Minimal Lagrangian Pseudo-Riemannian Isotropic Submanifolds

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Contents

- The problem treated in the present work has its beginning in the paper of Franki Dillen and Luc Vrancken- *Lorentzian Isotropic Lagrangian immersions*, where they study Lagrangian isotropic immersions of a **Lorentzian** manifold into a complex Lorentzian space form. They show that such submanifolds are always H-umbilical warped product immersions.

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- But this does not hold for **the indefinite case**, so the authors find another approach, using lightlike vectors.

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for any $X(p) \in T_pM$, we say that M has isotropic second fundamental form.
If λ is independent of the point p, the submanifold is called constant isotropic.

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  Amongst others they showed that the following conditions are equivalent:
  1. $< h(x,x), h(x,x) >= \lambda(p) < x,x >^2$, for any tangent vector,
  2. $< h(x,x), h(x,x) >= \lambda(p) < x,x >^2$, for any spacelike vector,
  3. $< h(x,x), h(x,x) >= \lambda(p) < x,x >^2$, for any timelike vector.

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  3. $< h(x,x), h(x,x) >= \lambda(p) < x,x >^2$, for any timelike vector.

  Note however that they also showed that the same characterisation is not true for lightlike vectors. The fact that

$$< h(x,x), h(x,x) >= 0$$

  for any lightlike vector does not imply the immersion to be isotropic.

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- When $n < 14$, they give a classification of such submanifolds .
- In the present work, we show that the dimension of M^n, can only be 5, 8, 14 or 26 and we determine all the components of the second fundamental form.

- Results

Proposition

Let M^n be an n-dimensional ($n \geq 3$) minimal Lagrangian isotropic submanifold in an indefinite complex space form $\tilde{M}^n_s(4\tilde{c})$ and suppose M^n is not totally geodesic. Then, the dimension of M^n satisfies $n = 3r + 2$, for $r > 0$.

Proposition

Let $n \geq 3$ and M^n be an n-dimensional minimal Lagrangian λ-isotropic submanifold in an indefinite complex space form $\tilde{M}^n(4\tilde{c})$. Then M^n is constant isotropic.

- Results

Proposition

Let M^n be an n-dimensional Lagrangian submanifold in an indefinite complex space form $\tilde{M}^n(4\tilde{c})$. If M^n is constant isotropic, then M^n has parallel second fundamental form ($\nabla h = 0$).

Proposition

Let $n \geq 3$ and M^n be an n-dimensional minimal Lagrangian λ-isotropic submanifold in an indefinite complex space form $\tilde{M}^n(4\tilde{c}), (\tilde{c} = 0, \pm 1)$, M^n not totally geodesic. Then M^n is a locally symmetric space and $\tilde{c} = 1, \lambda = \frac{1}{2}$ or $\tilde{c} = -1, \lambda = -\frac{1}{2}$.

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  2. For any null vector v, $h(v,v)$ is orthogonal to Jv. However there exists a null vector u such that $h(u,u)$ and Ju are linearly independent
  3. There exists a null vector v such that $< h(v,v), Jv > \neq 0$.
- In the first case, as it is valid for any null vector, it is possible to deduce that the submanifold is totally geodesic.
- The second case leads to a contradiction.
- The present work is dealing with the third case.

- work in a neighbourhood of a point $p \in M$; choose a local frame in T_pM, $\{f_1, f_2, u_1, \cdots, u_r, w_1^1, w_2^1, \cdots, w_1^r, w_2^r\}$, where

$$\begin{aligned} \langle f_1, f_1\rangle = -\langle f_2, f_2\rangle = 1, \\ \langle u_k, u_k\rangle = \langle w_1^k, w_1^k\rangle = -\langle w_2^k, w_2^k\rangle = 1 \end{aligned} \tag{1}$$

- Define U, W
  Start with $e_1 = v_0, e_2 = A_{Jv_0}v_0 = -Jh(v_0, v_0)$, where v_0 is a null vector such that v_0 and $A_{Jv_0}v_0$ are linearly independent, but not orthogonal.We can show that $\{e_1, e_2\}$ is invariant under A_{Je_1}.
  Let U be the eigenspace of the operator A_{Je_1} on the space $\{e_1, e_2\}^\perp$ with the eigenvalue λ;
  Let W be the eigenspace of the operator A_{Je_1} on the space $\{e_1, e_2\}^\perp$ with the eigenvalues $-\frac{1}{2}\lambda \pm \frac{\sqrt{3}}{2}\lambda i$:

$$U = span\{u_1, \ldots, u_r\}$$
$$W = span\{w_1^1, w_2^1, \ldots, w_1^r, w_2^r\}$$

  Finally, let $W := span\{\omega_1^1, \omega_2^1, \cdots, \omega_1^r, \omega_2^r\}$, for $\omega_1^k = Re(w_1^k)$ and $\omega_2^k = Im(w_1^k)$.

- Define $L : W \times W \to U$

$$L(w, \tilde{w}) := A_{Jw}\tilde{w} + \frac{1}{4\lambda^2}\langle A_{Jw}\tilde{w}, e_2\rangle e_1 + \frac{1}{4\lambda^2}\langle A_{Jw}\tilde{w}, e_1\rangle e_2,$$

- Define $T : W \to W$

$$Tw := \frac{2}{\sqrt{3}\lambda}\left(A_{Je_1} + \frac{1}{2}\lambda I\right) w, \forall w \in W$$

- Properties

$$\begin{array}{ll} T\omega_1^\alpha = \omega_2^\alpha & \langle Tw, Tv\rangle = -\langle w, v\rangle \\ T\omega_2^\alpha = -\omega_1^\alpha & \langle Tv, w\rangle = \langle v, Tw\rangle \\ T^2 w = -w & L(w, Tv) = -L(v, Tw) \\ & L(Tw, Tv) = L(v, w) \end{array}$$

- Look at the theory of composition of quadratic forms over fields- it has started in the 19th century with the search of the n-square identities of the type

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) = z_1^2 + z_2^2 + \cdots + z_n^2,$$

where $X = (x_1, \cdots, x_n)$, $Y = (y_1, \cdots, y_n)$ are systems of indeterminates and $z_k = z_k(X, Y)$ is a bilinear form in X and Y.

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- An easy example: for $n = 2$ we find the 'law of moduli' for complex numbers:

$$|x|^2|y|^2 = |z|^2 \Leftrightarrow (x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

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- In 1898, Hurwitz proved that there exists an n-square identity with complex coefficients if and only if $n = 1, 2, 4, 8$.

In order to apply the previous result, find a conveniently defined operator which preserves lengths.
Look at the eigenvales of T : $-i, i$ with the corresponding eigenspaces

$$W_1 := span\{\omega_1^\alpha + i\omega_2^\alpha\} = span\{v + iTv | v \in W, v \text{ real vector}\},$$
$$W_2 := span\{\omega_1^\beta - i\omega_2^\beta\} = span\{w - iTw | w \in W, w \text{ real vector}\},$$

respectively, where $W = span\{\omega_1^\alpha, \omega_2^\alpha\}$ as initially chosen.

Define

$$L : W_1 \times W_2 \to U^{\mathbb{C}},$$

with $U^{\mathbb{C}} = span\{u_1, \cdots, u_r\}$ over $\mathbb{C}$.

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with $U^{\mathbb{C}} = span\{u_1, \cdots, u_r\}$ over $\mathbb{C}$.

- $L(x, -)$ is bijective, for some non-null vector x.
- L satisfies

$$\langle L(x,y), L(x,y) \rangle = \frac{3\mu^2}{4} \langle x, x \rangle \langle y, y \rangle,$$

  which is an r-square quadratic equation. This gives that $r = 1, 2, 4, 8$, which implies that $n = 5, 8, 14, 26$.

- (?How to) Construct orthonormal bases for W and U, respectively:

$$W = span\{\omega_1^1, \omega_2^1, \omega_1^2, \omega_2^2, \omega_1^3, \omega_2^3, \omega_1^4, \omega_2^4\}$$
$$U = span\{u_1, u_2, u_3, u_4\},$$

with $\omega_1^\alpha, \omega_2^\alpha$ real, $\alpha = 1, 2, 3, 4$.

Depending on whether the vectors in the basis of U have length 1 or -1, we have the following cases:

	I	II	III	IV
$L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1)$	$\sqrt{3}\mu u_1$	$\sqrt{3}\mu u_1$	$\sqrt{3}\mu u_1$	$\sqrt{3}\mu u_1$
$L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$	$\sqrt{3}\mu i u_2$	$\sqrt{3}\mu i u_2$	$\sqrt{3}\mu i u_2$	$\sqrt{3}\mu u_2$
$L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3)$	$\sqrt{3}\mu i u_3$	$\sqrt{3}\mu i u_3$	$\sqrt{3}\mu u_3$	$\sqrt{3}\mu u_3$
$L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4)$	$\sqrt{3}\mu i u_4$	$\sqrt{3}\mu u_4$	$\sqrt{3}\mu u_4$	$\sqrt{3}\mu u_4$

	I.i	I.ii	II.i	II.ii
$L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1)$	$\sqrt{3}\mu u_1$	$\sqrt{3}\mu u_1$	$\sqrt{3}\mu u_1$	$\sqrt{3}\mu u_1$
$L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$	$\sqrt{3}\mu i u_2$	$\sqrt{3}\mu i u_2$	$\sqrt{3}\mu i u_2$	$\sqrt{3}\mu i u_2$
$L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3)$	$\sqrt{3}\mu i u_3$	$\sqrt{3}\mu i u_3$	$\sqrt{3}\mu i u_3$	$\sqrt{3}\mu i u_3$
$L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4)$	$\sqrt{3}\mu i u_4$	$\sqrt{3}\mu i u_4$	$\sqrt{3}\mu u_4$	$\sqrt{3}\mu u_4$
$L(\omega_1^2 + i\omega_2^1, \omega_1^1 - i\omega_2^1)$	$-\sqrt{3}\mu i u_2$	$-\sqrt{3}\mu i u_2$	$-\sqrt{3}\mu i u_2$	$-\sqrt{3}\mu i u_2$
$L(\omega_1^2 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$	βu_3	αu_1	βu_3	αu_1
$L(\omega_1^2 + i\omega_2^1, \omega_1^3 - i\omega_2^3)$	$b u_4$	$b u_4$	$b u_4$	$b u_4$
$L(\omega_1^2 + i\omega_2^1, \omega_1^4 - i\omega_2^4)$	$c u_1$	$d u_3$	$c u_1$	$d u_3$
$L(\omega_1^3 + i\omega_2^3, \omega_1^1 - i\omega_2^1)$	$-\sqrt{3}\mu i u_3$	$-\sqrt{3}\mu i u_3$	$-\sqrt{3}\mu i u_3$	$-\sqrt{3}\mu i u_3$
$L(\omega_1^3 + i\omega_2^3, \omega_1^2 - i\omega_2^2)$	$\bar{b} u_4$	$\bar{b} u_4$	$\bar{b} u_4$	$\bar{b} u_4$
$L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3)$	$f u_1$	$f u_1$	$f u_1$	$f u_1$
$L(\omega_1^3 + i\omega_2^3, \omega_1^4 - i\omega_2^4)$	$e u_2$	$e u_2$	$e u_2$	$e u_2$
$L(\omega_1^3 + i\omega_2^3, \omega_1^1 - i\omega_2^1)$	$-\sqrt{3}\mu i u_4$	$-\sqrt{3}\mu i u_4$	$-\sqrt{3}\mu u_4$	$-\sqrt{3}\mu u_4$
$L(\omega_1^3 + i\omega_2^3, \omega_1^2 - i\omega_2^2)$	$\bar{c} u_1$	$\bar{d} u_3$	$\bar{c} u_1$	$\bar{d} u_3$
$L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3)$	$\bar{e} u_2$	$\bar{e} u_2$	$\bar{e} u_2$	$\bar{e} u_2$
$L(\omega_1^3 + i\omega_2^3, \omega_1^4 - i\omega_2^4)$	$g u_2$	$g u_1$	$g u_3$	$g u_1$

	III.i	III.ii	IV.i	IV.ii
$L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1)$	$\sqrt{3}\mu u_1$	$\sqrt{3}\mu u_1$	$\sqrt{3}\mu u_1$	$\sqrt{3}\mu u_1$
$L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$	$\sqrt{3}\mu i u_2$	$\sqrt{3}\mu i u_2$	$\sqrt{3}\mu u_2$	$\sqrt{3}\mu u_2$
$L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3)$	$\sqrt{3}\mu u_3$	$\sqrt{3}\mu u_3$	$\sqrt{3}\mu u_3$	$\sqrt{3}\mu u_3$
$L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4)$	$\sqrt{3}\mu u_4$	$\sqrt{3}\mu u_4$	$\sqrt{3}\mu u_4$	$\sqrt{3}\mu u_4$
$L(\omega_1^2 + i\omega_2^1, \omega_1^1 - i\omega_2^1)$	$-\sqrt{3}\mu i u_2$	$-\sqrt{3}\mu i u_2$	$\sqrt{3}\mu i u_2$	$\sqrt{3}\mu i u_2$
$L(\omega_1^2 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$	βu_3	αu_1	βu_3	αu_1
$L(\omega_1^2 + i\omega_2^1, \omega_1^3 - i\omega_2^3)$	$b u_4$	$b u_4$	$b u_4$	$b u_4$
$L(\omega_1^2 + i\omega_2^1, \omega_1^4 - i\omega_2^4)$	$c u_1$	$d u_3$	$c u_1$	$d u_3$
$L(\omega_1^3 + i\omega_2^3, \omega_1^1 - i\omega_2^1)$	$\sqrt{3}\mu u_3$	$\sqrt{3}\mu u_3$	$\sqrt{3}\mu u_3$	$\sqrt{3}\mu u_3$
$L(\omega_1^3 + i\omega_2^3, \omega_1^2 - i\omega_2^2)$	$\bar{b} u_4$	$\bar{b} u_4$	$\bar{b} u_4$	$\bar{b} u_4$
$L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3)$	$f u_1$	$f u_1$	$f u_1$	$f u_1$
$L(\omega_1^3 + i\omega_2^3, \omega_1^4 - i\omega_2^4)$	$e u_2$	$e u_2$	$e u_2$	$e u_2$
$L(\omega_1^3 + i\omega_2^4, \omega_1^1 - i\omega_2^1)$	$\sqrt{3}\mu i u_4$	$\sqrt{3}\mu i u_4$	$\sqrt{3}\mu u_4$	$\sqrt{3}\mu u_4$
$L(\omega_1^3 + i\omega_2^4, \omega_1^2 - i\omega_2^2)$	$\bar{c} u_1$	$\bar{d} u_3$	$\bar{c} u_1$	$\bar{d} u_3$
$L(\omega_1^3 + i\omega_2^4, \omega_1^3 - i\omega_2^3)$	$\bar{e} u_2$	$\bar{e} u_2$	$\bar{e} u_2$	$\bar{e} u_2$
$L(\omega_1^3 + i\omega_2^4, \omega_1^4 - i\omega_2^4)$	$g u_2$	$g u_1$	$g u_3$	$g u_1$

For the case when $dimU = 8$, according to the signature of the metric on U, there are the following cases:

1	2	3	4	5	6	7	8
+	+	+	+	+	+	+	+
−	+	−	+	−	+	−	+
−	+	+	−	−	+	−	+
−	+	+	−	+	−	−	+
−	+	+	−	+	−	+	−
−	+	+	−	+	−	+	−
−	+	+	−	+	−	+	−
−	+	+	−	+	−	+	−

For the case when $dimU = 8$, according to the signature of the metric on U, there are the following cases:

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+	+	+	+	+	+	+	+
−	+	−	+	−	+	−	+
−	+	+	−	−	+	−	+
−	+	+	−	+	−	−	+
−	+	+	−	+	−	+	−
−	+	+	−	+	−	+	−
−	+	+	−	+	−	+	−
−	+	+	−	+	−	+	−

To deal with these cases, we take $L : W_1 \times W_2 \to U$, with

$$W_1 = span\{v + iTv | v \in W\},$$
$$W_2 = span\{w - iTw | w \in W\},$$

and construct convenient bases for W and U, respectively.

Straightforward computations lead to finding that only two cases hold, namely case 1 and case 7. For each case, we could entirely determine L.

- Case 1

$$
\begin{aligned}
L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^1)&=\sqrt{3}\mu u_1 & L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^1)&=-\sqrt{3}\mu i u_2\\
L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^2)&=\sqrt{3}\mu i u_2 & L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^2)&=\sqrt{3}\mu u_1\\
L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^3)&=\sqrt{3}\mu i u_3 & L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^3)&=-\sqrt{3}\mu i u_4\\
L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^4)&=\sqrt{3}\mu i u_4 & L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^4)&=\sqrt{3}\mu i u_3\\
L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^5)&=\sqrt{3}\mu i u_5 & L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^5)&=-\sqrt{3}\mu i u_6\\
L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^6)&=\sqrt{3}\mu i u_6 & L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^6)&=\sqrt{3}\mu i u_5\\
L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^7)&=\sqrt{3}\mu i u_7 & L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^7)&=\sqrt{3}\mu i u_8\\
L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^8)&=\sqrt{3}\mu i u_8 & L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^8)&=-\sqrt{3}\mu i u_7
\end{aligned}
$$

$$L(\omega_1^3+i\omega_2^3,\omega_1^1-i\omega_2^1)=-\sqrt{3}\mu u_3 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^1-i\omega_2^1)=-\sqrt{3}\mu i u_4$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^2-i\omega_2^2)=\sqrt{3}\mu i u_4 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^2-i\omega_2^2)=-\sqrt{3}\mu u_3$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^3-i\omega_2^3)=\sqrt{3}\mu u_1 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^3-i\omega_2^3)=\sqrt{3}\mu i u_2$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^4-i\omega_2^4)=-\sqrt{3}\mu i u_2 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^4-i\omega_2^4)=\sqrt{3}\mu u_1$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^5-i\omega_2^5)=-\sqrt{3}\mu i u_7 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^5-i\omega_2^5)=-\sqrt{3}\mu i u_8$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^6-i\omega_2^6)=-\sqrt{3}\mu i u_8 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^6-i\omega_2^6)=\sqrt{3}\mu i u_7$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^7-i\omega_2^7)=\sqrt{3}\mu i u_5 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^7-i\omega_2^7)=-\sqrt{3}\mu i u_6$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^8-i\omega_2^8)=\sqrt{3}\mu i u_6 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^8-i\omega_2^8)=\sqrt{3}\mu i u_5$$

$$L(\omega_1^5+i\omega_2^5,\omega_1^1-i\omega_2^1)=-\sqrt{3}\mu u_5 \qquad L(\omega_1^6+i\omega_2^6,\omega_1^1-i\omega_2^1)=-\sqrt{3}\mu i u_3$$
$$L(\omega_1^5+i\omega_2^5,\omega_1^2-i\omega_2^2)=\sqrt{3}\mu i u_6 \qquad L(\omega_1^6+i\omega_2^6,\omega_1^2-i\omega_2^2)=-\sqrt{3}\mu u_5$$
$$L(\omega_1^5+i\omega_2^5,\omega_1^3-i\omega_2^3)=\sqrt{3}\mu u_7 \qquad L(\omega_1^6+i\omega_2^6,\omega_1^3-i\omega_2^3)=\sqrt{3}\mu i u_8$$
$$L(\omega_1^5+i\omega_2^5,\omega_1^4-i\omega_2^4)=\sqrt{3}\mu i u_8 \qquad L(\omega_1^6+i\omega_2^6,\omega_1^4-i\omega_2^4)=-\sqrt{3}\mu u_7$$
$$L(\omega_1^5+i\omega_2^5,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu u_1 \qquad L(\omega_1^6+i\omega_2^6,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu i u_2$$
$$L(\omega_1^5+i\omega_2^5,\omega_1^6-i\omega_2^6)=-\sqrt{3}\mu i u_2 \qquad L(\omega_1^6+i\omega_2^6,\omega_1^6-i\omega_2^6)=\sqrt{3}\mu u_1$$
$$L(\omega_1^5+i\omega_2^5,\omega_1^7-i\omega_2^7)=-\sqrt{3}\mu i u_3 \qquad L(\omega_1^6+i\omega_2^6,\omega_1^7-i\omega_2^7)=\sqrt{3}\mu i u_4$$
$$L(\omega_1^5+i\omega_2^5,\omega_1^8-i\omega_2^8)=-\sqrt{3}\mu i u_4 \qquad L(\omega_1^6+i\omega_2^6,\omega_1^8-i\omega_2^8)=-\sqrt{3}\mu i u_3$$

$$
\begin{aligned}
L(\omega_1^7+i\omega_2^7,\omega_1^1-i\omega_2^1)&=-\sqrt{3}\mu i u_7 & L(\omega_1^8+i\omega_2^8,\omega_1^1-i\omega_2^1)&=-\sqrt{3}\mu i u_8\\
L(\omega_1^7+i\omega_2^7,\omega_1^2-i\omega_2^2)&=-\sqrt{3}\mu i u_8 & L(\omega_1^8+i\omega_2^8,\omega_1^2-i\omega_2^2)&=\sqrt{3}\mu i u_7\\
L(\omega_1^7+i\omega_2^7,\omega_1^3-i\omega_2^3)&=-\sqrt{3}\mu i u_5 & L(\omega_1^8+i\omega_2^8,\omega_1^3-i\omega_2^3)&=-\sqrt{3}\mu i u_6\\
L(\omega_1^7+i\omega_2^7,\omega_1^4-i\omega_2^4)&=\sqrt{3}\mu i u_6 & L(\omega_1^8+i\omega_2^8,\omega_1^4-i\omega_2^4)&=-\sqrt{3}\mu i u_5\\
L(\omega_1^7+i\omega_2^7,\omega_1^5-i\omega_2^5)&=\sqrt{3}\mu i u_3 & L(\omega_1^8+i\omega_2^8,\omega_1^5-i\omega_2^5)&=\sqrt{3}\mu i u_4\\
L(\omega_1^7+i\omega_2^7,\omega_1^6-i\omega_2^6)&=-\sqrt{3}\mu i u_4 & L(\omega_1^8+i\omega_2^8,\omega_1^6-i\omega_2^6)&=\sqrt{3}\mu i u_3\\
L(\omega_1^7+i\omega_2^7,\omega_1^7-i\omega_2^7)&=\sqrt{3}\mu u_1 & L(\omega_1^8+i\omega_2^8,\omega_1^7-i\omega_2^7)&=-\sqrt{3}\mu i u_2\\
L(\omega_1^7+i\omega_2^7,\omega_1^8-i\omega_2^8)&=\sqrt{3}\mu i u_2 & L(\omega_1^8+i\omega_2^8,\omega_1^8-i\omega_2^8)&=\sqrt{3}\mu u_1
\end{aligned}
$$

- Case 7

$$L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^1)=\sqrt{3}\mu u_1 \qquad L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^1)=-\sqrt{3}\mu i u_2$$
$$L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^2)=\sqrt{3}\mu i u_2 \qquad L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^2)=\sqrt{3}\mu u_1$$
$$L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^3)=\sqrt{3}\mu u_3 \qquad L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^3)=\sqrt{3}\mu u_4$$
$$L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^4)=\sqrt{3}\mu u_4 \qquad L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^4)=-\sqrt{3}\mu u_3$$
$$L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^5)=\sqrt{3}\mu u_5 \qquad L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^5)=\sqrt{3}\mu u_8$$
$$L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^6)=\sqrt{3}\mu i u_6 \qquad L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^6)=-\sqrt{3}\mu i u_7$$
$$L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^7)=\sqrt{3}\mu i u_7 \qquad L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^7)=\sqrt{3}\mu i u_6$$
$$L(\omega_1^1+i\omega_2^1,\omega_1^1-i\omega_2^8)=\sqrt{3}\mu u_8 \qquad L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^8)=-\sqrt{3}\mu u_5$$

$$L(\omega_1^3+i\omega_2^3,\omega_1^1-i\omega_2^1)=\sqrt{3}\mu u_3 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^1-i\omega_2^1)=\sqrt{3}\mu u_4$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^2-i\omega_2^2)=\sqrt{3}\mu u_4 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^2-i\omega_2^2)=-\sqrt{3}\mu u_3$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^3-i\omega_2^3)=-\sqrt{3}\mu u_1 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^3-i\omega_2^3)=\sqrt{3}\mu i u_2$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^4-i\omega_2^4)=-\sqrt{3}\mu i u_2 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^4-i\omega_2^4)=-\sqrt{3}\mu u_1$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu i u_6 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu i u_7$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^6-i\omega_2^6)=-\sqrt{3}\mu i u_5 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^6-i\omega_2^6)=\sqrt{3}\mu u_8$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^7-i\omega_2^7)=-\sqrt{3}\mu u_8 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^7-i\omega_2^7)=-\sqrt{3}\mu u_5$$
$$L(\omega_1^3+i\omega_2^3,\omega_1^8-i\omega_2^8)=\sqrt{3}\mu i u_7 \qquad L(\omega_1^4+i\omega_2^4,\omega_1^8-i\omega_2^8)=-\sqrt{3}\mu i u_6$$

$$
\begin{aligned}
L(\omega_1^5+i\omega_2^5,\omega_1^1-i\omega_2^1)&=\sqrt{3}\mu u_5 & L(\omega_1^6+i\omega_2^6,\omega_1^1-i\omega_2^1)&=-\sqrt{3}\mu i u_6\\
L(\omega_1^5+i\omega_2^5,\omega_1^2-i\omega_2^2)&=\sqrt{3}\mu u_8 & L(\omega_1^6+i\omega_2^6,\omega_1^2-i\omega_2^2)&=\sqrt{3}\mu i u_7\\
L(\omega_1^5+i\omega_2^5,\omega_1^3-i\omega_2^3)&=-\sqrt{3}\mu i u_6 & L(\omega_1^6+i\omega_2^6,\omega_1^3-i\omega_2^3)&=-\sqrt{3}\mu u_5\\
L(\omega_1^5+i\omega_2^5,\omega_1^4-i\omega_2^4)&=-\sqrt{3}\mu i u_7 & L(\omega_1^6+i\omega_2^6,\omega_1^4-i\omega_2^4)&=\sqrt{3}\mu u_8\\
L(\omega_1^5+i\omega_2^5,\omega_1^5-i\omega_2^5)&=-\sqrt{3}\mu u_1 & L(\omega_1^6+i\omega_2^6,\omega_1^5-i\omega_2^5)&=\sqrt{3}\mu u_3\\
L(\omega_1^5+i\omega_2^5,\omega_1^6-i\omega_2^6)&=\sqrt{3}\mu u_3 & L(\omega_1^6+i\omega_2^6,\omega_1^6-i\omega_2^6)&=\sqrt{3}\mu u_1\\
L(\omega_1^5+i\omega_2^5,\omega_1^7-i\omega_2^7)&=\sqrt{3}\mu u_4 & L(\omega_1^6+i\omega_2^6,\omega_1^7-i\omega_2^7)&=-\sqrt{3}\mu i u_2\\
L(\omega_1^5+i\omega_2^5,\omega_1^8-i\omega_2^8)&=-\sqrt{3}\mu i u_2 & L(\omega_1^6+i\omega_2^6,\omega_1^8-i\omega_2^8)&=-\sqrt{3}\mu u_4
\end{aligned}
$$

$$
\begin{aligned}
L(\omega_1^7+i\omega_2^7,\omega_1^1-i\omega_2^1)&=-\sqrt{3}\mu i u_7 & L(\omega_1^8+i\omega_2^8,\omega_1^1-i\omega_2^1)&=\sqrt{3}\mu u_8\\
L(\omega_1^7+i\omega_2^7,\omega_1^2-i\omega_2^2)&=-\sqrt{3}\mu i u_6 & L(\omega_1^8+i\omega_2^8,\omega_1^2-i\omega_2^2)&=-\sqrt{3}\mu u_5\\
L(\omega_1^7+i\omega_2^7,\omega_1^3-i\omega_2^3)&=-\sqrt{3}\mu u_8 & L(\omega_1^8+i\omega_2^8,\omega_1^3-i\omega_2^3)&=-\sqrt{3}\mu i u_7\\
L(\omega_1^7+i\omega_2^7,\omega_1^4-i\omega_2^4)&=-\sqrt{3}\mu u_5 & L(\omega_1^8+i\omega_2^8,\omega_1^4-i\omega_2^4)&=-\sqrt{3}\mu i u_6\\
L(\omega_1^7+i\omega_2^7,\omega_1^5-i\omega_2^5)&=\sqrt{3}\mu u_4 & L(\omega_1^8+i\omega_2^8,\omega_1^5-i\omega_2^5)&=\sqrt{3}\mu i u_2\\
L(\omega_1^7+i\omega_2^7,\omega_1^6-i\omega_2^6)&=\sqrt{3}\mu i u_2 & L(\omega_1^8+i\omega_2^8,\omega_1^6-i\omega_2^6)&=-\sqrt{3}\mu u_4\\
L(\omega_1^7+i\omega_2^7,\omega_1^7-i\omega_2^7)&=\sqrt{3}\mu u_1 & L(\omega_1^8+i\omega_2^8,\omega_1^7-i\omega_2^7)&=\sqrt{3}\mu u_3\\
L(\omega_1^7+i\omega_2^7,\omega_1^8-i\omega_2^8)&=\sqrt{3}\mu u_3 & L(\omega_1^8+i\omega_2^8,\omega_1^8-i\omega_2^8)&=-\sqrt{3}\mu u_1
\end{aligned}
$$

- Once we have determined L, we can determine the second fundamental form. By the Gauss equation, the theorem of Cartan is verified and therefore, any two minimal Lagrangian isotropic submanifolds of a complex space form should be isometric.
- By the existence theorem 2.4. in the studied paper, we get that the immersions are congruent.
- This implies that, in order to complete the classification theorem, it is sufficient to present an example which has the desired properties.

Thank You!