X) \leq (\geq) -\frac{h_c'}{h_c}(u)(1+\langle X, \nabla u\rangle^2)$$

for every unit tangent vector field $X \in T\Sigma$.

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for every unit tangent vector field $X \in T\Sigma$.

- Therefore,

$$h_c(u)\sum_{i=1}^m \overline{\nabla}^2 r(E_i, E_i) \leq (\geq) - h_c'(u)(m + |\nabla u|^2),$$

which, jointly with the expression above, gives the following inequality for the Laplacian of v

$$\Delta v \leq (\geq) - m h_c'(u) + m h_c(u)\langle \mathbf{H}, \overline{\nabla} r\rangle.$$

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- Before stating our main results, we need to introduce some terminology about:
  - Our analytical tool: the weak maximum principle.
  - Our geometric objects: trapped submanifolds.

The Omori-Yau maximum principle

- Following the terminology introduced by Pigola, Rigoli and Setti (2005), the Omori-Yau maximum principle is said to hold on an n-dimensional Riemannian manifold Σ if, for any smooth function $u \in \mathcal{C}^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ with the properties

$$\text{(i)}\ \ u(p_k) > u^* - \frac{1}{k}, \ \ \text{(ii)}\ \ |\nabla u(p_k)| < \frac{1}{k}, \text{ and (iii)}\ \ \Delta u(p_k) < \frac{1}{k}.$$

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- Equivalently, for any $u \in \mathcal{C}^2(\Sigma)$ with $u_* = \inf_\Sigma u > -\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ satisfying

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- In this sense, the classical maximum principle given by Omori (1967) and Yau (1975) stays that the Omori-Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below.

The weak maximum principle

- The *weak* maximum principle is said to hold on Σ if, for any $u \in \mathcal{C}^2(\Sigma)$ with $u^* < +\infty$ there is a sequence $\{p_k\}_{k\in\mathbb{N}}$ in Σ with
$$\text{(i)} \quad u(p_k) > u^* - \frac{1}{k}, \quad \text{and} \quad \text{(iii)} \quad \Delta u(p_k) < \frac{1}{k}.$$
- Pigola, Rigoli and Setti (2003) proved that the weak maximum principle holds on Σ if and only if Σ is stochastically complete.
- Σ is said to be stochastically complete if its Brownian motion is stochastically complete, i.e, the probability of a particle to be found in the state space is constantly equal to 1. In other words,
$$\int_\Sigma p(x,y,t)dy = 1 \text{ for any } (x,t) \in \Sigma \times (0,+\infty),$$
where $p(x,y,t)$ is the heat kernel of the Laplacian operator[2].

[2] For any open $\Omega \subset \Sigma$, $\int_\Omega p(x,y,t)dy$ is the probability that a random path starting at x lies in Ω at finite time t. Hence $\int_\Sigma p(x,y,t)dy < 1$ means that there is a positive probability that a random path will reach infinity in finite time t.

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- In particular, every parabolic manifold is stochastically complete. Hence, the weak max principle holds on every parabolic manifold.

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Trapped submanifolds in a spacetime

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- Analogously, Σ is said to be weakly past trapped if $\mathbf{H}$ is causal and past-pointing on Σ.

Weakly trapped submanifolds in the chronological future of a point. Case $K_M(\Pi) \geq c$

Weakly trapped submanifolds in the chronological future of a point. Case $K_M(\Pi) \geq c$

Theorem 1

Let M be a spacetime with a reference point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$, and assume $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes in M.

1. If $c \geq 0$ there exists no stochastically complete, weakly past trapped submanifold contained in $\mathcal{I}^+(p)$.
2. If $c < 0$ and Σ is a stochastically complete, weakly past trapped submanifold contained in $\mathcal{I}^+(p) \cap B^+(p, \pi/\sqrt{-c})$, then
$$u_* = \inf_{\Sigma} u \geq \frac{\pi}{2\sqrt{-c}},$$
where u denotes the Lorentzian distance d_p along the hypersurface. In other words, Σ is contained in $B^+(p, \pi/\sqrt{-c}) \cap O^+(p, \pi/2\sqrt{-c})$.

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- Here, for $\delta > 0$, the subsets $B^+(p, \delta)$ and $O^+(p, \delta)$ denote the future inner ball and the future outer ball of radius δ, that is,

$$\begin{aligned} B^+(p,\delta) &= \{q \in I^+(p) : d_p(q) < \delta\} \\ O^+(p,\delta) &= \{q \in I^+(p) : d_p(q) > \delta\}. \end{aligned}$$

Proof of Theorem 1

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$$-\frac{1}{k} < \Delta v(p_k) \leq -mh'(u(p_k)) + mh(u(p_k))\langle \mathbf{H}, \overline{\nabla} r\rangle(p_k),$$

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and, making $k \to \infty$ here we get $h_c'(u_*) \leq 0$.
- The result then follows by observing that, when $c \geq 0$ then $h_c'(t) > 0$, and if $c < 0$ then $h_c'(t) \leq 0$ when $\pi/2\sqrt{-c} \leq t < \pi/\sqrt{-c}$.

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Theorem 2

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$$\sup_{\Sigma} |\mathbf{H}_0| \geq \frac{h_c'}{h_c}(u_*),$$

where $\mathbf{H}_0$ stands for the spacelike component of the lightlike vector field $\mathbf{H}$ which is orthogonal to $\overline{\nabla} r$, and $u_* = \inf_\Sigma u$. In particular, if $u_* = 0$ then $\sup_M |\mathbf{H}_0| = +\infty$.

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Corollary 1

Under the assumptions of Theorem 2, if $|\mathbf{H}_0|$ is bounded from above on Σ, then there exists some $\delta > 0$ such that $\Sigma \subset O^+(p, \delta)$, where $O^+(p, \delta)$ denotes the future outer ball of radius δ.

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Proof of Theorem 2

- Applying again the weak maximum principle on Σ to the function $v = \phi_c(u)$, with $v_* = \inf_\Sigma v = \phi_c(u_*)$, we have

$$-\frac{1}{k} < \Delta v(p_k) \leq -mh_c'(u(p_k)) + mh_c(u(p_k)) \sup_\Sigma |\mathbf{H}_0|,$$

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- The last assertion follows from the fact that $h_c(0) = 0$ and $h_c'(0) = 1$.

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Theorem 3

Let M be a spacetime with a reference point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$, and assume $K_M(\Pi) \le c$, $c \in \mathbb{R}$, for all timelike planes in M. Let Σ be a stochastically complete, marginally future trapped submanifold contained in $\mathcal{I}^+(p) \cap B^+(p, \delta)$ for some $\delta > 0$ (with $\delta \le \pi/\sqrt{-c}$ when $c < 0$). Then

$$\inf_{\Sigma} |\mathbf{H}_0| \le \frac{h_c'}{h_c}(u^*),$$

where $\mathbf{H}_0$ stands for the spacelike component of the lightlike vector field $\mathbf{H}$ which is orthogonal to $\overline{\nabla} r$, and $u^* = \sup_{\Sigma} u$.

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- Since $K_M(\Pi) \leq c$ and $\langle \mathbf{H}, \overline{\nabla} r \rangle = |\mathbf{H}_0| > 0$ on Σ, we have

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- Applying the weak maximum principle on Σ to the function $v = \phi_c(u)$, with $v^* = \sup_\Sigma v = \phi_c(u^*)$, we have

$$\frac{1}{k} > \Delta v(p_k) \geq -mh_c'(u(p_k)) + mh_c(u(p_k)) \inf_\Sigma |\mathbf{H}_0|,$$

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- Since $K_M(\Pi) \leq c$ and $\langle \mathbf{H}, \overline{\nabla} r \rangle = |\mathbf{H}_0| > 0$ on Σ, we have

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- If $\inf_\Sigma |\mathbf{H}_0| = -\infty$ then there is nothing to prove.
- Otherwise, let us write

$$\Delta v \geq -mh_c'(u) + mh_c(u)|\mathbf{H}_0| \geq -mh_c'(u) + mh_c(u) \inf_\Sigma |\mathbf{H}_0|.$$

- Applying the weak maximum principle on Σ to the function $v = \phi_c(u)$, with $v^* = \sup_\Sigma v = \phi_c(u^*)$, we have

$$\frac{1}{k} > \Delta v(p_k) \geq -mh_c'(u(p_k)) + mh_c(u(p_k)) \inf_\Sigma |\mathbf{H}_0|,$$

  for $\{p_k\} \subset \Sigma$ with $\lim_{k\to\infty} v(p_k) = v^*$ and $\lim_{k\to\infty} u(p_k) = u^*$.
- Making $k \to +\infty$ we conclude that

$$\inf_\Sigma |\mathbf{H}_0| \leq \frac{h_c'(u^*)}{h_c(u^*)}.$$

Marginally future trapped submanifolds in Lorentzian space forms

- In particular, when the ambient spacetime is a Lorentzian space form, by putting together Theorems 1, 2 and 3 we obtain the following consequence.

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Let M^n_c be a Lorentzian space form of constant sectional curvature c and let $p \in M^n_c$. Let Σ be a stochastically complete, marginally trapped submanifold of M^n_c which is contained in $\mathcal{I}^+(p) \cap B^+(p,\delta)$ for some $\delta > 0$ (with $\delta \leq \pi/2\sqrt{-c}$ if $c < 0$). Then

$$\inf_{\Sigma} |\mathbf{H}_0| \leq \frac{h_c'(u^*)}{h_c(u^*)} \leq \frac{h_c'(u_*)}{h_c(u_*)} \leq \sup_{\Sigma} |\mathbf{H}_0|,$$

where $u_* = \inf_\Sigma u$ and $u^* = \sup_\Sigma u$.

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$$\inf_\Sigma |\mathbf{H}_0| \leq \frac{h'_c(u^*)}{h_c(u^*)} \leq \frac{h'_c(u_*)}{h_c(u_*)} \leq \sup_\Sigma |\mathbf{H}_0|,$$

where $u_* = \inf_\Sigma u$ and $u^* = \sup_\Sigma u$.

- The estimates are sharp as proved by considering Σ as a constant mean curvature hypersurface of a level set of the Lorentzian distance in M^n_c.

Lorentzian distance function from an achronal hypersurface

- Given $S \subset M^n$ an achronal spacelike hypersurface, one can define the Lorentzian distance function from S, $d_S : M \to [0, +\infty]$, by

$$d_S(q) := \sup\{d(p, q) : p \in S\}.$$

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$$s(p) = \sup\{t \geq 0 : d_S(\gamma_p(t)) = t\},$$

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- Then, we can define
$$\tilde{\mathcal{I}}^+(S) = \{t\eta_p : \text{ for all } p \in S \text{ and } 0 < t < s(p)\}$$
and consider the subset $\mathcal{I}^+(S) \subset M$ given by
$$\mathcal{I}^+(S) = \exp_S(\mathrm{int}(\tilde{\mathcal{I}}^+(S))) \subset I^+(S),$$
where $\exp_S$ denotes the exponential map with respect to the hypersurface S.

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Lemma 4 (Erkekoglu, García-Río and Kupeli, 2003)

Let S be an achronal spacelike hyersurface in a spacetime M.

1. If S is compact and (M, g) is globally hyperbolic, then $s(p) > 0$ for all $p \in S$ and $\mathcal{I}^+(S) \neq \emptyset$.
2. If $\mathcal{I}^+(S) \neq \emptyset$, then d_S is smooth on $\mathcal{I}^+(S)$ and its gradient $\overline{\nabla} d_S$ is a past-directed timelike (geodesic) unit vector field on $\mathcal{I}^+(S)$.

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That's all !!
Muchas gracias.