Elastic curves and Willmore Hopf-Tori

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Variational Problems & Geometric PDE's
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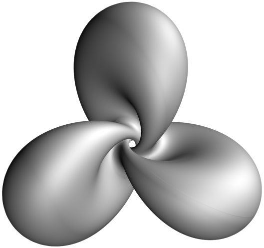

A brief history

- 1691 J. Bernoulli: first characterization of planar elastica
- 1742 D. Bernoulli: proposed variational techniques
- 1744 Euler: characterization using variational methods
- 1984 Langer-Singer: classified elastica on 2-sphere
- 1985 Pinkall: connection between elastica and Willmore tori
- 1991 Goldstein-Petrich: relationship to mKdV
- 2003/04 Arroyo-Garay-Mencia: closed elastica and generalizations
- 2008 Bohle-Peters-Pinkall: constrained elastica, and constrained Willmore Hopf-tori
- 2012 L.Heller, J.Zentgraf: PhD theses on constrained Willmore Hopf-tori etc.

Outline of talk:

- Joint work with F. Pedit and N. Schmitt
- Finite type curves on $\mathbb{S}^2$
- Polynomial Killing fields and spectral curves
- Whitham deformation
- Constrained elastic curves
- Moduli space of constrained Willmore Hopf-tori

- If $h : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf-fibration, then $h^{-1}(\gamma)$ for a closed curve $\gamma : \mathbb{R} \to \mathbb{S}^2$ is called a *Hopf-torus*. The curve is called its *profile curve*
- A Hopf-torus is a constrained Willmore Hopf-torus iff its profile curve is a closed constrained elastic curve (area and length constraint). Its geodesic curvature satisfies

$$\kappa'^2 + \frac{1}{4}\kappa^4 + a\,\kappa^2 + b\,\kappa + c = 0\,.$$

- When $b = 0$ we speak of elastic curves (area constraint).
- The torus is a Willmore Hopf-torus iff the profile curve is a closed free elastic curve (no constraint). Its geodesic curvature satisfies

$$\kappa'^2 + \frac{1}{4}\kappa^4 + \frac{1}{2}\,\kappa^2 + c = 0\,.$$

Constrained Willmore Hopf-tori

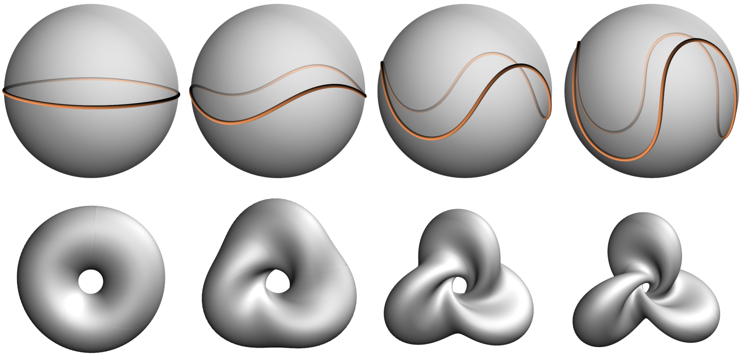

Figure: A sequence of 3-lobed constrained Willmore Hopf-tori, starting at the Clifford torus, and ending at a Willmore Hopf-torus

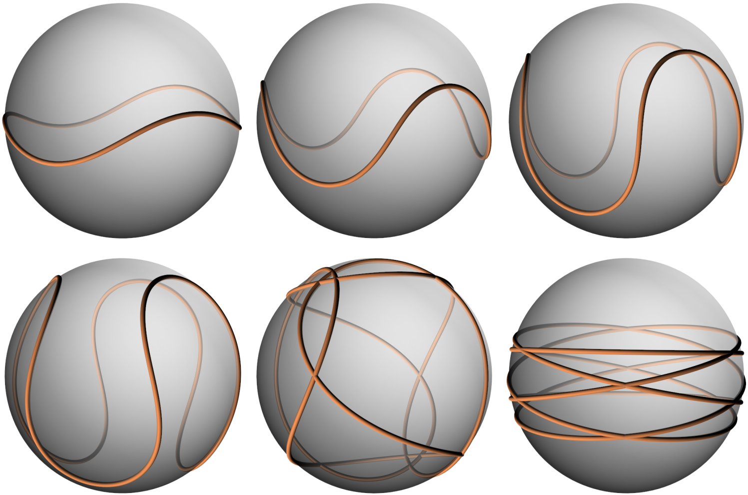

Figure: A deformation through constrained elastic curves, starting near a 1-wrapped equator and ending near a 5-wrapped equator

Theorem

The Willmore energy along each deformation family containing a Willmore Hopf-torus attains its maximum at this torus. The 2-lobed Willmore Hopf-torus has the least energy among the Willmore Hopf-tori.

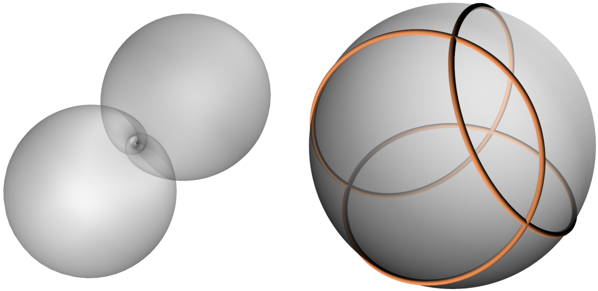

Figure: The 2-lobed Willmore torus, and its free elastic profile curve.

Curves in $\mathbb{S}^2$

Let $\gamma : \mathbb{R} \to \mathbb{S}^2$ be a smooth immersed arc-length parametrized curve with geodesic curvature κ. The frame $G : \mathbb{R} \to \mathrm{SO}_3$ of γ with columns

$$G = (\gamma, \gamma', \gamma \times \gamma')$$

satisfies

$$G' = G \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\kappa \\ 0 & \kappa & 0 \end{pmatrix} .$$

Here *prime* denotes differentiation with respect to arc-length.

mKdV hierarchy

The mKdV hierarchy is an infinite hierarchy of evolution equations with first three equations

$$\begin{aligned}
\dot{\kappa} &= -\kappa' \\
\dot{\kappa} &= -k''' - \tfrac{3}{2}\kappa^2\kappa' \\
\dot{\kappa} &= -k''''' - \tfrac{15}{8}\kappa^4\kappa' - \tfrac{5}{2}\kappa^2\kappa''' - 10\kappa\kappa'\kappa'' \\
&\vdots
\end{aligned}$$

A curve is called *of finite type* if κ is stationary under all but fintely many of the flows in this hierarchy.

Reconstruction

Via the double cover $\mathrm{SU}_2 \to \mathrm{SO}_3$ the *extended frame* equation can be written as

$$F_\lambda' = F_\lambda \begin{pmatrix} \mathbb{i}\lambda & \kappa \\ -\kappa & -\mathbb{i}\lambda \end{pmatrix} \qquad \text{where } \lambda \in \mathbb{R}\,.$$

The associate family of curves $\mathbb{R} \to \mathrm{SU}_2 \cap \mathbb{R}^3 \cong \mathbb{S}^2$ given by

$$\gamma_\lambda = F_\lambda \, F_{-\lambda}^{-1}$$

has geodesic curvature $\lambda^{-1}\kappa$, and constant speed $|\lambda|$.
Call the evaluation points the *sym points* (w.l.o.g. take $\lambda = 1$).

Polynomial Killing fields

A *polynomial Killing field* of a curve on $\mathbb{S}^2$ is a polynomial

$$X = \sum_{k=0}^{d} X_k \lambda^k \quad \text{where } X_d = \begin{pmatrix} \mathbb{i} & 0 \\ 0 & -\mathbb{i} \end{pmatrix}$$

with $X_k : \mathbb{R} \to \mathrm{su}_2$ smooth that satisfies a Lax equation

$$X' = \tfrac{1}{2}[X, V(X)] \quad \text{with} \quad V(X) := X_{d-1} + X_d \lambda \,,$$

and is *twisted* (or *anti-twisted*), so that

$$X(-\lambda) = \pm \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) X(\lambda) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) .$$

Finite type curves

A *finite type curve* $\mathbb{R} \to \mathbb{S}^2$ is one whose extended frame satisfies

$$F_\lambda' = \tfrac{1}{2} F_\lambda V(X)\,, \quad F_\lambda(0) = \mathbb{1}$$

for some polynomial Killing field X.
Let $X(0) = X_0$. Then

$$F_\lambda^{-1} X_0 F_\lambda = X\,,$$

so $\det X = \det X_0$ is independent of arc-length. By the twistedness condition $\det X$ is an even polynomial.

Spectral curve

- Suppose X is a polynomial Killing field of degree d, and $X_0 = X(0)$.
- Since $X_0 \in \mathrm{su}_2$ conclude that $\det X$ is a real, even polynomial of degree $2d$.
- Assume all the roots of $\det X$ are simple.
- The *spectral curve* Σ of X is the hyperelliptic curve over $\mathbb{C}$ with branchpoints only at the roots of $\det X$. It is a compact Riemann surface of genus $g = d - 1$.
- The genus g of Σ is called *spectral genus*.

Closing conditions

Let X be a periodic polynomial Killing field with period ρ, and extended frame F_λ. Then the corresponding curve is closed if and only if the *monodromy* $M_\lambda = F_\lambda(\rho)$ satisfies

$$M_1 = M_{-1} = \mathbb{1} \quad \text{or} \quad M_1 = M_{-1} = -\mathbb{1}\,.$$

From $F_\lambda^{-1} X_0 F_\lambda = X$ and $X_0 = X(\rho)$ it follows that

$$[X_0,\, M_\lambda] = 0\,.$$

An eigenvalue μ of M_λ is a meromorphic function on Σ whose only poles are simple poles over the two points $\infty^\pm$.

Whitham deformation

Hence $d\log\mu$ is an abelian differential of the 2^{nd} kind on Σ, so of the form

$$d\log\mu = \frac{B}{\sqrt{A}}\,d\lambda$$

where $A = \det X$ and $\lambda \mapsto B(\lambda)$ is a real polynomial of degree $d = g + 1$.
Suppose A and B depend on an additional parameter $t \in \mathbb{R}$. Can write

$$\frac{d}{dt}\log\mu = \frac{C}{\sqrt{A}}$$

where $\lambda \mapsto C(\lambda)$ is a polynomial of degree at most g.

The deformation in the parameter t is subject to the integrability condition

$$\frac{d^2}{d\lambda\, dt} \log \mu = \frac{d^2}{dt\, d\lambda} \log \mu \,,$$

or equivalently in terms of the polynomials A, B and C that

$$2\,A\,\dot{B} - \dot{A}\,B = 2\,AC' - A'C\,. \qquad (3.1)$$

The variables in the flow are the $2g+2$ coefficients of A, and the $g+1$ coefficients of B. The $g+1$ coefficients of C can be chosen freely.

To preserve closing conditions $\mu(\lambda) = \mu(-\lambda) = \pm 1$ at $\lambda = 1$ during the deformation, also require

$$\frac{d}{dt}\log\mu\Big|_{\lambda=\pm1} = \left(\left(\frac{d}{d\lambda}\log\mu\right)\dot{\lambda} + \frac{d}{dt}\log\mu\right)\Big|_{\lambda=\pm1} = 0\,,$$

or equivalently

$$\dot{\lambda}\Big|_{\lambda=\pm1} = -\left.\frac{C}{B}\right|_{\lambda=\pm1}\,. \tag{3.2}$$

Thus we may fix the sym points at $\lambda = \pm 1$ during the flow by picking C to have fixed zeroes there. Equations (3.1) and (3.2) define a meromorphic vector field on the parameter space of coefficients of A and B.

Increasing the spectral genus

Lemma

Let X be a polynomial Killing field. Then $(\lambda^2 - a^2)\, X$ for $a \in \mathbb{R}$ is a polynomial Killing field, and both induce the same curve.

Opening double points:

- Suppose $a \in \mathbb{R}$ is a double point on Σ.
- Then $\mu(a) = \pm 1$ and write

$$d \log \mu = \frac{(\lambda^2 - a^2)\, B}{(\lambda^2 - a^2)\sqrt{A}}\, d\lambda\,.$$

- Now can pick initial condition for the flow at $t = 0$ such that when $t > 0$ the roots $\pm a$ move vertically off the real axis.

Circles

The simplest polynomial Killing field is

$$X = \begin{pmatrix} \mathbb{i}\lambda & \kappa \\ -\kappa & -\mathbb{i}\lambda \end{pmatrix}$$

with $\kappa \in \mathbb{R}$ constant. The corresponding curves are circles with (geodesic) curvature κ, and the branchpoints of the spectral curve are located at $\pm\mathbb{i}\kappa$. Since $\log\mu = \pi\mathbb{i}\sqrt{\lambda^2+\kappa^2}$, doublepoints are $a \in \mathbb{R}$ with $a^2+\kappa^2 \in \mathbb{Z}^2$.

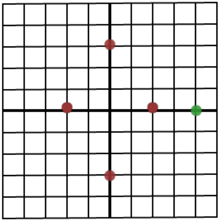

Figure: The spectral curve of a bifurcating circle.

The branchpoints of the spectral curve of an elastic curve are the roots of the real even quartic polynomial

$$m_4 + m_2\lambda^2 + \lambda^4 .$$

With discriminant $\Delta = \frac{1}{4}m_2 - m_4$, away from degenerate situations, there are two cases

wave-like if $\Delta < 0$: distinct branchpoints $\pm p_1, \pm p_2$ with $p_2 = \bar{p}_1$

orbit-like if $\Delta > 0$: distinct branchpoints on $\mathbb{i}\mathbb{R}^{\times}$.

In the case $\Delta = 0$ the spectral curve is *singular*.

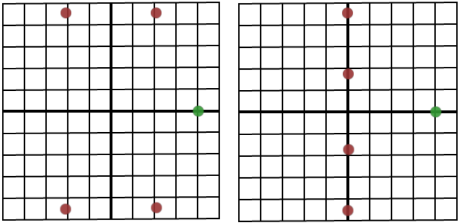

Figure: Spectral curves of wave-like and orbit-like elastic curves.

Flowing off a great circle by opening up a pair of branchpoints increases the spectral genus from $g = 0$ to $g = 1$. The flow through closed elastic curves of spectral genus $g = 1$ are given by

$$\dot{p}_1 = (1 - p_1^2)(p_2 + p_1\beta)\,,$$
$$\dot{p}_2 = (1 - p_2^2)(p_1 + p_2\beta)\,.$$

Qualitative analysis of these equations yield a picture of the moduli space of closed elastic curves on $\mathbb{S}^2$.

Properties of elastic curves

- Wave-like elastica bifurcate off great circles.
- Free elastic curves (no area or length constraint) are wave-like.
- Orbit-like elastica bifurcate off non-great, at least twice wrapped circles. They are never embedded, and always end in a singular limit.
- Some wave-like families deform between ω_1–wrapped and ω_2–wrapped equators. Denote the the lobe number by l. Then for $l/\omega_1 \in (1/2,\, 1) \cap \mathbb{Q}$ have

$$\omega_1 + \omega_2 = 2l\,.$$

- Free elastica are only contained in families with $l/\omega_1 \in (1/2,\, 1/\sqrt{2})$.

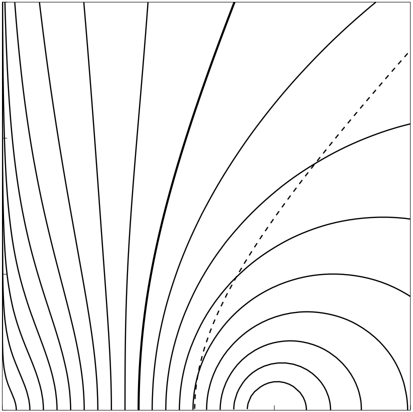

Figure: Closed wave-like elastica lie on a discrete dense set of curves. The position of the branchpoint in the first quadrant is shown. All 1-parameter family of closed wave-like elastica start at a multiwrapped equator (along the real axis). The semicircle-like curves represent families which end at multiwrapped equators, those to the left of the thick line end in a singular limit. The thick curve marks the boundary between these two types. The dashed curve represents the free elastica.

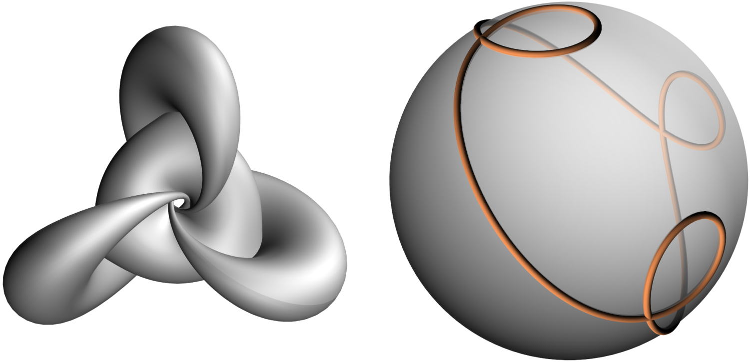

Figure: Orbit-like example with orbit-like elastic profile curve.

Constrained elastica on $\mathbb{S}^2$ arise from anti-twisted polynomial Killing fields of degree 3:

$$X = \begin{pmatrix} \mathbb{i}A & B \\ -B^* & -\mathbb{i}A \end{pmatrix}$$

where

$$\begin{aligned} A &= \tfrac{1}{2}(m_2 - \kappa^2)\lambda + \lambda^3 \\ B &= -(\kappa'' + \tfrac{1}{2}\kappa^3 - \tfrac{1}{2}m_2\kappa) + \mathbb{i}\kappa'\lambda + \kappa\lambda^2 \,. \end{aligned}$$

The branchpoints of the spectral curve are the zeros of

$$\det X = m_6 + m_4\lambda^2 + m_2\lambda^4 + \lambda^6$$

where the curvature κ satisfies

$$\kappa'^2 + \tfrac{1}{4}\kappa^4 - \tfrac{1}{2}m_2\kappa^2 + 2\sqrt{m_6}\kappa + \tfrac{1}{4}m_2^2 - m_4 = 0 \,.$$

Assuming that none of the zeros of $\det X$ are real, three classes of curves in $\mathbb{S}^2$ with curvature κ can be distinguished:

- constrained elastic: area and length constraint. no conditions;
- elastic: area constraint, but no length constraint, $m_6 = 0$;
- free elastic: no area or length constraint, $m_6 = 0$ and $m_2 = -1$.

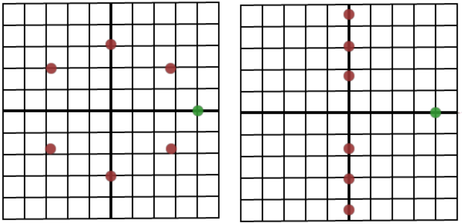

Figure: Spectral curves of wave-like and orbit-like constrained elastica.

Constrained elastica all arise from a simple 'translational' Whitham flow of elastica.

Theorem

The moduli space of closed constrained elastica is the cartesian product of the 1-dimensional moduli space of closed elastica with the half-open interval $[0, 1)$*.*

Corollary

The moduli space of closed constrained elastica has two connected components, one for each homotopy type.

Lemma

The conformal type of a constrained Willmore Hopf-torus is

$$\tau = \frac{\mathcal{A}}{4\pi} + \frac{\mathbb{i}\mathcal{L}}{4\pi} = \begin{cases} \frac{\omega}{2} + \frac{\mathbb{i}\mathcal{L}}{4\pi} & \textit{for wave-like}\ , \\ \frac{\omega - l}{2} + \frac{\mathbb{i}\mathcal{L}}{4\pi} & \textit{for orbit-like} \end{cases}$$

Proof.

Computing $\mathcal{A}$ using Gauss-Bonnet gives

$$\mathcal{A} = 2\pi\omega - l\int_0^\rho \kappa$$

where ρ is the period of κ. A computation gives

$$\int_0^\rho \kappa \ = \begin{cases} 0 & \text{for wave-like}\ , \\ 2\pi & \text{for orbit-like} \end{cases}$$

□

Theorem

Amongst the constrained Willmore Hopf-tori with conformal type near the Clifford torus, the 2-lobed constrained Willmore Hopf-tori have the least Willmore energy amongst the constrained Willmore Hopf-tori.

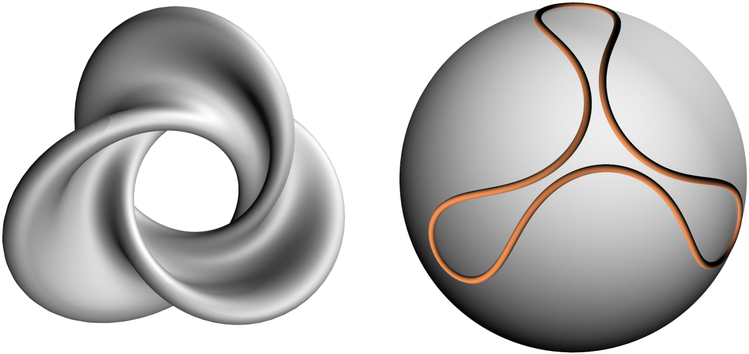

Figure: Spectral genus $g = 2$ constrained wave-like elastic profile curve, and corresponding constrained Willmore-Hopf-torus.

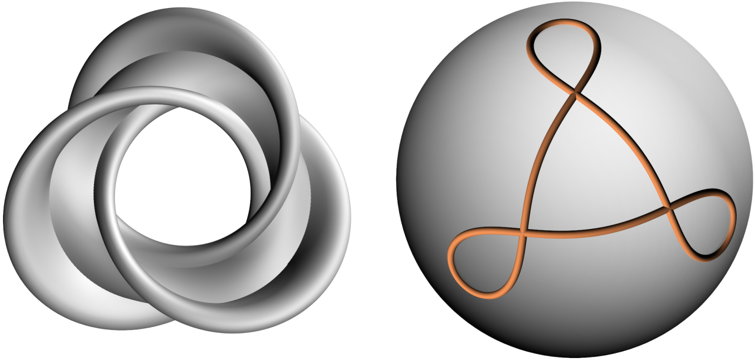

Figure: Nonembedded example. Spectral genus $g = 2$ constrained wave-like elastic profile curve, and corresponding constrained Willmore-Hopf-torus.

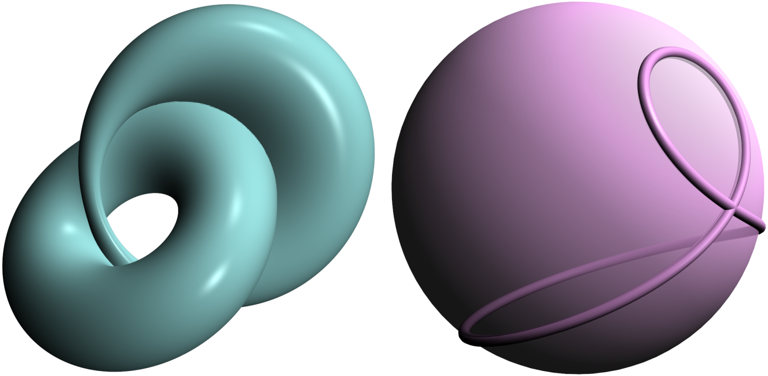

Figure: Willmore Hopf-torus with a singular spectral curve. The profile curve is a dressed circle.