Almost complex surfaces in the nearly Kähler $S^3 \times S^3$

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Abstract

We will discuss almost complex surfaces in $S^3 \times S^3$. The main results to discuss are the following:

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1. Classification of totally geodesic almost complex surfaces.
2. Construction of a holomorphic differential
3. An almost complex topological 2-sphere in $S^3 \times S^3$ is totally geodesic.

Contents

Nearly Kähler manifolds and almost complex submanifolds

Nearly Kähler manifolds
Definitions

- An almost Hermitian manifold (M, g, J) is a manifold M with metric g and almost complex structure J (endomorphism s.t. $J^2 = -\mathrm{Id}$) satisfying

$$g(JX, JY) = g(X, Y), \qquad X, Y \in TM.$$

- A nearly Kähler manifold is an almost Hermitian manifold (M, g, J) with the extra assumption that ∇J is skew-symmetric:

$$(\nabla_X J)Y + (\nabla_Y J)X = 0, \qquad X, Y \in TM.$$

Nearly Kähler manifolds

Identities

- For convenience, we will write $G(X,Y) = (\nabla_X J)Y$.
- Some identities:

$$G(X,Y) + G(Y,X) = 0,$$
$$G(X,JY) + JG(X,Y) = 0,$$
$$g(G(X,Y),Z) \text{ and } g(G(X,Y),JZ) \text{ are totally anti symmetric.}$$
$$\overline{\nabla} J = 0.$$

Here $\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{2} JG(X,Y)$ is the canonical Hermitian connection.

Almost complex submanifolds

- An almost complex submanifold of a nearly Kähler manifold is a submanifold N such that

$$JTN = TN \quad \text{or} \quad J \text{ maps tangent vectors on tangent ones.}$$

- Useful identities:

$$\nabla_X JX = J\nabla_X X, \qquad h(X, JY) = Jh(X, Y),$$
$$A_{J\xi}X = JA_\xi X = -A_\xi JX, \qquad G(X, \xi) = \nabla^\perp_X J\xi - J\nabla^\perp_X \xi.$$

for tangent X, Y and normal ξ.

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All these spaces are compact.

The nearly Kähler $S^3 \times S^3$
The nearly Kähler structure

- The almost complex structure J on $S^3 \times S^3$ is defined by

$$JZ_{(p,q)} = \frac{1}{\sqrt{3}} \left(2pq^{-1}Y - X, -2qp^{-1}X + Y\right).$$

- If $\langle \cdot\,, \cdot \rangle$ is the product metric on $S^3 \times S^3$, the metric g is given by

$$\begin{aligned} g(Z, Z') &= \frac{\sqrt{3}}{4} \left(\langle Z, Z' \rangle + \langle JZ, JZ' \rangle\right) \\ &= \frac{2}{\sqrt{3}} \left(\langle X, X' \rangle + \langle Y, Y' \rangle\right) \\ &\quad - \frac{1}{\sqrt{3}} \left(\langle p^{-1}X, q^{-1}Y' \rangle + \langle p^{-1}X', q^{-1}Y \rangle\right). \end{aligned}$$

The nearly Kähler $S^3 \times S^3$

Isometries

- For unit quaternions a, b and c the map

$$F(p,q) = (apc^{-1}, bqc^{-1})$$

is an isometry.

The nearly Kähler $S^3 \times S^3$
An almost product structure P

- We define the almost product structure P as

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- Properties:

$$\begin{aligned} P^2 &= \mathrm{Id}, \\ PJ &= -JP, \\ g(PZ,PZ') &= g(Z,Z'). \end{aligned}$$

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However, $\nabla P \neq 0$.

The nearly Kähler $S^3 \times S^3$

Curvature tensor

- The Riemann curvature tensor of $S^3 \times S^3$ is

$$\begin{aligned}\tilde{R}(U,V)W = {} & \frac{5}{6\sqrt{3}}\big(g(V,W)U - g(U,W)V\big) \\ & + \frac{1}{6\sqrt{3}}\big(g(JV,W)JU - g(JU,W)JV - 2g(JU,V)JW\big) \\ & + \frac{2}{3\sqrt{3}}\big(g(PV,W)PU - g(PU,W)PV \\ & \qquad + g(JPV,W)JPU - g(JPU,W)JPV\big).\end{aligned}$$

Some results

Totally geodesic surfaces

The curvature of a totally geodesic almost complex curve is either 0 or $\frac{4}{3\sqrt{3}}$.

Some results

- Let $\phi\colon M \to S^3 \times S^3 \colon (u,v) \mapsto \big(p(u,v), q(u,v)\big)$ be a.c. immersion, with isothermal coordinates (u,v). Write

$$\phi_u = (p_u, q_u) \quad \text{and } \phi_v = (p_v, q_v).$$

  Then $J\phi_u = \phi_v$, since the coordinates are isothermal.
- There are local functions $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\delta}$ from M to $\mathbb{R}^3$ such that

$$\begin{aligned} p_u &= p\tilde{\alpha}, & p_v &= p\tilde{\beta}, \\ q_u &= q\tilde{\gamma}, & q_v &= q\tilde{\delta}. \end{aligned}$$

Some results

- Then $\phi_v = J\phi_u$ gives

$$\tilde{\gamma} = \frac{\sqrt{3}}{3}\tilde{\beta} + \frac{1}{2}\tilde{\alpha},$$
$$\tilde{\delta} = \frac{1}{2}\tilde{\beta} - \frac{\sqrt{3}}{2}\tilde{\alpha}.$$

- The integrability conditions $p_{uv} = p_{vu}$ and $q_{uv} = q_{vu}$ yield

$$\tilde{\alpha}_v - \tilde{\beta}_u = 2\tilde{\alpha} \times \tilde{\beta},$$
$$\tilde{\alpha}_u + \tilde{\beta}_v = \frac{2}{\sqrt{3}}\tilde{\alpha} \times \tilde{\beta}.$$

- Now write

$$\alpha = \cos\theta\tilde{\alpha} + \sin\theta\tilde{\beta},$$
$$\beta = -\sin\theta\tilde{\alpha} + \cos\theta\tilde{\beta},$$

  where $\theta = 2\pi/3$.

- The two previous equations become

$$\alpha_v = \beta_u,$$
$$\alpha_u + \beta_v = -\frac{4}{\sqrt{3}}\alpha \times \beta.$$

Some results

- Hence there is a function ε such that

Differential equation

$$\varepsilon_{uu} + \varepsilon_{vv} = -\frac{4}{\sqrt{3}}\varepsilon_u \times \varepsilon_v,$$
$$\varepsilon_u = \alpha, \qquad \varepsilon_v = \beta.$$

Some results
A holomorphic differential

Lemma

The following Cauchy-Riemann equations hold:

$$(\alpha \cdot \beta)_u = \frac{1}{2}(\alpha \cdot \alpha - \beta \cdot \beta)_v,$$
$$(\alpha \cdot \beta)_v = -\frac{1}{2}(\alpha \cdot \alpha - \beta \cdot \beta)_u.$$

Corollary

The differential $\Lambda\, dz^2 = g(P\phi_z, \phi_z)\, dz^2$ *is a holomorphic globally defined differential.*

Some results

Lemma

$PTM \subset T^{\perp}M$ if and only if $\alpha \cdot \alpha = \beta \cdot \beta$ and $\alpha \cdot \beta = 0$.

Proof.

$PTM \subset T^{\perp}M$ if and only if $g(P\phi_u, \phi_u) = 0$ and $g(P\phi_u, \phi_v) = 0$. The equations $\phi_u = \big(p\alpha, q((\sqrt{3}/2)\beta + 1/2\alpha\big)$ and $\phi_v = \big(p\beta, q(1/2\beta - (\sqrt{3}/2)\alpha)\big)$ and the definition of P then give the lemma. □

Some results

Corollary

If M is an almost complex surface, then $PTM \subset T^{\perp}M$ if and only if $\Lambda\, dz^2 = 0$.

Corollary

If we have an almost complex 2-sphere S^2 in $S^3 \times S^3$, then $PTM \subset T^{\perp}M$.

Some results

Key results

Proof of the key results.

- Since $\varepsilon_u = \alpha$ and $\varepsilon_v = \beta$ and $\Lambda\, dz^2 = 0$, the coordinates are isothermal. We also have

$$2H\varepsilon_u \times \varepsilon_v = \varepsilon_{uu} + \varepsilon_{vv} = -\frac{4}{\sqrt{3}}\varepsilon_u \times \varepsilon_v$$

  from the DE on a previous slide.

- If $g = \lambda(du^2 + dv^2)$ is the induced metric on M, then

$$g(\phi_u, \phi_u) = \frac{\sqrt{3}}{2}(\alpha \cdot \alpha + \beta \cdot \beta),$$

  which is equal to $\sqrt{3}\,\alpha \cdot \alpha$ since $\alpha \cdot \alpha = \beta \cdot \beta$ if $\Lambda\, dz^2 = 0$. Since $\varepsilon_u = \alpha$, $\varepsilon_v = \beta$ we have $g = \sqrt{3}g'$.

□

Proofs of the main theorems

Theorem

Theorem

Every almost complex topological 2-sphere in $S^3 \times S^3$ is totally geodesic.

Proof.

By a corollary, $\Lambda\, dz^2 \equiv 0$, so we have a CMC 2-sphere in $\mathbb{R}^3$. This is a round sphere (by a theorem of Alexandrov), hence it is totally umbilical. Therefore the Gauss curvature of the CMC 2-sphere is $H^2 = 4/3$. Hence the Gauss curvature of the almost complex sphere in $S^3 \times S^3$ is $4/(3\sqrt{3})$. The Gauss equation then says

$$2\|h(v,v)\|^2 = \frac{4}{3\sqrt{3}} - K = 0.$$

□