Genuine infinitesimal bendings of Euclidean submanifolds

Miguel Ibieta Jimenez

Joint work with **M. Dajczer**

Introduction

- A basic problem in submanifold theory is to determine if a given isometric immersion

$$f\colon M^n \to \mathbb{R}^{n+p}$$

of a Riemannian manifold M^n into Euclidean space $\mathbb{R}^{n+p}$, is unique up to isometries of the ambient space.

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  of a Riemannian manifold M^n into Euclidean space $\mathbb{R}^{n+p}$, is unique up to isometries of the ambient space.

- If it is unique we say that f is *isometrically rigid*, otherwise we say it is *isometrically deformable*.

- An *isometric bending* of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ is a smooth variation
$$\mathcal{F}\colon (-\epsilon,\epsilon)\times M^n \to \mathbb{R}^{n+p}$$
of $\mathcal{F}(0,\cdot) = f(\cdot)$ such that $f_t = \mathcal{F}(t,\cdot)\colon M^n \to \mathbb{R}^{n+p}$ is an isometric immersion for any $t \in (-\epsilon,\epsilon)$.

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- It is said to be trivial if
$$\mathcal{F}(t,x) = C(t)f(x) + v(t)$$
where $C(t)$ is an orthogonal transformation of $\mathbb{R}^{n+p}$ and $v(t) \in \mathbb{R}^{n+p}$ for each $t \in (-\epsilon,\epsilon)$.

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- Global case: Sacksteder (1962), Dajczer and Gromoll (1990).

Theorem (Dajczer-Gromoll, 1995)

An isometrically deformable compact Euclidean submanifold of dimension at least five and codimension two, satisfies that along connected components of an open dense subset it is either isometrically rigid or is contained in a deformable hypersurface (with possible singularities) and any isometric deformation of the former is given by an isometric deformation of the latter.

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Infinitesimal bendings

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Definition

Given an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, a section $\mathcal{T}$ of $f^*T\mathbb{R}^{n+p}$ is called an *infinitesimal bending* of f if

$$\langle \tilde{\nabla}_X \mathcal{T}, f_* Y\rangle + \langle \tilde{\nabla}_Y \mathcal{T}, f_* X\rangle = 0$$

for any $X, Y \in \mathfrak{X}(M)$, where $\tilde{\nabla}$ denotes the Euclidean connection.

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$$\mathcal{T}(x) = \mathcal{D}f(x) + w$$

where $\mathcal{D}$ is a skew-symmetric linear endomorphism of $\mathbb{R}^{n+p}$ and $w \in \mathbb{R}^{n+p}$.

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- An isometric immersion is said to be *infinitesimally bendable* if it admits a non-trivial infinitesimal bending, otherwise we say it is *infnfinitesimally rigid*.

Infinitesimal bendings of hypersurfaces

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- Infinitesimal bendings of hypersurfaces (dimension at least 3) were considered by Sbrana (1908) and later by Schouten (1928).

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- From their work it follows that the class of infinitesimally bendable hypersurfaces is much larger than the class of isometrically bendable ones.
- Infinitesimal bendings of compact hypersurfaces were considered by Dajczer and Rodríguez (1990).

Notation

- The second fundamental form of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ is denoted by

$$\alpha(X,Y) = (\tilde{\nabla}_X f_* Y)_{N_f M},$$

where $N_f M$ denotes the normal bundle of f.

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- The *relative nullity* subspace $\Delta(x)$ of f at $x \in M^n$ is the kernel of the second fundamental form, that is

$$\Delta(x) = \{X \in T_x M : \alpha(X, Y) = 0 \ \text{ for all } \ Y \in T_x M\}.$$

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$$\Delta(x) = \{X \in T_x M : \alpha(X,Y) = 0 \;\text{ for all }\; Y \in T_x M\}.$$

- The dimension $\nu(x)$ of $\Delta(x)$ is called the *index of relative* nullity of f at $x \in M^n$. It is a standard fact that on open subsets of M^n where $\nu > 0$ is constant Δ defines a smooth totally geodesic distribution whose leaves are mapped by f to open subsets of affine subspaces of $\mathbb{R}^{n+p}$.

- An isometric immersion $G\colon M^n \times \mathbb{R}^k \to \mathbb{R}^{n+p} \times \mathbb{R}^k$ of the Riemannian product $M^n \times \mathbb{R}^k$ is called a *k-cylinder* (cylinder) over the isometric immersion $g\colon M^n \to \mathbb{R}^{n+p}$, if it factors as

$$G = g \times Id\colon M^n \times \mathbb{R}^k \to \mathbb{R}^{n+p} \times \mathbb{R}^k$$

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- A connected component of the subset of a ruled hypersurface M^n where the rulings are all complete is called a *ruled strip*.

Theorem

Let $f\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 4$, be an isometric immersion of a complete Riemannian manifold. Assume that there is no open subset of M^n where f is either totally geodesic or a cylinder over a hypersurface in $\mathbb{R}^4$ with complete one-dimensional leaves of relative nullity. Then f admits non-trivial infinitesimal bendings only along ruled strips.

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Corollary

Let $f\colon M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold satisfying the hypothesis of the previous theorem. If $\mathcal{T}$ is a non-trivial infinitesimal bending of f, then $\mathcal{T}$ is the variational vector field of an isometric bending.

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- If $\tilde{\mathcal{T}}$ is an infinitesimal bending of an isometric immersion

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and $j\colon M^n \to \tilde{M}^{n+\ell}$ is an embedding, then $\mathcal{T} = \tilde{\mathcal{T}}|_{j(M)}$ is an infinitesimal bending of $f = F \circ j$.

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- This motivates the following definition, where a more general situation is considered since some singularities are allowed.

Definition

A smooth map $F\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$, $0 < \ell < p$, from a differentiable manifold into Euclidean space is said to be an *singular extension* of a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ if there exists an embedding $j\colon M^n \to \tilde{M}^{n+\ell}$, $0 < \ell < p$, such that F is an immersion along $\tilde{M}^{n+\ell} \setminus j(M)$ and $f = F \circ j$.

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Definition

An infinitesimal bending $\mathcal{T}$ of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ *extends in the singular sense* if there is a singular extension $F\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$ of f and a smooth map

$$\tilde{\mathcal{T}}\colon \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$$

such that $\tilde{\mathcal{T}}$ is an infinitesimal bending of $F_{\tilde{M}\setminus j(M)}$ and $\mathcal{T} = \tilde{\mathcal{T}}|_{j(M)}$.

Genuine infinitesimal bendings

Definitions

An infinitesimal bending $\mathcal{T}$ of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, is called a *genuine infinitesimal bending* if $\mathcal{T}$ does not extend in the singular sense when restricted to any open subset of M^n.

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If an immersion admits such a bending, we say it is *genuinely infinitesimally bendable*.

And f is said to be *genuinely infinitesimally rigid* if given any infinitesimal bending $\mathcal{T}$ of f there is an open dense subset of M^n such that $\mathcal{T}$ restricted to any connected component extends in the singular sense.

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Problem

Find necessary conditions for a submanifold to be genuinely infinitesimally bendable.

Examples

- The first normal space of $f\colon M^n \to \mathbb{R}^{n+p}$ at $x \in M^n$ is

$$N_1(x) = \mathrm{span}\{\alpha(X,Y) : X, Y \in T_xM\}.$$

Then

$$\mathcal{T} = f_* Z + \delta$$

is an infinitesimal bending if $Z \in \mathfrak{X}(M)$ is a Killing field and $\delta \in \Gamma(N_1^\perp)$ is a smooth normal vector field.

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is an infinitesimal bending if $Z \in \mathfrak{X}(M)$ is a Killing field and $\delta \in \Gamma(N_1^\perp)$ is a smooth normal vector field.

- If $\mathcal{T} = \mathcal{D}f(x) + w$ is a trivial infinitesimal bending of

$$f\colon M^n \to \mathbb{R}^{n+p}, \quad p \geq 2.$$

Then, as expected, $\mathcal{T}$ always extends in the singular sense.

The associated tensor

- Let $\mathcal{T}$ be an infinitesimal bending of a isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$. We define $L \in \Gamma(\mathrm{End}(TM, f^*T\mathbb{R}^{n+p}))$ by

$$LX = \tilde{\nabla}_X \mathcal{T},$$

where $\tilde{\nabla}$ is the Levi-Civita connection in $\mathbb{R}^{n+p}$.

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$$LX = \tilde{\nabla}_X \mathcal{T},$$
where $\tilde{\nabla}$ is the Levi-Civita connection in $\mathbb{R}^{n+p}$.
- Hence the equation of an infinitesimal bending can be written as
$$\langle LX, f_*Y\rangle + \langle LY, f_*X\rangle = 0$$
for any $X, Y \in \mathfrak{X}(M)$.

- We define the symmetric tensor $\mathcal{B}\colon TM \times TM \to f^*T\mathbb{R}^{n+p}$ as

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- Then we define the symmetric tensors $\mathcal{Y}\colon TM \times TM \to f_*TM$ and $\beta\colon TM \times TM \to N_fM$ by taking tangent and normal components of $\mathcal{B}$

$$\mathcal{B}(X, Y) = \mathcal{Y}(X, Y) + \beta(X, Y).$$

Proposition

The tensor $\beta\colon TM \times TM \to N_fM$ *satisfies*

$$\langle\beta(X,W),\alpha(Y,Z)\rangle + \langle\alpha(X,W),\beta(Y,Z)\rangle = \langle\beta(X,Z),\alpha(Y,W)\rangle + \langle\alpha(X,Z),\beta(Y,W)\rangle$$

and

$$(\nabla^\perp_X\beta)(Y,Z) - (\nabla^\perp_Y\beta)(X,Z) = \alpha(Y,\mathcal{Y}(X,Z)) - \alpha(X,\mathcal{Y}(Y,Z)) - (LR(X,Y)Z)_{N_fM}$$

for any $X,Y,Z,W \in \mathfrak{X}(M)$.

Here R *is the curvature tensor of* M^n *and* $\nabla^\perp$ *is the normal connection.*

Local results

The first local result

Definition

An isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ is said to be *r-ruled* if there exists an r-dimensional smooth totally geodesic tangent distribution whose leaves (rulings) are mapped diffeomorphically by f to open subsets of affine subspaces of $\mathbb{R}^{n+p}$.

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Theorem

Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p \geq 4$, be an isometric immersion and let $\mathcal{T}$ be an infinitesimal bending of f. Then along each connected component of an open and dense subset either $\mathcal{T}$ extends in the singular sense or f is r-ruled with $r \geq n - 2p$.

Corollary

Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p \geq 4$, be a genuinely infinitesimally bendable isometric immersion. Then f is r-ruled with $r \geq n - 2p$ along connected components of an open dense subset of M^n.

Corollary

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Corollary

Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p \geq 4$, be an isometric immersion. If M^n has positive Ricci curvature then f is genuinely infinitesimally rigid.

Proof of the first local result

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- Consider the map

$$F(x,t) = f(x) + tf_*Z(x),$$

where $Z \in \mathfrak{X}(V)$ is nowhere vanishing defined on an open subset $V \subset M^n$.

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where $Z \in \mathfrak{X}(V)$ is nowhere vanishing defined on an open subset $V \subset M^n$.

- Then define $\tilde{\mathcal{T}}\colon V \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+p}$ by

$$\tilde{\mathcal{T}}(x,t) = \mathcal{T}(x) + tLZ(x).$$

- The maps F and $\tilde{\mathfrak{T}}$ satisfy:

$$\langle F_*\partial_t, \tilde{\nabla}_{\partial_t}\tilde{\mathfrak{T}}\rangle = \langle f_*Z, LZ\rangle = 0,$$

and

$$\langle \tilde{\nabla}_{\partial_t}\tilde{\mathfrak{T}}, F_*X\rangle + \langle \tilde{\nabla}_X\tilde{\mathfrak{T}}, F_*\partial_t\rangle = 0.$$

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- Hence, for $\tilde{\mathfrak{T}}$ and F to verify the equation of an infinitesimal bending we need the following expression to vanish:

$$\begin{aligned}\langle F_* X, \tilde{\nabla}_X \tilde{\mathfrak{T}}\rangle =& \langle f_* X + t\tilde{\nabla}_X Z, LX + t\tilde{\nabla}_X LZ\rangle \\ =& \langle \alpha(X,Z), \beta(X,Z)\rangle,\end{aligned}$$

for all $X \in \mathfrak{X}(V)$.

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for all $X \in \mathfrak{X}(V)$.

- Therefore we look for directions Z such that the right side of the expression above vanishes for all $X \in \mathfrak{X}(V)$.

Flat bilinear forms

- Let V and U be finite dimensional real vector spaces and let $W^{p,q}$ be a real vector space of dimension $p+q$ endowed with an indefinite inner product of type (p,q).
- A bilinear form $\phi\colon V\times U\to W^{p,q}$ is said to be *flat* if

$$\langle\phi(X,Z),\phi(Y,T)\rangle-\langle\phi(X,T),\phi(Y,Z)\rangle=0$$

for all $X,Y\in V$ and $T,Z\in U$.

Flat bilinear forms

A vector $Y \in V$ is called a (left) *regular element* of ϕ if

$$\dim \phi_Y(U) = \max\{\dim \phi_X(U) \colon X \in V\}$$

where $\phi_Y(W) = \phi(Y, W)$ for any $W \in U$.

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Lemma

Let $\phi \colon V \times U \to W$ be a flat bilinear form. If $Y \in RE(\phi)$ then

$$\phi(X, \ker \phi_Y) \subset \phi_Y(U) \cap \phi_Y(U)^\perp$$

for any $X \in V$.

- Let $\theta\colon TM \times TM \to N_fM \oplus N_fM$ be defined at any point of M^n by

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Lemma

The bilinear form θ is flat with respect to the inner product in $N_f M \oplus N_f M$ given by

$$\langle\langle (\xi_1,\eta_1),(\xi_2,\eta_2)\rangle\rangle_{N_f M \oplus N_f M} = \langle \xi_1,\xi_2\rangle_{N_f M} - \langle \eta_1,\eta_2\rangle_{N_f M}.$$

- Let $U \subset M^n$ be an open subset where $Y \in \mathfrak{X}(U)$ satisfies $Y \in RE(\theta)$ and $D = \ker \theta_Y$ has dimension d at any point.

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for any $X \in \mathfrak{X}(U)$ and $Z \in \Gamma(D)$.

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- Equivalently

$$\langle \alpha(X,Z), \beta(X,Z) \rangle = 0$$

for any $X \in \mathfrak{X}(U)$ and $Z \in \Gamma(D)$.

- Therefore, if there is an open subset where a nowhere vanishing $Z \in \Gamma(D)$ induces a singular extension of f, the infinitesimal bending $\mathcal{T}$ extends in the singular sense.

- Therefore, if there is an open subset where a nowhere vanishing $Z \in \Gamma(D)$ induces a singular extension of f, the infinitesimal bending $\mathcal{T}$ extends in the singular sense.
- We conclude the proof using the following result

Proposition (Florit and Guimarães, 2018)

Let $f\colon M^n \to \mathbb{R}^{n+p}$ be an isometric immersion and let D be a smooth tangent distribution of dimension $d > 0$. Assume that there does not exist an open subset $U \subset M^n$ and $Z \in \Gamma(D|_U)$ such that the map $F\colon U \times \mathbb{R} \to \mathbb{R}^{n+p}$ given by

$$F(x,t) = f(x) + tf_*Z(x)$$

is a singular extension of f on some open neighborhood of $U \times \{0\}$. Then for any $x \in M^n$ there is an open neighborhood V of the origin in $D(x)$ such that $f_*(x)V \subset f(M)$. Hence f is d-ruled along each connected component of an open dense subset of M^n.

- Let $F\colon \tilde{M}^{n+1} \to \mathbb{R}^{n+p}$ be an isometric immersion and let $\tilde{\mathcal{T}}$ be an infinitesimal bending of F. Given an isometric embedding $j\colon M^n \to \tilde{M}^{n+1}$ call

$$f = F \circ j\colon M^n \to \mathbb{R}^{n+p}.$$

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- Moreover

$$\langle \beta(X,Y), F_*\eta\rangle + \langle \alpha(X,Y), \tilde{L}\eta\rangle = 0$$

  for $\eta \in \Gamma(N_jM)$ of unit length, $X, Y \in \mathfrak{X}(M)$ and $\tilde{L}$ is associated to $\tilde{\mathfrak{T}}$.

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- A condition of this type may guarantee that an infinitesimal bending is not genuine. In fact, this was already proved by Florit (1996) in a special case.

The condition $(*)$

- An infinitesimal bending of an isometric immersion

$$f\colon M^n \to \mathbb{R}^{n+p}, \ \ p \geq 2,$$

satisfies the *condition* $(*)$ if there is $\eta \in \Gamma(N_f M)$ of unit length and $\xi \in \Gamma(R)$, where R is determined by the orthogonal splitting $N_f M = P \oplus R$ and $P = \text{span}\{\eta\}$, such that

$$\langle \beta(X,Y), \eta \rangle + \langle \alpha(X,Y), \xi \rangle = 0,$$

for any $X, Y \in \mathfrak{X}(M)$.

The second local result

Theorem

Let $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, be an isometric immersion and let $\mathcal{T}$ be a genuine infinitesimal bending of f that satisfies the condition $(*)$. Then f is r-ruled with $r \geq n - 2p + 3$ on connected components of an open dense subset of M^n.

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The proof of the result above follows the same ideas as the proof of the first local result. However an additional effort is needed in order to find a suitable flat bilinear form.

Lemma

Let $\phi\colon V^n \times V^n \to W^{p,q}$, $p \leq 5$ *and* $p+q < n$, *be a symmetric flat bilinear form and set*

$$\mathcal{N}(\phi) = \{X \in V : \phi(X,Y) = 0 \ \text{ for all } \ Y \in V\}.$$

If $\dim \mathcal{N}(\phi) \leq n-p-q-1$ *then there is an orthogonal decomposition*

$$W^{p,q} = W_1^{\ell,\ell} \oplus W_2^{p-\ell,q-\ell},\ 1 \leq \ell \leq p,$$

such that the W_j*-components* ϕ_j *of* ϕ *satisfy:*

(i) ϕ_1 *is nonzero and satisfies*

$$\langle \phi_1(X,Y), \phi_1(Z,T) \rangle = 0$$

for all $X, Y, Z, T \in V$.

(ii) ϕ_2 *is flat and* $\dim \mathcal{N}(\phi_2) \geq n-p-q+2\ell$.

The previous lemma allows us to be more precise in the case of low codimension:

Theorem

Let $f\colon M^n \to \mathbb{R}^{n+p}$, $n > 2p$, be a genuinely infinitesimally bendable isometric immersion. If $2 \leq p \leq 5$, then one of the following holds along any connected component, say U, of an open dense subset of M^n:

(i) *$f|_U$ is ν-ruled by leaves of relative nullity with $\nu \geq n - 2p$.*

(ii) *$f|_U$ has $\nu < n - 2p$ at any point and is r-ruled with $r \geq n - 2p + 3$.*

Global result

The nullity of θ

Let Δ^* be the kernel of the bilinear form θ. Then

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Proposition

Let $\mathcal{T}$ be an infinitesimal bending of $f : M^n \to \mathbb{R}^{n+p}$ and let θ be the flat bilinear form associated to $\mathcal{T}$. Denote

$$\nu^*(x) = \dim \Delta^*(x)$$

at $x \in M^n$. Then, on any open subset of M^n where ν^* is constant the distribution Δ^* is totally geodesic and its leaves are mapped by f onto open subsets of affine subspaces of $\mathbb{R}^{n+p}$.

Lemma

Let $f: M^n \to \mathbb{R}^{n+p}$, $p \leq 5$ *and* $n > 2p$ *be an isometric immersion of a compact Riemannian manifold and let* $\mathcal{T}$ *be an infinitesimal bending of* f*. Then, at any* $x \in M^n$ *there is a pair of vectors* $\zeta_1, \zeta_2 \in N_f M(x)$ *of unit length such that* $(\zeta_1, \zeta_2) \in (\mathcal{S}(\theta))^{\perp}(x)$ *where*

$$\mathcal{S}(\theta)(x) = span\,\{\theta(X, Y) : X, Y \in T_x M\}.$$

Moreover, on any connected component of an open dense subset of M^n *the pair* ζ_1, ζ_2 *at* $x \in M^n$ *extend to smooth vector fields* ζ_1 *and* ζ_2 *parallel along* Δ^* *that satisfy the same conditions.*

Compact case

Theorem

Let $f : M^n \to \mathbb{R}^{n+2}$, $n \geq 5$, be an isometric immersion of a compact Riemannian manifold with no open flat subset. For any infinitesimal bending $\mathcal{T}$ of f one of the following holds along any connected component, say U, of an open dense subset of M^n:

(i) *The infinitesimal bending $\mathcal{T}|_U$ extends in the singular sense.*

(ii) *There is an orthogonal splitting $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \oplus \text{span}\{e\}$ so that $f(U) \subset \mathbb{R}^{n+1}$ and $\mathcal{T}|_U = \mathcal{T}_1 + \mathcal{T}_2$ is a sum of infinitesimal bendings that extend in the singular sense where $\mathcal{T}_1 \in \mathbb{R}^{n+1}$ and $\mathcal{T}_2 = \phi e$ for $\phi \in C^\infty(U)$.*

Proof

- It follows from the last lemma that, on connected components of an open dense subset of M^n there are $\zeta_1, \zeta_2 \in \Gamma(N_f M)$ with $\|\zeta_1\| = \|\zeta_2\| = 1$ and parallel along the leaves of Δ^* such that

  $$\langle\!\langle \theta(X, Y), (\zeta_1, \zeta_2) \rangle\!\rangle = 0$$

  for any $X, Y \in \mathfrak{X}(M)$.

Proof

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- Hence we have that

$$\langle \beta(X, Y), \zeta_1 + \zeta_2 \rangle + \langle \alpha(X, Y), \zeta_1 - \zeta_2 \rangle = 0.$$

Proof

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- Hence we have that

$$\langle \beta(X, Y), \zeta_1 + \zeta_2 \rangle + \langle \alpha(X, Y), \zeta_1 - \zeta_2 \rangle = 0.$$

- On an open subset $U \subset M^n$ where ζ_1, ζ_2 are smooth and

$$\zeta_1 + \zeta_2 \neq 0$$

  we have that $\mathcal{T}$ verifies the condition $(*)$.

- If $\mathcal{T}$ is genuine on an open subset $V \subset U$, our local results imply that f is $(n-1)$-ruled.

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- A careful analysis of the rulings and the hypothesis on M^n lead to a contradiction. Hence $\mathcal{T}$ extends in the singular sense on U.

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- On an open subset $U' \subset M^n$ where ζ_1, ζ_2 are smooth and

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we have that $f|_{U'}$ reduces codimension $f(U') \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$.

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- Thus, we have an orthogonal decomposition of $\mathcal{T}|_{U'}$ as in part (ii) of the statement and $\mathcal{T}_1, \mathcal{T}_2$ extend in the singular sense as follows:
  - (i) $\bar{\mathcal{T}}_1(x,t) = \mathcal{T}_1(x)$ to $F\colon U \times \mathbb{R} \to \mathbb{R}^{n+2}$ where $F(x,t) = f(x) + te$.
  - (ii) For instance locally as $\bar{\mathcal{T}}_2(x,t) = \mathcal{T}_2(x)$ to $F\colon U \times I \to \mathbb{R}^{n+2}$ where $F(x,t) = f(x) + tN$ being N is a unit normal field to $f|_U$ in $\mathbb{R}^{n+1}$. □

What comes next?

Ongoing projects

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Gracias