Minimal surfaces in $\mathbf{H}^2 \times \mathbf{R}$ with finite total curvature and related problems

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Introduction

Theorem (Hauswirth-Rosenberg, 2006)

$\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ *compl. or. min. surf.*
$|\int_\Sigma K| < +\infty$, $\quad K =$ *Gauss curv. of* Σ

- $\Sigma \overset{conf}{\cong} \mathbb{M} - \{p_1, \cdots, p_k\}$.
- $Q=$ *Hopf diff. of* $\Sigma \to \mathbb{H}^2$ *extiends meromorphically to* $\mathbb{M}$; $Q(z) = z^{2m_i}(dz)^2$ *at* p_i, $m_i \geq 0$.
- $N_3 \to 0$ *at* p_i.
- $\int_\Sigma K = 2\pi(2 - 2g - 2k - \sum_{i=1}^k m_i)$.

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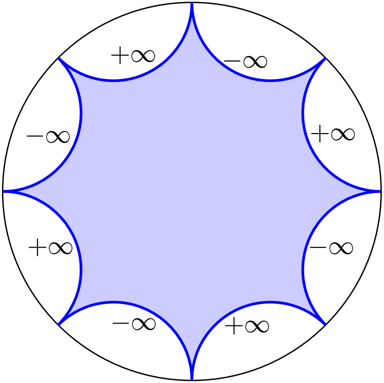
Introduction

Examples: Scherk graphs over ideal polygons with $2k$ edges, $k \geq 2$ (J-S condition) $\rightsquigarrow \int_\Sigma K = 2\pi(1-k)$



Introduction

Question [Hauswirth-Rosenberg]:

Are there non-symply connected examples of f.t.c.?

An annulus Σ with $\int_\Sigma K = -4\pi$?

New examples

k-noids

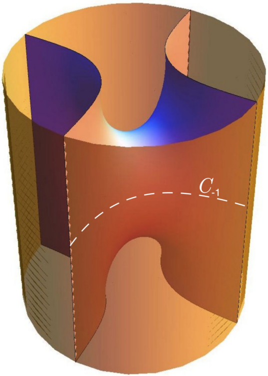
Theorem (Pyo, Morabito - __)

For any $k \geq 2$, $\exists \Sigma_k \subset \mathbb{H}^2 \times \mathbb{R}$ PEMS with genus 0, k vertical planar ends and

$$\int_\Sigma K = 4\pi(1-k).$$

($\exists$ a $(2k-3)$-parameter family)

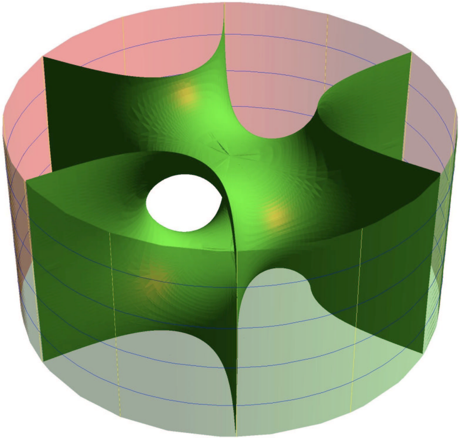
New examples
k-noids



Parameter = dist. between the asymptotic vertical planes

New examples

k-noids



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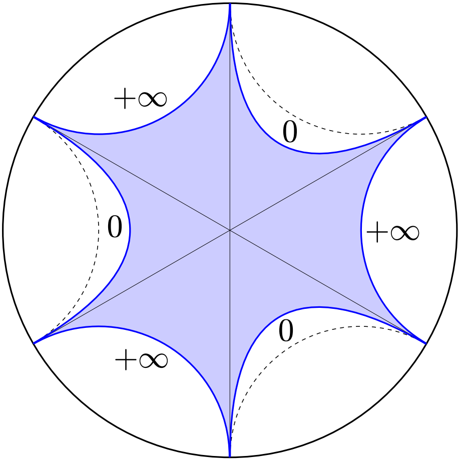

New examples
k-noids

We can take limits of k-noids, $k \to +\infty$

Question [Ros]: Is there a PEMS for any genus 0 topology?

Theorem (Martín - __)

$\forall \Sigma =$ *planar domain*, $\exists f : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ *prop. min. embedding.*
Finite topology $\Rightarrow$ *f.t.c.*

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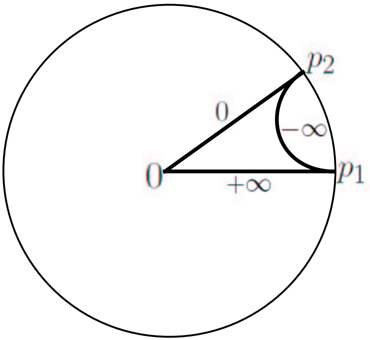
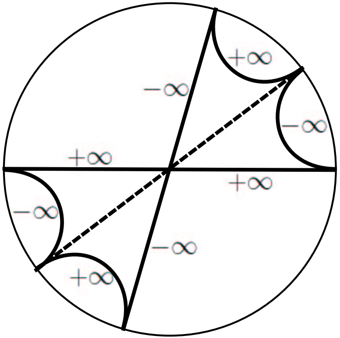
New examples
Twisted-Scherk examples

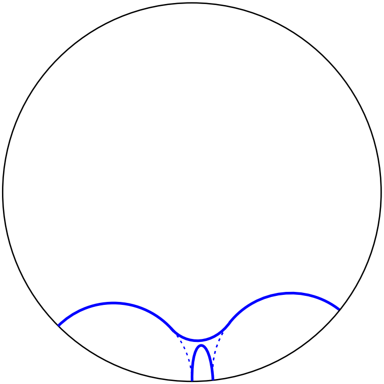
Question: Are the geodesics defining the ends "ordered"?





New examples
Gluing

Question: Examples with higher genus?



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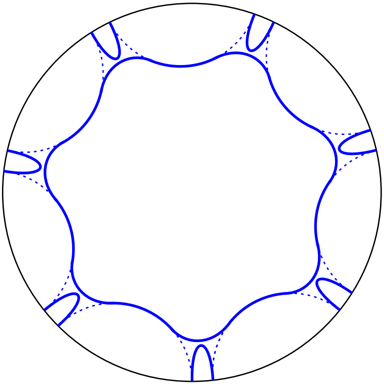

New examples
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Theorem (Martín - Mazzeo - __)

For any $g \geq 0$ and $k > 1$ large, $\exists \Sigma_{g,k} \subset \mathbb{H}^2 \times \mathbb{R}$ PEMS with f.t.c., genus g and k vertical planar ends.

Moreover, the c.c. of

$$\mathcal{M}_{g,k} = \left\{ \begin{array}{c} \Sigma \subset \mathbb{H}^2 \times \mathbb{R} \text{ PEMS with f.t.c.,} \\ \text{genus } g \text{ and } k \text{ vertical planar ends} \end{array} \right\}$$

containing $\Sigma_{g,k}$ is a real analytic space of dimension $2k - 3$.

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Classification results

$\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ with f.t.c. $\Rightarrow \int_\Sigma K = 2\pi(2 - 2g - 2k - \sum_{i=1}^k m_i)$

Theorem (Hauswirth - Sa Earp - Toubiana)

Σ = min. surf. in $\mathbb{H}^2 \times \mathbb{R}$ with $\int_\Sigma K = 0 \Rightarrow \Sigma$ = vert. plane

Theorem (Pyo - __)

Σ = min. surf. in $\mathbb{H}^2 \times \mathbb{R}$ with f.t.c. $\int_\Sigma K = -2\pi$

$\Rightarrow \Sigma$ = a Scherk minimal graph over an ideal quadrilateral

Theorem (Hauswirth - Nelli- Sa Earp - Toubiana)

Σ = min. surf. in $\mathbb{H}^2 \times \mathbb{R}$ with f.t.c. and 2 vertical planar ends

$\Rightarrow \Sigma$ = horizontal catenoid

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Embedded Calabi-Yau problem

Question: When is a compl. emb. min. surf. proper?

Theorem (Colding-Minicozzi)

Any compl. emb. min. surf. with fin. top. in $\mathbb{R}^3$ must be proper.

Generalizations $\rightsquigarrow$ Meeks-Rosenberg, Meeks-Pérez-Ros

Theorem (Coskunuzer)

There exists a compl. non-proper emb. min. disk in $\mathbb{H}^3$.

Theorem (__ - Tinaglia)

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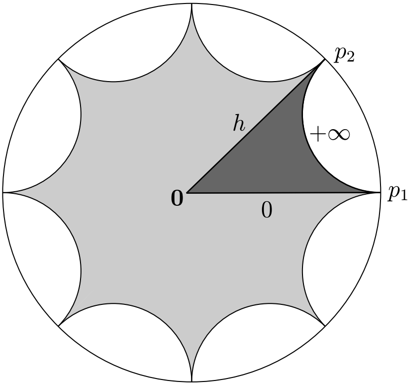

- *Saddle Towers and minimal k-noids in* $\mathbb{H}^2 \times \mathbb{R}$
(joint work with Filippo Morabito),
J. Inst. Math. Jussieu, 11 (2), pp 333-349 (2012).
- *Minimal surfaces with limit ends in* $\mathbb{H}^2 \times \mathbb{R}$,
to appear in J. Reine Angew. Math. (Crelle's Journal).
- *Non-simply connected minimal planar domains in* $\mathbb{H}^2 \times \mathbb{R}$
(joint work with Francisco Martín), to appear in Trans. AMS.
- *Minimal surfaces with positive genus and finite total curvature in* $\mathbb{H}^2 \times \mathbb{R}$ (joint work with Francisco Martín and Rafe Mazzeo), preprint.
- *Simply-connected minimal surfaces with finite total curvature in* $\mathbb{H}^2 \times \mathbb{R}$ (joint work with Juncheol Pyo),
to appear in Int. Math. Res. Not. (IMRN).
- *Non-proper complete minimal surfaces embedded in* $\mathbb{H}^2 \times \mathbb{R}$
(joint work with Giuseppe Tinaglia), preprint.