Large isoperimetric regions in the product of a compact manifold with Euclidean space

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In this talk I shall consider the following

Problem

Given a compact Riemannian manifold M without boundary, and an isoperimetric region of "large" volume $E \subset M \times \mathbb{R}^k$, prove that $E = M \times B$, where $B \subset \mathbb{R}^k$ is an Euclidean ball

More formal statement

There exists $v_0 > 0$, depending on M, such that any isoperimetric region $E \subset M \times \mathbb{R}^k$ with volume $|E| \geq v_0$ is of the form $M \times B$, where $B \subset \mathbb{R}^k$ is an Euclidean ball

Remark

$M \times B$ is a tubular neighborhood of $M \times \{x_0\}$, where $x_0 \in \mathbb{R}^k$

Problem actually solved

M. Ritoré, E. Vernadakis, Large isoperimetric regions in the product of a compact manifold with Euclidean space, arXiv:1312.1581, 5 Dec 2013
Also available at http://www.ugr.es/~ritore

Problem posed by W.-Y. Hsiang

- Gonzalo (1991, Ph.D. Thesis UC berkeley)
- Duzaar-Steffen (1996), $M \times \mathbb{R}$
  - Morgan observed that the monotonicity formula and some properties of the isoperimetric profile can be used to prove this result
  - Ritoré-Vernadakis (2014), $C \times \mathbb{R}$, $C \subset \mathbb{R}^n$ is a convex body with non-smooth boundary
- Pedrosa-Ritoré (1999), $\mathbb{S}^1 \times \mathbb{R}^k$, obtained a classification of isoperimetric regions, from which the result follows
- Eichmair-Menzger (2013), classification of large isoperimetric regions in asymptotically flat manifolds
- Gonzalo (25 Dec 2013), gave a proof of the result using soap bubble geometry

Background

$N = M \times \mathbb{R}^k$, where M is a compact m-dimensional Riemannian manifold without boundary. $n = m + k$. Given $E \subset N$, $|E|$ will be its volume and $P(E)$ its perimeter

Anisotropic dilations

$$\varphi_t(p, x) = (p, tx),\ p \in M,\ x \in \mathbb{R}^k,\ t > 0$$

Volume

$$|\varphi_t(E)| = t^k |E|$$

Perimeter

$$t^{k-1} P(E) \geq P(\varphi_t(E)) \geq t^k P(E), \qquad t \leq 1$$

Equality holds at the left side inequality when the normal ξ at regular points of ∂E is tangent to the $\mathbb{R}^k$ factor, as for tubes

Symmetrization

Given $E \subset N$, there exists $\mathrm{sym}(E)$ such that

$$\mathrm{sym}(E) \cap (\{p\} \times \mathbb{R}^k) = \{p\} \times B_p,$$

where $B_p \subset \mathbb{R}^k$ is a ball centered at $0 \in \mathbb{R}^k$ with the same k-volume as $E \cap (\{p\} \times \mathbb{R}^k)$. It satisfies

- $|\mathrm{sym}(E)| = |E|$
- $P(\mathrm{sym}(E)) \leq P(E)$

Equality characterized in some cases (Cheblík-Cianchi-Fusco, Ann. Math. (2005) for Steiner's)

Normalized sets

$E \subset N$ is normalized if $E = \mathrm{sym}(E)$

Tubes

For tubes $T(x,r) = M \times D(x,r)$ we have

$$P(T(r)) = k\,(\omega_k H^m(M))^{1/k}|T(r)|^{(k-1)/k}$$

The isoperimetric profile

It is defined by $I(v) = \inf\{P(E); E \subset N, |E| = v\}$

- I is non-decreasing and continuous (use the anisotropic dilations)
- $I(v) \leq k\,(\omega_k H^m(M))^{1/k} v^{(k-1)/k}$ (compare with tubes)

Existence and regularity of isoperimetric regions

Existence proven by Morgan. Regularity is well-known (Gonzalez-Massari-Tamanini)

Simple isoperimetric inequalities

- (on M) given $0 < v_0 < H^m(M)$, there exists $a(v_0) > 0$ s. t. $H^{m-1}(\partial E) \geq a(v_0)\, H^m(E)$ for $E \subset M$ with $0 < H^m(E) < v_0$
- (on N) given $v_0 > 0$, there exists $c(v_0) > 0$ such that $I(v) \geq c(v_0)\, v^{(n-1)/n}$ for any $0 < v < v_0$

Sketch of the proof

Take $E_i \subset N$ normalized isoperimetric regions with $|E_i| \to \infty$. Scale down so that $|\varphi_{t_i}(E_i)| = |T|$, where T is a tube

- A geometric argument shows that $\varphi_{t_i}(E_i) \xrightarrow{L^1} T$. Equivalently $|\varphi_{t_i}(E_i) \triangle T| \to 0$
- Improve the convergence to Hausdorff's using uniform density estimates for isoperimetric regions of large volume
- Tubes are strictly $O(k)$-stable for large radius. White (1994) $\Rightarrow \varphi_{t_i}(E_i) = T$ for large i. Hence E_i is a tube. This proves the result for normalized isoperimetric regions
- If E is a general (not normalized) isoperimetric region, replace it by a normalized one using symmetrization. Prove that, if $\mathrm{sym}(E)$ is a tube and E is isoperimetric, then E is a tube

Proposition

Let $\{E_i\}_{i\in\mathbb{N}}$ normalized $|E_i| \to \infty$. Scale down so that $|\varphi_{t_i}(E_i)| = v_0$ for all $i \in \mathbb{N}$. T =normalized tube of volume v_0.

If $\varphi_{t_i}(E_i) \overset{L^1}{\not\to} T$, there exists $c > 0$, only depending on $\{E_i\}_{i\in\mathbb{N}}$, so that, passing to a subsequence we get,

$$H^{n-1}(\partial E_i) \geq c|E_i|.$$

This implies

L^1-convergence of scaled isoperimetric regions

Let $\{E_i\}_{i\in\mathbb{N}}$ normalized isoperimetric sets with $|E_i| \to \infty$. Scale down so that $|\varphi_{t_i}(E_i)| = v_0$ for all $i \in \mathbb{N}$. T =normalized tube of volume v_0. Then $\varphi_{t_i}(E_i) \overset{L^1}{\to} T$

Proof of Corollary

$$c\,|E_i| \leq P(E_i) \leq k\left(\omega_k H^m(M)\right)^{1/k} |E_i|^{(k-1)/k}$$

Proof of Proposition

Given $E \subset N$ normalized, let $T(E)$ be the normalized tube of the same volume as E and $E^+ = E \setminus T(E)$.

The fact that $\varphi_{t_i}(E_i) \overset{L^1}{\not\to} T$ implies

1. $\limsup_{i\to\infty} \frac{|E_i^+|}{|E_i|} > c_1 > 0$
2. $\liminf_{i\to\infty} H^m((\varphi_{t_i}(E_i) \cap \partial T)^*) < H^m(M)$

Hence

$$\begin{aligned}
H^{n-1}(\partial E_i) &\geq H^{n-1}(\partial E_i \cap (N \setminus T(r_i))) \\
&\geq \int_{r_i}^{\infty} H^{n-2}(\partial E_i \cap \partial T(s))\, ds \\
&\geq \int_{r_i}^{\infty} H^{n-2}(\partial(E_i \cap \partial T(s)))\, ds \\
&= \int_{r_i}^{\infty} H^{m-1}(\partial(E_i \cap \partial T(s))^*)\, H^{k-1}(\partial D(s))\, ds \\
&\geq \int_{r_i}^{\infty} a\, H^{m}((E_i \cap \partial T(s))^*)\, H^{k-1}(\partial D(s))\, ds \\
&= a \int_{r_i}^{\infty} H^{n-1}(E_i \cap \partial T(s))\, ds = a\, |E_i^+| > a\, c_1 |E_i|,
\end{aligned}$$

Improvement of convergence

Define

$$h(E,x,R) = \frac{\min\left\{|E \cap T(x,R)|, |T(x,R) \setminus E|\right\}}{R^n}$$

Density estimates

$E \subset N$ isoperimetric region $|E| > v_0$. $\tau > 1$ such that $|\Omega| = |\varphi_\tau^{-1}(E)| = v_0$. Choose

$$0 < \varepsilon < \left\{ v_0, \left(\frac{c(v_0)\, v_0^{1/k}}{2H^m(M)}\right)^n, \left(\frac{c(v_0)}{8n}\right)^n \right\},$$

For any $x \in \mathbb{R}^k$ and $R \leq 1$ so that $h(\Omega, x, R) \leq \varepsilon$, we get

$$h(\Omega, x, R/2) = 0.$$

Hausdorff convergence of scaled isoperimetric regions

$E_i \subset N$ normalized isoperimetric with $|E_i| \to \infty$. Scale down so that $|\Omega_i| = |\varphi_{t_i}(E_i)| = v_0$. Then for every $r > 0$, $\partial\Omega_i \subset (\partial T)_r$, for large enough $i \in \mathbb{N}$, where T is the tube of volume v_0.

Theorem (White)

Let T be a normalized tube so that $\Sigma = \partial T$ is a strictly $O(k)$-stable cylinder. Then there exists $r > 0$ so that any $O(k)$-invariant finite perimeter set E with $|E| = |T|$ and $\partial E \subset (\partial T)_r$ has larger perimeter than T unless $E = T$.

Strict stability of tubes with large volume

The cylinder $\Sigma(s)$ is strictly $O(k)$-stable if and only if

$$s^2 > \frac{k}{\lambda_1(M)},$$

where $\lambda_1(M)$ is the first positive eigenvalue of the Laplacian in M.

Proof of the Theorem

1. Get a sequence $\{E_i\}_{i\in\mathbb{N}}$ of isoperimetric regions with $|E_i| \to \infty$. Replace each set E_i by $\mathrm{sym}(E_i)$.
2. The previous results imply that $\mathrm{sym}(E_i)$ are normalized tubes for i large enough. This implies that there exists a constant $v_0 > 0$ such that $I(v) = k\,(\omega_k H^m(M))^{1/k}\, v^{(k-1)/k}$ for $v \geq v_0$.
3. In particular $I(t^k v) = t^{k-1} I(v)$, whenever $t \leq 1$, $t^k v \geq v_0$.
4. Let $E \subset N$ be isoperimetric with $|E| > v_0$. Then

$$I(t^k|E|) \leq P(\varphi_t(E)) \leq t^{k-1}P(E) = t^{k-1}I(|E|)$$

   and equality holds. This implies that the normal ξ to the regular part of ∂E is tangent to the $\mathbb{R}^k$ factor.
5. Since E is isoperimetric and has the same perimeter as the tube of the same volume, Federer's coarea formula implies that $E \cap (\{p\} \times \mathbb{R}^k)$ is a disc H^m-a.e. $p \in M$.
6. Hence E is a tube.

Final comments

- An equivariant version of a result of Morgan and Ros implies (modulo L^1-convergence) our result for small dimension
- It is an open problem to prove a similar result in $M \times \mathbb{H}^n$
- The result follows when ∂M is smooth enough
- It would be interesting to find an explicit dependence of v_0 in terms of the geometry of M

Thanks for your attention!