

The work of Jesse Douglas on Minimal Surfaces

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Area

Let Σ be a surface, for example, the unit disc $\mathbb{B} \subset \mathbb{C}$. The area, $\mathcal{A}(\mathbf{r})$, of a map

$$\mathbf{r}: \Sigma \rightarrow \mathbb{R}^n$$

is given by the formula

$$\mathcal{A}(\mathbf{r}) := \int_{\Sigma} (\|\mathbf{r}_x\|^2 \|\mathbf{r}_y\|^2 - \langle \mathbf{r}_x, \mathbf{r}_y \rangle^2)^{1/2} dx dy.$$

The map \mathbf{r} is **minimal** if it is stationary for the area functional \mathcal{A} and its image is then a **minimal surface**.

First Variation of Area

Let

$$\mathbf{r}_t: \Sigma \rightarrow \mathbb{R}^n, \quad t \in (-\varepsilon, \varepsilon), \quad \varepsilon > 0$$

be a 1-parameter family of maps such that

$$\mathbf{r}_0 = \mathbf{r} \quad \text{and} \quad \mathbf{r}_t|_{\partial\Sigma} = \mathbf{r}|_{\partial\Sigma} \quad \forall t.$$

This is a [variation](#) of \mathbf{r} and the associated variation vector field \mathbf{s} is defined by

$$\mathbf{s} := \left. \frac{\partial \mathbf{r}_t}{\partial t} \right|_{t=0}.$$

$$(\delta\mathcal{A})(\mathbf{s}) := \left. \frac{\partial\mathcal{A}(\mathbf{r}_t)}{\partial t} \right|_{t=0} = - \int_{\Sigma} \mathbf{H} \cdot \mathbf{s} d\mathcal{A}$$

where \mathbf{H} is the [mean curvature vector](#) of \mathbf{r} .

In 1776, Meusnier gave an (incomplete) argument which established the vanishing of the mean curvature of a surface of least area. But he did not have the first variation of area formula as written above.

In any case, a minimal surface is characterised differential geometrically by having zero mean curvature.

A minimal surface need *not* minimize area! (e. g. thin catenoid.)

Conformal representation

According to a fundamental theorem (on the existence of isothermal coordinates) in the theory of surfaces, an immersion $\mathbf{r}: \Sigma \rightarrow \mathbb{R}^n$ can always be precomposed with a diffeomorphism of Σ so as to make it conformal, i. e., so as to satisfy

$$\|\mathbf{r}_x\|^2 = \|\mathbf{r}_y\|^2 \quad \text{and} \quad \langle \mathbf{r}_x, \mathbf{r}_y \rangle = 0.$$

The surface Σ can then be viewed as a Riemann surface with local complex coordinate $z = x + iy$.

With respect to isothermal coordinates, the mean curvature vanishes if, and only if, each component of \mathbf{r} is a harmonic function.

Plateau Problem as formulated at the time of Weierstrass:

Given a Jordan curve $\Gamma \subset \mathbb{R}^3$, find

$$\mathbf{r}: \mathbb{B} \rightarrow \mathbb{R}^3$$

such that

- (i) \mathbf{r} is harmonic and conformal,
- (ii) $\mathbf{r}|_{\partial\mathbb{B}}: \partial\mathbb{B} \rightarrow \Gamma$ is a homeomorphism.

The solution of this problem that is usually presented is based on the following *fundamental relation between energy and area*:

$$g(\mathbf{r}) := \left(\|\mathbf{r}_x\|^2 \|\mathbf{r}_y\|^2 - \langle \mathbf{r}_x, \mathbf{r}_y \rangle^2 \right)^{1/2} \leq \frac{1}{2} (\|\mathbf{r}_x\|^2 + \|\mathbf{r}_y\|^2) =: e(\mathbf{r})$$

with equality if, and only if, \mathbf{r} is conformal.

The Dirichlet energy $\mathcal{E}(\mathbf{r})$ of \mathbf{r} is defined by

$$\mathcal{E}(\mathbf{r}) := \int_{\mathbb{B}} e(\mathbf{r}) \, dx \, dy .$$

The conformal representation (if it exists) of an area minimizer has least Dirichlet energy among all maps of \mathbb{B} whose restriction to $\partial\mathbb{B}$ parameterises Γ .

This suggests the following procedure for establishing the existence of a solution to the Plateau problem.

One can write down the harmonic extension (energy minimiser) $\mathbf{r}_{\mathbf{g}}$ of a parameterisation $\mathbf{g}: \partial\mathbb{B} \rightarrow \Gamma$ by means of the Poisson integral formula.

One then seeks a special parameterisation \mathbf{g}^* such that

$$\mathcal{E}(\mathbf{r}_{\mathbf{g}^*}) \leq \mathcal{E}(\mathbf{r}_{\mathbf{g}})$$

for all parameterisations \mathbf{g} of Γ . The existence of \mathbf{g}^* is established using a 3-point condition (to overcome the conformal invariance of \mathcal{E}) and the Courant-Lebesgue Lemma (to obtain equicontinuity of an \mathcal{E} -minimizing sequence $\mathbf{r}_{\mathbf{g}_j}$). This solution of the Plateau Problem is often attributed to Douglas. This is wrong! It is Courant's solution!

Douglas's first full account of his solution of the Plateau problem appears in his 59-page paper 'Solution of the Problem of Plateau', published in the *Transactions of the American Mathematical Society* in January 1931. In this paper he sought \mathbf{g}^* as the minimiser of his famous A -functional:

$$A(\mathbf{g}) := \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\sum_{i=1}^n [g_i(\theta) - g_i(\varphi)]^2}{4 \sin^2 \frac{\theta - \varphi}{2}} d\theta d\varphi. \quad (1)$$

How did Douglas anticipate that the harmonic extension of a minimiser of his A -functional would be conformal?

In Part III of his paper, Douglas shows that $A(\mathbf{g})$ is equal to the Dirichlet energy of the harmonic extension $\mathbf{r}_{\mathbf{g}}$ of \mathbf{g} . It has always been assumed that this was Douglas's starting point but his announcements in the *Bulletin*, which have essentially been forgotten, indicate otherwise.

In these announcements, Douglas sought \mathbf{g}^* as a solution of an integral equation which, as we shall see, is very natural to derive.

In an Abstract published in 1927 Douglas claimed that, if

$$t \mapsto \mathbf{g}(t): \mathbb{R} \cup \{\infty\} \rightarrow \Gamma \subset \mathbb{R}^3$$

is a parameterisation of Γ and if

$$\varphi: \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}$$

is a homeomorphism which solves the following integral equation,

$$\int_{\Gamma} \frac{K(t, \tau)}{\varphi(t) - \varphi(\tau)} d\tau = 0, \quad \text{where } K(t, \tau) = \mathbf{g}'(t) \cdot \mathbf{g}'(\tau) \quad (2)$$

then the harmonic extension of $\mathbf{g} \circ \varphi^{-1}$ to the upper half plane

defined by means of Poisson's integral is the required conformal harmonic representation of a minimal surface spanning Γ .

If Γ is parameterised by a map from the unit circle C (and a surface S spanning Γ is parameterised by a map from the disc) then the integral equation (2) becomes

$$\int_0^{2\pi} K(t, \tau) \cot\left(\frac{1}{2}\varphi(t) - \frac{1}{2}\varphi(\tau)\right) d\tau = 0. \quad (3)$$

Douglas never published a proof of his claim but it is not hard to guess what his proof might have been.

But first, what is the connection between (3) and the A -functional?

Douglas was stuck for a while on how to solve (3) for a general contour.

His important breakthrough, which he announced in an Abstract

published in the July-August 1928 issue of the *Bulletin of the*

American Mathematical Society, came when he realised that (3) is

the Euler-Lagrange equation of the first version of his

later-to-be-famous A -functional:

$$A(\varphi) := - \int_0^{2\pi} \int_0^{2\pi} K(t, \tau) \log \sin \frac{1}{2} |\varphi(t) - \varphi(\tau)| dt d\tau . \quad (4)$$

Furthermore, he stated that he could use Fréchet's compactness theory of curves to assert the existence of a minimizer φ^* of the A -functional,

at least in the case that $K(t, \tau)$ is positive for all values of t and τ . This positivity requirement on K rendered this variational problem inapplicable to the Plateau problem but he overcame this problem when he discovered, late in 1929, that he could employ the functional (1) instead of (4). This was essentially achieved by integrating by parts and completing the square.

Douglas was a National Research Fellow between 1926 and 1930.

During this time he travelled widely, visiting Princeton and Harvard in 1927, Chicago in 1928, and Paris from 1928 to 1930, with trips to Göttingen, Hamburg, and Rome.

He described a solution of the Plateau problem at Hadamard's seminar in Paris on January 18, 1929, at the Courant-Herglotz seminar in Göttingen on June 4, 11, and 18, 1929, and at Blaschke's seminar in Hamburg on July 26, 1929.

Hadamard was impressed but the Göttingen gang was not. At this time, Douglas would have been proposing to solve (3) by minimising (4) but the German analysts were not convinced he had sorted out all the details.

He spoke again at Hadamard's seminar on December 17, 1929. This presentation was in the form of his 59-page paper in the January 1931 issue of the *Transactions*.

Derivation of (3)

Let Γ be a contour in \mathbb{R}^n parameterised by

$$\sigma \mapsto \mathbf{g}(\sigma) : [0, 2\pi] \rightarrow \mathbb{R}^n.$$

The harmonic surface $\mathbf{r} : \overline{\mathbb{B}} \rightarrow \mathbb{R}^n$ whose restriction to $\partial\mathbb{B}$ parameterises Γ by \mathbf{g} is given by

$$\mathbf{r}(\rho e^{i\theta}) = \int_0^{2\pi} K(\rho, \theta - \sigma) \mathbf{g}(\sigma) d\sigma,$$

where K is the Poisson kernel of $\overline{\mathbb{B}}$, i. e.,

$$K(\rho, \alpha) = \frac{1 - \rho^2}{1 - 2\rho \cos \alpha + \rho^2} = \operatorname{Re} \frac{e^{i\sigma} + w}{e^{i\sigma} - w},$$
$$w = \rho e^{i\theta}, \quad \alpha := \theta - \sigma.$$

It is convenient to think of \mathbf{g} as a 2π -periodic map

$$\mathbf{g}: \mathbb{R} \rightarrow \Gamma \subset \mathbb{R}^n.$$

If $\mathbf{f}: \mathbb{R} \rightarrow \mathbb{R}^n$ is another 2π -periodic parameterisation of Γ , then there exists a 2π -periodic homeomorphism

$$\varphi: \mathbb{R} \rightarrow \mathbb{R} \text{ such that } \mathbf{f} \circ \varphi = \mathbf{g}.$$

We shall denote $\mathbf{g} \circ \varphi^{-1}$ by \mathbf{g}_φ and we shall denote by \mathbf{r}_φ the harmonic surface whose restriction to $\partial\mathbb{B}$ parameterises Γ by \mathbf{g}_φ . Thus

$$\mathbf{r}_\varphi(\rho e^{i\theta}) = \int_0^{2\pi} K(\rho, \theta - \sigma) \mathbf{g}(\varphi^{-1}(\sigma)) d\sigma. \quad (5)$$

We seek φ so that \mathbf{r}_φ is conformal, i. e.,

$$\left\| \frac{\partial \mathbf{r}_\varphi}{\partial \rho} \right\|^2 = \frac{1}{\rho^2} \left\| \frac{\partial \mathbf{r}_\varphi}{\partial \theta} \right\|^2 \quad (6)$$

and

$$\frac{\partial \mathbf{r}_\varphi}{\partial \rho} \cdot \frac{\partial \mathbf{r}_\varphi}{\partial \theta} = 0. \quad (7)$$

We start by calculating the left hand side of (7):

$$\frac{\partial \mathbf{r}_\varphi}{\partial \rho} \cdot \frac{\partial \mathbf{r}_\varphi}{\partial \theta} = \int_0^{2\pi} \int_0^{2\pi} \left(\frac{\partial}{\partial \rho} K(\rho, \theta - \sigma) \right) \left(\frac{\partial}{\partial \theta} K(\rho, \theta - \lambda) \right) \mathbf{g}(\varphi^{-1}(\sigma)) \cdot \mathbf{g}(\varphi^{-1}(\lambda)) d\sigma d\lambda. \quad (8)$$

Now

$$\frac{\partial}{\partial \rho} K(\rho, \theta - \sigma) = \frac{1}{\rho} \frac{\partial}{\partial \theta} K^*(\rho, \theta - \sigma)$$

where K^* is the harmonic conjugate of K , i. e.,

$$K^*(\rho, \alpha) = \frac{2\rho \sin \alpha}{1 - 2\rho \cos \alpha + \rho^2}.$$

Therefore, (8) can be rewritten as:

$$\frac{\partial \mathbf{r}_\varphi}{\partial \rho} \cdot \frac{\partial \mathbf{r}_\varphi}{\partial \theta} = \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\rho} \left(\frac{\partial}{\partial \sigma} K^*(\rho, \theta - \sigma) \right) \left(\frac{\partial}{\partial \lambda} K(\rho, \theta - \lambda) \right) \mathbf{g}(\varphi^{-1}(\sigma)) \cdot \mathbf{g}(\varphi^{-1}(\lambda)) d\sigma d\lambda. \quad (9)$$

Assuming φ to be differentiable and $\varphi' > 0$, integration by parts in (9) yields:

$$\frac{\partial \mathbf{r}_\varphi}{\partial \rho} \cdot \frac{\partial \mathbf{r}_\varphi}{\partial \theta} = \int_0^{2\pi} \int_0^{2\pi} \frac{2 \sin(\theta - \sigma)}{1 - 2\rho \cos(\theta - \sigma) + \rho^2} K(\rho, \theta - \lambda) \frac{\mathbf{g}'(\varphi^{-1}(\sigma)) \cdot \mathbf{g}'(\varphi^{-1}(\lambda))}{\varphi'(\varphi^{-1}(\sigma))\varphi'(\varphi^{-1}(\lambda))} d\sigma d\lambda. \quad (10)$$

By changing variables $t = \varphi^{-1}(\sigma)$, $\mu = \varphi^{-1}(\lambda)$ in (10) we obtain:

$$\frac{\partial \mathbf{r}_\varphi}{\partial \rho} \cdot \frac{\partial \mathbf{r}_\varphi}{\partial \theta} = \int_0^{2\pi} \int_0^{2\pi} \frac{2 \sin(\theta - \varphi(t))}{1 - 2\rho \cos(\theta - \varphi(t)) + \rho^2} K(\rho, \theta - \varphi(\mu)) \mathbf{g}'(t) \cdot \mathbf{g}'(\mu) dt d\mu. \quad (11)$$

Now $\mathbf{F}_\varphi(\rho e^{i\theta}) := \frac{\partial \mathbf{r}_\varphi}{\partial \rho} - \frac{i}{\rho} \frac{\partial \mathbf{r}_\varphi}{\partial \theta}$ is a holomorphic \mathbb{C}^n -valued map because

\mathbf{r}_φ is harmonic. Therefore,

$$\mathbf{F}_\varphi \cdot \mathbf{F}_\varphi = \left\| \frac{\partial \mathbf{r}_\varphi}{\partial \rho} \right\|^2 - \frac{1}{\rho^2} \left\| \frac{\partial \mathbf{r}_\varphi}{\partial \theta} \right\|^2 - \frac{2i}{\rho} \frac{\partial \mathbf{r}_\varphi}{\partial \rho} \cdot \frac{\partial \mathbf{r}_\varphi}{\partial \theta}$$

is a holomorphic function. It follows that (7) holds everywhere on \mathbb{B} if, and only if, it holds on $\partial\mathbb{B}$. Thus we let $\rho \uparrow 1$ in (11) and, using the fact that

$$\lim_{\rho \uparrow 1} K(\rho, \theta - \alpha) = \delta_\theta(\alpha),$$

we obtain:

$$\left(\frac{\partial \mathbf{r}_\varphi}{\partial \rho} \cdot \frac{\partial \mathbf{r}_\varphi}{\partial \theta} \right) (e^{i\theta}) = \int_0^{2\pi} \frac{\sin(\theta - \varphi(t))}{1 - \cos(\theta - \varphi(t))} \mathbf{g}'(t) \cdot \mathbf{g}'(\varphi^{-1}(\theta)) dt.$$

The change of variables $\tau = \varphi^{-1}(\theta)$ in the above equation enables us

to rewrite it as:

$$\left(\frac{\partial \mathbf{r}_\varphi}{\partial \rho} \cdot \frac{\partial \mathbf{r}_\varphi}{\partial \theta} \right) (e^{i\varphi(\tau)}) = \int_0^{2\pi} \cot\left(\frac{1}{2}\varphi(\tau) - \frac{1}{2}\varphi(t)\right) \mathbf{g}'(t) \cdot \mathbf{g}'(\tau) dt. \quad (12)$$

The integrand on the right is not integrable but, if φ is a 2π -periodic diffeomorphism of \mathbb{R} to itself then a principal value may be assigned to it in a straightforward way.

Once a continuously differentiable 2π -periodic parameterisation $\mathbf{g}: \mathbb{R} \rightarrow \mathbb{R}^n$ of Γ has been fixed, the problem of finding a 2π -periodic diffeomorphism $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ so that \mathbf{r}_φ is conformal now has become to seek φ which satisfies the integral equation

$$\int_0^{2\pi} \cot\left(\frac{1}{2}\varphi(\tau) - \frac{1}{2}\varphi(t)\right) \mathbf{g}'(t) \cdot \mathbf{g}'(\tau) dt = 0. \quad (13)$$

By construction, if φ satisfies (13) then \mathbf{r}_φ satisfies (7).

One still has to check whether \mathbf{r}_φ also satisfies (6), but this turns out to be easy.

If φ satisfies (13), then so does ψ defined by $\psi(\theta) := \varphi(\theta) - \pi/4$.

Furthermore, $\mathbf{r}_\psi(\rho e^{i\theta}) = \mathbf{r}_\varphi(\rho e^{i(\theta+\pi/4)})$ and therefore,

$$(\mathbf{F}_\psi \cdot \mathbf{F}_\psi)(\rho e^{i\theta}) = i(\mathbf{F}_\varphi \cdot \mathbf{F}_\varphi)(\rho e^{i(\theta+\pi/4)}).$$

But $\mathbf{F}_\varphi \cdot \mathbf{F}_\varphi$ and $\mathbf{F}_\psi \cdot \mathbf{F}_\psi$ are both real constants. Therefore $\mathbf{F}_\varphi \cdot \mathbf{F}_\varphi$ and $\mathbf{F}_\psi \cdot \mathbf{F}_\psi$ are both zero and (6) is satisfied.

Douglas made no mention of the integral equation (3) in his 1931 paper. He showed directly that the harmonic extension of an A -minimiser is conformal. This is the stage at which Douglas connected the A -functional with Dirichlet's integral. He was then able to express A in terms of the Fourier coefficients of \mathbf{g} . The conformality of $\mathbf{r}_{\mathbf{g}^*}$ was established by a variation of these Fourier coefficients.

The A -functional is invariant under Möbius transformations of \mathbb{B} . Douglas showed this by a direct calculation, *not* by exploiting the well-known conformal invariance of Dirichlet's integral.

The relation between the A -functional and Dirichlet's integral led Radó and Courant to criticise Douglas's method as being unnecessarily complicated and not much more than an implementation of Dirichlet's principle.

Douglas repeatedly refuted these claims. In 1931, Dirichlet's principle was not yet firmly established and Douglas was keen to emphasize that his A -functional, being a 1-dimensional integral which did not involve any derivatives, did not suffer from all the difficulties that plagued Dirichlet's integral at the time. As Douglas remarked, the Dirichlet integral could not be shown to attain its lower bound, whereas his A -functional, being lower semi-continuous on a sequentially compact space, necessarily attained its minimum. More on the exchanges between Radó and Douglas and Courant in the forthcoming book with Jeremy Gray.

Radó's solution

Radó based his solution of the Plateau problem (published in July 1930) on the uniformisation theorem of Koebe.

Given a polygonal contour $\Gamma \subset \mathbb{R}^3$, define

$$\lambda := \inf\{\text{Area}(\Pi) \mid \Pi \text{ is a polyhedral surface spanning } \Gamma\}.$$

Then, for each $\sigma > 0$, there exists a polyhedron Π_σ spanning Γ whose area is less than $\lambda + \sigma$. By the uniformisation theorem of Koebe, Π_σ admits an isothermic parameterisation $\bar{\mathbf{r}}_\sigma: \bar{\mathbb{B}} \rightarrow \mathbb{R}^3$.

Let \mathbf{r}_σ be the harmonic extension of $\bar{\mathbf{r}}_\sigma$ restricted to $\partial\mathbb{B}$. By a lemma on harmonic surfaces, Radó asserted the existence of a polyhedron Π_σ^* whose area differs from that of $\mathbf{r}_\sigma(\bar{\mathbb{B}})$ by no more than σ . The following chain of inequalities results:

$$\begin{aligned} \lambda + \sigma &> \iint_{\bar{\mathbb{B}}} \sqrt{\bar{E}\bar{G} - \bar{F}^2} = \frac{1}{2} \iint_{\bar{\mathbb{B}}} \bar{E} + \bar{G} \\ &\geq \frac{1}{2} \iint_{\bar{\mathbb{B}}} E + G \geq \iint_{\bar{\mathbb{B}}} \sqrt{EG - F^2} > \text{Area}(\Pi_\sigma^*) - \sigma \geq \lambda - \sigma. \end{aligned}$$

Therefore, by taking σ sufficiently small, one can find, for each $\varepsilon > 0$, a harmonic vector $\mathbf{r}_\varepsilon: \mathbb{B} \rightarrow \mathbb{R}^3$ which

(i) extends continuously to the closed unit disc $\overline{\mathbb{B}}$ so that its restriction to $\partial\mathbb{B}$ parameterises Γ and

(ii) is approximately conformal in the sense that

$$\iint_{\overline{\mathbb{B}}} |F| < \varepsilon \quad \text{and} \quad \iint_{\overline{\mathbb{B}}} (E^{1/2} - G^{1/2})^2 < \varepsilon. \quad (14)$$

Let (ε_n) be a sequence of positive numbers decreasing to 0 and denote $\mathbf{r}_{\varepsilon_n}$ more simply by \mathbf{r}_n .

Radó showed that a subsequence of (\mathbf{r}_n) converges, up to reparameterisation by Möbius transformations of \mathbb{B} , to a generalised minimal surface spanning Γ . This limiting argument is one of Radó's major achievements.

Approximation Theorem *Let Γ_n be a sequence of simple closed curves of uniformly bounded length, for each of which the Plateau Problem is solvable. If Γ_n converges (in the sense of Fréchet) to a simple closed curve Γ then the Plateau problem for Γ is solvable.*

The proof of this Approximation Theorem was not only *much* simpler than Garnier's limiting argument but, together with the approximation

procedure just described, it also provided a solution to Plateau's problem for any rectifiable contour.

Radó's use of polyhedral surfaces is highly reminiscent of Lebesgue's definition of area of a surface as the infimum, over all sequences, of \liminf of the areas of a sequence of polyhedra tending to the surface.

It is surprising that Radó did not connect his method of proof with Lebesgue's definition of area until he gave a colloquium on his result at Harvard. Once Radó made this connection, he wrote another paper, also published in 1930, in which he showed that his solution of the Plateau problem has least area among discs spanning Γ .

Radó's strategy marked a total departure from the Riemann-Weierstrass-Darboux programme and it was strikingly original at the time.

Main deficiency of Douglas's solution

Douglas was justifiably proud that his solution avoided the use of Koebe's theorem. However, he did have to resort to this theorem in order to prove that it had least area. He tried to overcome this deficiency but could not. That was done later by Morrey. A more recent and more elementary proof has been given by Hildebrandt and von der Mosel.

Advantages of Douglas's solution

There are several.

- (i) **The Riemann-Carathéodory-Osgood Theorem.** Douglas's proof works for $\Gamma \subset \mathbb{R}^n$, *for any* n , in particular for $n = 2$. (A little work using the argument principle is required to establish univalence of the map.)
- (ii) **Infinite area** Douglas indicated that there are contours for which every spanning surface has infinite area. Nevertheless, he could prove the existence of a minimal surface spanning such a contour Γ as a limit of minimal surfaces spanning polygonal contours

which converge to Γ . Douglas was very cross that Radó regarded the Plateau problem as meaningless for contours which could only bound surfaces of infinite area. He compared the situation to that in Dirichlet's problem, for which Hadamard had earlier constructed continuous boundary values for which the boundary value problem is solvable, even though the Dirichlet functional is identically $+\infty$.

(iii) **Higher Connectivity and Higher Genus** Even before working out all the details for the disc case, Douglas was considering the Plateau problem for surfaces of higher connectivity and higher

genus. For instance, at the February 1927 meeting of the American Mathematical Society, he wrote down two integral equations that have to be satisfied to solve the Plateau problem for the case of two contours in \mathbb{R}^n , n arbitrary. There is a third equation that has to be solved; it determines the conformal type of the annulus. As Douglas pointed out, this form of the Plateau problem had only been raised in very special cases before (Riemann's investigation of two parallel circles, and two polygons in parallel planes) so his was the first general account. Douglas was doing this work *before* Teichmüller theory had been developed. He used

theta functions to write down the appropriate functional and to encode the conformal moduli of Riemann surfaces.

It is also amusing to note that Douglas anticipated a result reproved by Frank Morgan 50 years later!

The Fields medal

If priority is assigned on the basis of published papers alone, then Radó was the first to put into print a comprehensible solution of the Plateau problem in anything like generality. Douglas's announcements give the impression that he was occasionally cavalier about what he could achieve. Most of the time he delivered on his claims, but it may not always be appropriate to use the timing of his claims to determine priority issues.

In our forthcoming book, Jeremy and I conclude that Radó and Douglas share equal credit for solving the Plateau problem for disc-like

surfaces spanning a single contour which bounds at least one disc-like surface of finite area. Radó deserves full credit for solving the least area problem. Douglas, however, was the first to solve the Plateau problem in complete generality, that is, for an arbitrary contour, including ones that bound only surfaces of infinite area. He was also the only one to consider more general types of surface than the disc, to which Radó's attentions were exclusively confined. Radó's method for solving the Plateau problem shifts almost all the difficulty onto problems in conformal mapping whose solution for higher topological types was certainly not available at the time. By contrast, Douglas's method even

helped solve some of these problems in conformal mapping. Thus, Douglas's contributions to the Plateau problem are more major, broader and deeper than those of Radó. Douglas's ideas, as developed later by Courant (who brought the Dirichlet integral back to the forefront), have remained important in the theory of minimal surfaces up to the present.

Douglas was awarded one of the first Fields Medals for his work on the Plateau problem. He did not collect the medal at the ceremony in Oslo; Wiener collected it on his behalf, even though Douglas did attend the International Congress!

In the address, Carathéodory described a method for finding a minimal surface that is due to Radó (different from the one sketched above) but gave the impression that it was due to Douglas!

Brief biography of Jesse Douglas

Born in New York City on 3 July 1897. His father, Louis, was an immigrant from Poland via Canada (the family name was changed at Canadian Immigration).

Educated at public schools in New York City. Attended City College New York, graduated B.Sc.cum laude in February 1916 at the age of 18. Won Belden prize for excellence in pure mathematics. University scholar, and later fellow at Columbia University.

PhD in 1920 with a thesis under Edward Kasner, who inspired in him his love of differential geometry. Also influenced by the algebraist and number theorist Frank Nelson Cole.

Instructor at Columbia from 1920 to 1926.

Travelled widely as National Research Fellow for four years, visiting Princeton and Harvard in 1927, Chicago in 1928, and Paris from 1928 to 1930, with trips to Gttingen, Hamburg, and Rome.

Hadamard much impressed and wrote to the President of Cornell University (7 March 1929):

“I hear that Mr. Jesse Douglas would like to come to your university. Perhaps it may not be useless to let you know how satisfied and interested I have been in his stay at Paris and his collaboration to my seminary at the College de France. His exposés on Plateau’s problem are among the best I have had; the solution he has found is of a remarkable elegance [sic] and simplicity, almost unexpected for this difficult problem, one of the most beautiful [sic] in Mathematical

Physics; moreover he presents it with a perfect clearness, a clever thing at my seminary, where details must be left aside and only the general development of ideas brought in full light. Mr. Douglas lies perfectly succeeded in that difficult achievement, and every auditor has carried from his lecture a perfectly clear understanding of the question. “He would be a first rate recruit for any mathematical staff . . .”

Assistant Professor at MIT in January 1930, where he took leave of absence for a term in 1932.

Associate Professor at MIT in 1934, and almost at once took leave of absence again to be a Research Worker at the Institute for Advanced Study in Princeton from 1934 to 1935.

In December 1934 there was a chance Douglas might obtain a professorship at Columbia, and C.J. Keyser wrote to President Butler partly in these terms:

1. “Douglas’s academic record as student and as teacher both of undergraduates and of the most advanced students is one of the highest excellence.
2. His personal appearance is pleasing; his personality virile and wholesome.
3. There is in him none of the narrowness of the mere specialist. On the contrary, he is a man of general culture notable for its range and fineness. Suffice it to say in this connection that his discourse, spoken or written, is distinguished by its clarity and dignity; that his lectures, which are always inspiring, are models alike in content

and in form; that he has lectured in Paris in French; in Göttingen and Hamburg in German; and in Rome in Italian.”

Noting that Douglas was later to teach at Yeshiva, it seems likely that the extra-ordinary remark in (2) was written in the genteel anti-Semitic code of the day, and signals that Douglas was socially acceptable.

Awarded the first Fields Medal, together with Lars Ahlfors, in 1936.

Ahlfors went on to enjoy a successful, high-profile career at Harvard, but Douglas has all but disappeared from the record.

There is, for example, no biography of him in the Biographies of Members of the National Academy of Sciences, of which he was elected a Fellow in 1946.

Took leave of absence again from MIT in 1936, and on 1 July 1938, he resigned. Struik, in his reminiscence of MIT, wrote that Douglas “had his own lifestyle which did not include coming to class on a regular schedule, so that Phillips (the Head of Department), who stuck to the Runkle discipline of conscientious teaching, had to let him go, to my and others’ regret.”

Guggenheim fellow at Columbia in 1940 and 1941. Marston Morse tried hard to hire him on a tenured basis at this time, but failed.

Taught at Brooklyn College from 1942 to 1946. His teaching during these war years went well and he received a Distinguished Service

Award. Lt. Col. Henry A. Robinson of the Mathematics Department of the U.S. Military Academy wrote to Douglas:

“I just wish to drop you an informal note, and state you have done an excellent job with your pupils. They have spoken of you in glowing terms. They are all in classes with doctors and they compete quite well. I do not know who are your pupils, but the following have given praise to you: Gerard Washnitzer (who has now gone to the army), E. S. Krendel, Julius Jackson, Samuel Karp, and Albert Blank. It is usually a rare thing for students so young to give praise to their teacher. Hence I wanted to pass the information on to you, and to

congratulate you on inspiring so many to higher study.”

There is a blank in Douglas’s career from 1946 to 1950, and the later part of Douglass life seems to have been troubled. His marriage ended in divorce in 1950.

In 1955 he moved to City College, New York. In May 1954, Paul Smith, from Columbia, wrote to Garrison, the Department Head at City College: “Why not appoint Jesse Douglas? He came to us after a protracted illness, and there was simply no major position vacant in our staff at that time. The result was that he has never found at College the sort of position he ought to have . . . [he] is in good health and vigor — teaches 18 hours a week . . . travels to various institutions, does research.”

Norman Schaumberger describes a man who could be a good teacher when he chose, and who liked to get up late so he often taught the elementary classes, which were held in the evenings.

Took sick leave in 1959-60, in 1961-62, and in May 1965, but this time he did not recover and he died in hospital from a heart attack on 7 October 1965 at the age of 68.

Douglas harboured dislikes to the end of his life, not only of Radó and Courant, but of J.F. Ritt at Columbia, and he could be cutting to people who did not work hard. To one student who asked what he had to do to get an A grade, Douglas is said to have replied “Get better

parents”. But, the following anecdote balances the picture: Some students went to the Head of the Mathematics Department of City College to complain about one of their teachers. They were assured that whatever the mans failings as an instructor, he was a distinguished mathematician with a fine research record, and it was a privilege to be taught by him. The students were not impressed. “We dont want someone like that”, they said, “we want a real mathematician, someone like Douglas!”