ON THE ENTROPIES OF HYPERSURFACES WITH POSITIVE H

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The talk is organized as follows

Question (doCarmo1980's)

Is a noncompact complete stable constant mean curvature hypersurface of $\mathbb{R}^{n+1}$, for $n \geq 3$, necessarily minimal ?

Some positive answers

- $n = 2$: *[DaSilveira89], [LopezRos89], [Palmer86]...*
- $n = 3, 4$: *[Cheng06], [ElbertNelliRosenberg07] , using a Bonnet-Myers's type method.*

we give a positive answer to do Carmo's question if:

- The volume entropy of M is zero (the ambient manifold being arbitrary).
- The hypersurface M has bounded curvature and is properly embedded (the ambient manifold being a simply-connected manifold with bounded geometry and with zero volume entropy)
- The entropy of the total curvature of M is zero and $n \leq 5$ (the ambient manifold being a space-form).

Stability of Schrödinger operators

Quadratic form associated to $L := \Delta_M + V$

$$Q(f,f) := -\int_M fLf = \int_M |\nabla f|^2 dv - \int_M Vf^2 dv \quad f \in C_0^\infty(M)$$

Let $\Omega \subset M$ be rel. compact.

$$i_{L\Omega} := \#\text{neg. eigenvals of } Lu + \lambda u = 0, u|_{\partial\Omega} = 0$$

$$Index(L) := \sup_{\Omega \subset M} i_{L\Omega}$$

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Let $\Omega \subset M$ be rel. compact.

$$Wi_{L\Omega} := \#\text{neg. eigenvals of } Lu + \lambda u = 0, u|_{\partial\Omega} = 0, \int_M udv = 0$$

$$WIndex(L) := \sup_{\Omega \subset M} Wi_{L\Omega}$$

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$$Index(L) := \sup_{\Omega \subset M} i_{L\Omega}$$

M is stable (w.r.t. L) if $\mathrm{Index}(L) = 0$.

M is weakly stable if $\mathrm{WIndex}(L) = 0$.

M has finite index if $\mathrm{Index}(L)$ or $\mathrm{WIndex}(L)$ is finite [BarbBer00].

M has finite index iff $M \setminus K$ is stable for some K.

[FischerColbrie85],[Devyver11]

Stability of Schrödinger operators

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When $V = |A|^2 + Ric_{\mathcal{N}}(\nu,\nu)$, and $M \subset \mathcal{N}$ has cmc, $Q(f,f)$ is the second derivative of the volume in the direction of $f\nu$, and $\mathrm{Index}(M)$ measures the # of linearly independent normal deformations with compact supp. of M that decrease area. [BarbosadoCarmoEschenburg88]

Entropies

Definition

- Entropy of w where w is a positive non-decreasing function.

$$\mu_w := \limsup_{r \longrightarrow \infty} \left(\frac{\ln w(r)}{r} \right).$$

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- Volume entropy of M

$$\mu_M := \mu_{w_1}$$

$w_1(r) := |B_p^M(r)|$, volume of a geodesic ball in M of rad. r around a fixed $p \in M$.

Entropies

Definition (2)

- Extrinsic volume entropy of $M \subset \mathcal{N}$

$$\mu_M^{\mathcal{N}} := \mu_{w_2}$$

$w_2(r) := |B_p^{\mathcal{N}}(r) \cap M_p|_M$
(M_p is the connected component of $M \cap B_p(r)$ containing p)

- Total p-curvature entropy of $M \subset \mathcal{N}, p > 0$:

$$\mu_{\mathcal{T}_p} := \mu_{w_3}$$

$w_3(r) := \int_{B_p(r)} |A_0|^{2p} dv_M$ where A_0 is the traceless shape operator.

Bottom of the essential spectrum of Δ_M

Definition

If $M' \subset M$ unbounded

$$\lambda_0(M') := \inf_{\Omega \subset M'} \lambda_0(\Omega) = \inf_{\substack{f \in C_0^\infty(M') \\ f \neq 0}} \left(\frac{\int_{M'} |\nabla f|^2}{\int_M f^2} \right).$$

$$\lambda_0^{ess}(M) := \sup_{K \subset M} \lambda_0(M \setminus K)$$

Volume entropy & essential spectrum

Theorem ([Brooks-Cheeger])

If M has infinite volume, then

$$\frac{h_M^2}{4} \leq \lambda_0(M) \leq \lambda_0(M \setminus K) \leq \lambda_0^{ess}(M) \leq \frac{\mu_M^2}{4}$$

where K is any compact subset of M and $h_M = \inf_{\Omega} \frac{|\Omega|}{|int(\Omega)|}$ (where Ω runs over all compact codimension one submanifolds of M dividing M into two components and $int(\Omega)$ denotes the bounded component).

Stable ends with suffic. large H have exp. volume growth

A positive answer to the question when $\mu_M = 0$

Theorem

There is no complete, noncompact, finite index $M \subset \mathcal{N}$, provided H satisfies $nH^2 + Ric(\nu,\nu) \geq \varepsilon > \frac{\mu_M^2}{4}$ for some positive ε.

Corollary

If M is CMC hypersurface of finite index immersed in a space of constant curvature k then $n(H^2 + k) \leq \frac{\mu_M^2}{4}$.
$\Longrightarrow$ *if $H^2 + k > 0$, the volume growth of M is exponential.*

Related results : [Higuchi], [Karp], [BarbosadoCarmo], [doCarmoZhou]...

Stable ends with suffic. large H have exp. volume growth

Proof.

M has finite index $\iff \exists K \subset M$ such that $M \setminus K$ is stable.

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$$0 \leq Q(f,f) = \int_{M \setminus K} |\nabla f|^2 - (|A|^2 + Ric(\nu,\nu))f^2 \;\forall f \in C_0^\infty(M \setminus K).$$

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But

$|A|^2 + Ric(\nu,\nu) \geq nH^2 + Ric(\nu,\nu) \geq \varepsilon > \frac{\mu_M^2}{4}$ (or $\lambda_0^{ess}(M)$) $\Longrightarrow$

$$0 \leq \int_{M \setminus K} |\nabla f|^2 - \varepsilon \int_{M \setminus K} f^2$$

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$$0 \leq \int_{M\setminus K} |\nabla f|^2 - \varepsilon \int_{M\setminus K} f^2$$

Hence $\lambda_0(M \setminus K) \geq \varepsilon > \frac{\mu_M^2}{4}$. This contradicts Brooks Theorem $\square$

Stable ends with suffic. large H have exp. volume growth

Let us find a bound for the H of a sub. with finite index, that is independent of M

Last result is not satisfying.

Remark

As extrinsic distance is bounded by intrinsic distance

$$B_p^M(R) \subset M \cap B_p^{\mathcal{N}}(R) \implies |B_p^M(R)|_M \leq |M \cap B_p^{\mathcal{N}}(R)|_M$$

Hence

$$\mu_M \leq j_M := \inf_{\{\mathcal{N}:M \text{ isometrically immersed in } \mathcal{N}\}} \mu_M^{\mathcal{N}} \tag{1}$$

Volume comparison

In the whole section we will suppose that :

Curvatures of M and $\mathcal{N}$ are bounded : $|R_{M,\mathcal{N}}|, |\nabla R_{M,\mathcal{N}}| \leq c_1$.

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Claim:

Suppose M is prop. embedded in a $\mathcal{N}$ with $\pi_1(\mathcal{N}) = \{1\}$ and s.t. M and $\mathcal{N}$ are of bounded curvature. If $H \geq \epsilon > 0$ then there is a constant c, depending on c_1 such that

$$|B_p^M(R)| \leq |B_p^{\mathcal{N}}(R) \cap M|_M \leq c |B_p^{\mathcal{N}}(R)|_{\mathcal{N}}$$

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$$|B_p^M(R)| \leq |B_p^{\mathcal{N}}(R) \cap M|_M \leq c|B_p^{\mathcal{N}}(R)|_{\mathcal{N}}$$

$\Longrightarrow \mu_M \leq \mu_M^{\mathcal{N}} \leq \mu_{\mathcal{N}}$

Extrinsic upperbounds for $\lambda_0^{ess}(M)$

M is prop. emb. in $\mathcal{N}$ and $H \geq \epsilon > 0$

$$\mu_M \leq \mu_M^{\mathcal{N}} \leq \mu_{\mathcal{N}} \overset{Brooks}{\Longrightarrow} \lambda_0^{ess}(M) \leq \frac{\mu_{\mathcal{N}}^2}{4}$$

Example

$Ricc_{\mathcal{N}} \geq 0$: Bishop comparison theorem $\Longrightarrow \mu_{\mathcal{N}} = 0$.
$\Longrightarrow \lambda_0^{ess}(M) = \lambda_0(M) = h_M = 0$ (and discrete L^2 spec$(\Delta) = \emptyset$).
$Ricc_{\mathcal{N}} \geq -nk$: (k is a positive constant). BCT $\Longrightarrow \mu_{\mathcal{N}} \leq n\sqrt{k}$
$\Longrightarrow \lambda_0^{ess}(M) \leq \frac{n^2 k}{4}$ (Compare with [Karp85]).

Upperbounds for the Cheeger constant are similarly obtained.

Properly embedded stable ends with bounded curvature

Theorem

There is no complete, noncompact, finite index M prop. emb. in a s.c $\mathcal{N}$ where M and $\mathcal{N}$ have bounded curvature provided $H \geq \varepsilon > 0$ and $nH^2 + Ric(\nu, \nu) > \frac{\mu_{\mathcal{N}}^2}{4}$ for some positive ε.

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Proof.

$\mu_M \leq \mu_M^{\mathcal{N}} \leq \mu_{\mathcal{N}} \overset{Brooks}{\Longrightarrow} \lambda_0^{ess}(M) \leq \frac{\mu_{\mathcal{N}}^2}{4} \overset{\text{finite index}}{\Longrightarrow} \lambda_0^{ess}(M) > \frac{\mu_{\mathcal{N}}^2}{4}$, contradiction. □

Properly embedded stable cmc ends of bounded curvature in $\mathbb{R}^n$ are minimal

Theorem

There is no complete, noncompact, finite index M prop. emb. in a s.c $\mathcal{N}$ where M and $\mathcal{N}$ have bounded curvature provided $H \geq \varepsilon > 0$ and $nH^2 + Ric(\nu, \nu) > \frac{\mu_{\mathcal{N}}^2}{4}$ for some positive ε.

Corollary

If M is a prop. emb. CMC hypersurface of bounded curvature in a space form of curvature $k \geq 0$ of finite index, then $k = 0$ and M is minimal.

A uniform bound for the mean curvature of prop. emb. stable cmc ends of bounded curvature in $\mathbb{H}^n$

Theorem

There is no complete, noncompact, finite index M prop. emb. in a s.c $\mathcal{N}$ where M and $\mathcal{N}$ have bounded curvature provided $H \geq \varepsilon > 0$ and $nH^2 + Ric(\nu, \nu) > \frac{\mu_{\mathcal{N}}^2}{4}$ for some positive ε.

Corollary

If M is a prop. emb. CMC hypersurface of bounded curvature in a space form of curvature $-k < 0$ of finite index, then $H^2 \leq \frac{kn}{4} + k$

Definition of Halftubes

We consider the normal exponential map of M defined by

$$\begin{aligned} exp: \quad M \times \mathbb{R} &\longrightarrow \mathcal{N} \\ (p, r) &\mapsto exp_p(r\nu) \end{aligned}$$

Let

$$T^+(r_0) := exp\,(M \times (0, r_0))$$

be the halftube in $\mathcal{N}$ around M of radius r_0.

Embedded halftube theorem

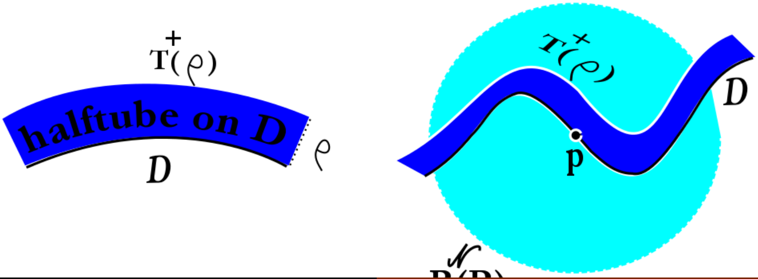
Theorem

Let M be prop. embedded in a s.c. $\mathcal{N}$, where M and $\mathcal{N}$ are spaces of bounded curvature. Suppose $H \geq \epsilon > 0$.
If $\partial M = \emptyset$, there exists an embedded half-tube $T^+(\rho), \rho > 0$ contained in the mean-convex side of M.
If ∂M is compact, then there exists a compact subset K of M and an embedded half-tube $T^+(\rho'), \rho' > 0$ of $M \setminus K$ contained in the mean-convex side of M

See also [MeeksTinaglia]

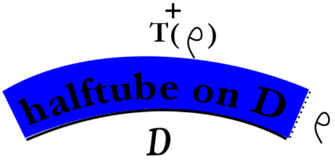
Embedded halftube theorem $\Longrightarrow$ volume comparison

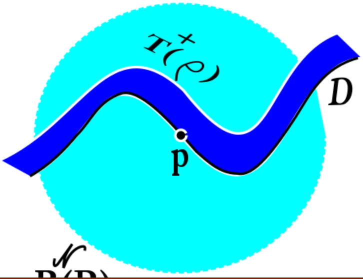
Weyl tube formula [Gray82] & expansion w.r.t. $\rho \Longrightarrow$
$T^+(D) \sim \rho |D|_M$ (fig 1)



Embedded halftube theorem $\Longrightarrow$ volume comparison

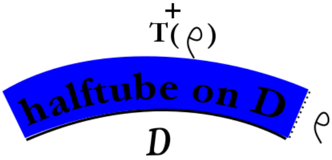
Weyl tube formula [Gray82] & expansion w.r.t. ρ $\Longrightarrow$
$T^+(D) \sim \rho|D|_M$ (fig 1)
Apply to fig2 : $|T^+(\rho) \cap B_p^{\mathcal{N}}(R)| \sim \rho|M \cap B_p^{\mathcal{N}}(R)|_M$

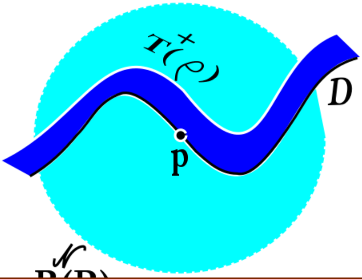




Embedded halftube theorem $\implies$ volume comparison

Weyl tube formula [Gray82] & expansion w.r.t. $\rho \implies$
$T^+(D) \sim \rho |D|_M$ (fig 1)
Apply to fig2 : $|T^+(\rho) \cap B_p^{\mathcal{N}}(R)| \sim \rho |M \cap B_p^{\mathcal{N}}(R)|_M$ EHT
$\implies |T^+(\rho) \cap B_p^{\mathcal{N}}(R)|_{\mathcal{N}} \leq |B_p^{\mathcal{N}}(R)| \implies |B_R^{\mathcal{N}} \cap M|_M \sim \frac{1}{\rho}|T^+(\rho) \cap B_R^{\mathcal{N}}| \leq \frac{1}{\rho}|B_r^{\mathcal{N}}|$





Sketch of proof of the embedded halftube theorem

Two cases :

1. either $M \cap T^+(\rho) = \emptyset$ for some ρ
2. or $\forall$ suff. small $\rho > 0$, $T^+(\rho) \cap M \neq \emptyset$.

Sketch of proof of the EHT : case 1

$M \cap T^+(\rho) = \emptyset \implies T^+(\rho/2)$ embedded

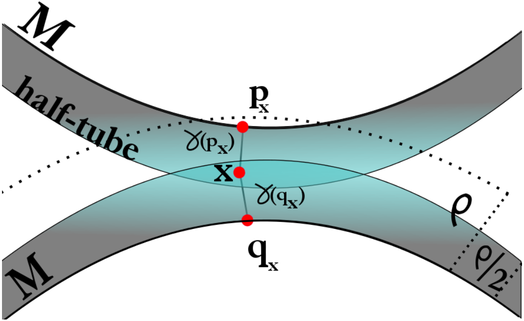
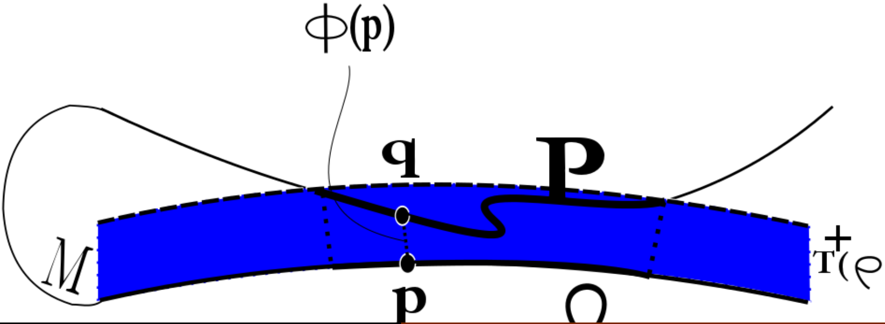


Figure : Self-intersection of a halftube of M in $\mathcal{N}$

Sketch of proof of the EHT : case 2

Suppose $\forall$ suff. small $\rho > 0$, $T^+(\rho) \cap M \neq \emptyset$.; then contradiction

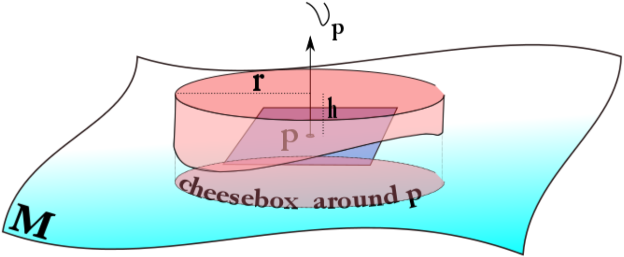
$T^+(\rho) \cap M$ defines a graph of a section ϕ of the normal bundle of M on $\Omega \subset M$. $\Omega := \{p \in M : t \leq \rho\}$ and $\phi(p) = t$.
$P := \{(p, \phi(p)) : p \in \Omega\}$. M properly embedded $\Longrightarrow \phi(p) > 0$.



Step 1 : $\|\phi\|_1 := \left(\sup_{p\in\Omega}|\phi| + \sup_{p\in\Omega}|\nabla\phi|\right) \leq O(\rho)$

Cheesebox argument at $p \in M$

$\exists\rho$ M is loc. a graph G_p defined on $D(p,\rho)$ of T_pM.
$G_p \subset D(p,\rho)\times[-h/2,h/2]$, of radius ρ and height $h = c\rho^2$
G_p cuts the boundary of the cheesebox in $\partial D(p,\rho)\times[-c\rho^2, c\rho^2]$.



Step 1 : $\|\phi\|_1 \leq O(\rho)$

Compare cheeseboxes at $p \in M$ and $q \in P$

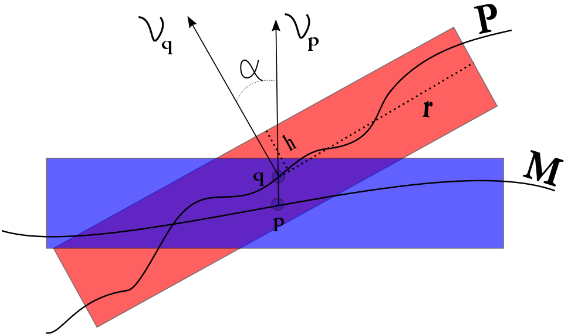
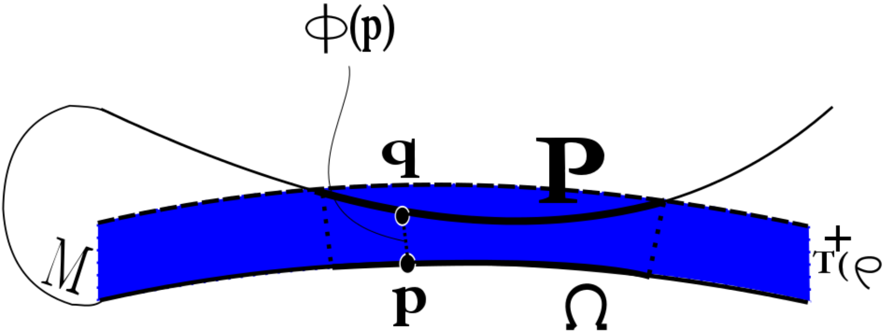


Figure : Intersection of the cheeseboxes at $p \in M$ and at $q \in P$ with $\mathbb{R}\nu_p \oplus \mathbb{R}\nu_q$

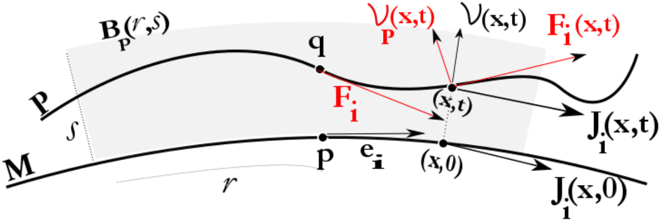
$\|\phi\|_1 \leq O(\rho) \implies$ P is a graph



Step 2 : ϕ satisfies a unif. elliptic quasilin. PDE

0. Mean curvature equation of the section P defined by ϕ

$$nH_P(q) := -div_P(\nu_P)(q) = -\sum_{i=1}^{n} g_{\mathcal{N}}(\nabla_{F_i}\nu_P, F_i)(q),$$



Step 2.1 Fermi fields : Local frame on P

$$F_i\left(x,\phi\left(x\right)\right) := \partial_t|exp\left(\gamma_i\left(t\right),\phi\left(\gamma_i\left(t\right)\right)\right), \quad i = 1,\cdots,n$$

$$\Longrightarrow F_i\left(x,\phi\left(x\right)\right) = J_i\left(x,\phi\left(x\right)\right) + \phi_i\left(x\right)\nu(x,\phi(s)),\ \left(\phi_i\left(p\right) = \nabla_{e_i}\phi\right)$$

$$\left\{g_{\mathcal{N}}\left(\nu_P,F_i\right) = 0\right\}_{i=1,\cdots,n}, g_{\mathcal{N}}\left(J_i,J_{n+1}\right) = 0, g_{\mathcal{N}}\left(J_{n+1},J_{n+1}\right) = 1,$$

$$\Longrightarrow \nu_P := \frac{1}{W}\left(\nu - \sum_{i=1}^{n} g^{ij}\phi_i J_j\right). \tag{2}$$

$\left(g_{ij} := g_{\mathcal{N}}\left(J_i,J_j\right), W := \|\nu - \sum_{i=1}^{n} g^{ij}\phi_i J_j\| = \sqrt{1+\|\nabla\phi\|^2}\right).$

Step 2.2 Mean curvature equation in Fermi coordinates

Geodesic coordinates

$$\begin{cases} \tilde{g}_{ij} = g_{\mathcal{N}}(F_i, F_j), \tilde{g}^{ij} = g^{ij} - \frac{1}{W^2} g^{k,n+1} g^{l,n+1} \phi_{,k}\phi_{,l} \\ F_i = P_i^j e_j, P_i^j = \delta_i^j + \phi_i \delta_{n+1}^j \\ (\nabla_{J_\alpha} J_\beta)^\gamma = \Gamma_{\alpha,\beta}^\gamma J_\gamma. \end{cases}$$

$$\Longrightarrow \quad \begin{aligned} nHW = \; & \tilde{g}^{ij}(\phi_{,ij} + (\Gamma_{n+1,n+1}^{n+1} - \Gamma_{n+1,n+1}^{m}\phi_{,m})\phi_{,i}\phi_{,j} \\ & -\Gamma_{n+1,(j}^{k}\phi_{,k}\phi_{,i)} + \Gamma_{n+1,(j}^{n+1}\phi_{,i)} - \Gamma_{ij}^{k}\phi_{,k} + \Gamma_{ij}^{n+1}). \end{aligned}$$

Minimal surface equation [ColdingMinicozzi99]

Step 2.2 Mean curvature equation in Fermi coordinates

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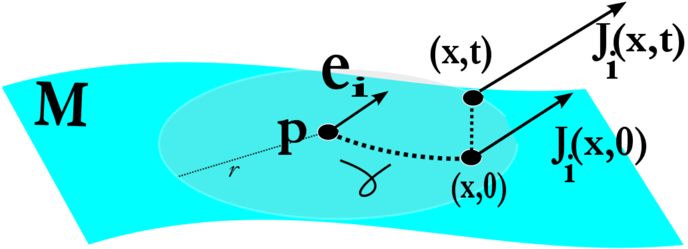
Minimal surface equation [ColdingMinicozzi99]

Remark

This is not the mean curvature equation in harmonic coordinates

Step 2.3 Expansion $H(\phi)$ w.r.t. ρ

$$nH_P = nH + \Delta_M \phi + O(\rho).$$



Step 2.4 Proof : Jacobi fields

We extend the vectors $\{e_i\}_{i\neq 1}$ to Jacobi fields $\{J_i\}_{i=1,\cdots n}$ along γ such that $J_1 = T$ on γ. The Jacobi fields $\{J_i\}_{i\neq 1}$ are defined according to the evolution equation

$$J_i''(y) + R_M(y)(J_i, T)T(y) = 0, \quad i \neq 1 \quad \forall y \in \gamma \subset exp(D(p, \rho)) \tag{3}$$

(where $J_i''(y) := \nabla_T \nabla_T J_i(y)$ and R_M is the curvature tensor of M), with the second order initial conditions at p :

$$\begin{cases} J_i(p) & = e_i \\ \nabla_T J_i(p) & = 0. \end{cases} \tag{4}$$

Step 2.5 Proof : Estimates of Jacobi fields

Since the curvature of M is bounded, Rauch Comparison Theorem

$$\Longrightarrow J_i(x) = J_i(p) + O(\rho^2), \qquad x \in \gamma \subset exp(D(p,\rho)),$$

where $O(\rho^2)$ depends on bounds of the curvature tensor R_M of M.

Step 2.5 Proof : Estimates of Jacobi fields

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where $O(\rho^2)$ depends on bounds of the curvature tensor R_M of M.

$$\Longrightarrow g_{ij}(x) := g_M\left(J_i(x), J_j(x)\right) = \delta_{ij} + 0(\rho^2) \qquad x \in \gamma \subset exp\left(D\left(p,\rho\right)\right)$$

Step 2.6 Proof : Fermi fields

Evolution equation:

$$J_i''(x,s) + R_{\mathcal{N}}(x,s)(\nu, J_i)\nu = 0 \quad x \in \gamma, s \in [0,\sigma] \tag{5}$$

together with the second order initial conditions

$$\begin{cases} J_i(x,0) & = J_i(x) \\ \nabla_\nu J_i(x,0) & = A(x)J_i(x) \quad i = 1, \cdots n \end{cases} \tag{6}$$

($J_i(x,s) := J_i\,(exp_x\,(s\nu_M))$, $\nu(x,s)$ is the unit tangent to the normal geodesics at $exp_x(s\nu_M)$, $R_{\mathcal{N}}(x,s) := R_{\mathcal{N}}\,(exp_x\,(s\nu_M))$, $J_i''(x,s) := \nabla_\nu \nabla_\nu J_i(x,s)$- and $A(x)$ is the shape operator of M at $x \in \gamma$).

Step 2.6 Proof : Estimates of Fermi fields

Compare $J_i(x,s)$ with $J_i(p,0)$.
$||R_{\mathcal{N}}|| \leq c$ and Rauch comparison theorem for Fermi fields

$$\Longrightarrow J_i(x,s) = J_i(x,0) + O(\sigma) \quad x \in \gamma, s \in [0,\sigma]. \tag{7}$$

where $O(\sigma)$ depends on the curvature of the ambient space (from the term $R_{\mathcal{N}}$ in Fermi's equation (5)) and also on the principal curvatures of M at x as one can see from the second initial condition of (6).

Step 2.7 Proof : Estimates of $(\nabla_{J_{n+1}} J_\beta)$

Riccati equation :

$$\begin{cases} J_i'(x,s) := \nabla_\nu J_j(x,s) & = A(x,s)J_i(x,s) \\ A'(x,s) := \nabla_\nu A(x,s) & = -A^2(x,s) + R_\nu(x,s). \end{cases} \qquad (8)$$

($R_\nu(x,s)J := R_{\mathcal{N}}(x,s)(J,\nu)\nu$.) Riccati's comparison argument : and the fact that $s \leq \sigma \leq c\rho^2$

$$\Longrightarrow \begin{cases} J_i'(x,s) = J_i'(x,0) + O(\rho^2), & x \in exp\,(D(p,\rho),0) \subset M,\ 0 \leq s \\ A'(x,s) = A(x,0) + O(\rho^2), & x \in exp\,(D(p,\rho),0) \subset M,\ 0 \leq s \end{cases} \qquad (9)$$

$O(\rho^2)$ depends on bounds of the shape operator A of M and bounds of $R_{\mathcal{N}}$.

Step 2.8 Proof : estimates of $(\nabla_{J_\alpha} J_\beta)$

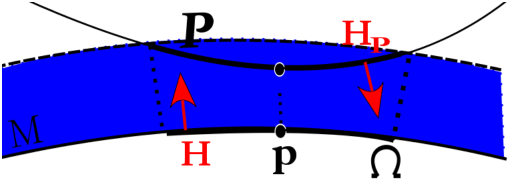
Evolution equation for $\nabla_{J_i} J_j(x,s)$: derivate Fermi equation with second order initial conditions : $\nabla_{J_i} J_j(x)$ and $\nabla_\nu \left(\nabla_{J_i} J_j\right)(x)$.
Riccati comparison argument yields again

$$\nabla_{J_i} J_j(x,s) = \nabla_{J_i} J_j(p) + O(\rho), \qquad x \in exp(D(p,\rho)), 0 \le s \le c\rho^2 \tag{10}$$

where $O(\rho)$ depends on the curvatures R_M and $R_{\mathcal{N}}$ and their first derivatives. □

Step 3 : Consequences of $nH_P = nH + \Delta\phi + O(\rho)$

$H \geq \varepsilon \Longrightarrow H_P \leq -\varepsilon. \Longrightarrow \Delta\phi \leq -2\varepsilon.$



Step 3 : Consequences of $nH_P = nH + \Delta\phi + O(\rho)$

$H \geq \varepsilon \implies H_P \leq -\varepsilon. \implies \Delta\phi \leq -2\varepsilon.$
$\Delta\psi_R \geq -\varepsilon$ on $B_R \cap \Omega$.. Max. principle $\implies \phi \geq \psi_R$.
$R \mapsto \infty \implies T^+(\beta)$ is embedded. □

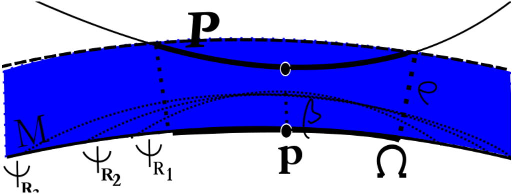


Figure : $\Psi_R := \beta(1 - \frac{r^2}{R^2})$, $\Delta r_p \leq c \implies \Delta_M \Psi_R \geq -\epsilon$ in the weak sense

Remarks on the EHT

$$(v, u) \in \mathbb{R} \times [0, \epsilon], ds^2 = j^2(u, v)du^2 + dv^2, j(u, v) = e^{f(u,v)}$$

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$f(u,v) = ln\left(\cos\left(a\left(u\right)v\right)\right) \implies f_v = -atan(av)$

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for small v : $\phi'' + \left(\left(a^2\right)_u \phi\left(\phi' + 2\phi'^2\right) - a^2\right)\phi = j^2\left(1 + (\frac{\phi'}{j})^2\right)^{3/2}$
$\exists$ embedded $\kappa = 1$ curve in surface with $1 \geq K \geq 0$ with no embedded half-tube.

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$\exists$ embedded $\kappa = 1$ curve in surface with $1 \geq K \geq 0$ with no embedded half-tube.
But : K is only piecewise continuous

Remarks on Schauder estimates

A counterexample to Schauder estimates for elliptic op. with unbounded coeff.

[Priola01]

$$\begin{cases} \psi - \frac{1}{2}\psi_{,xx} - \frac{1}{2}\psi_{,yy} + y\psi_{,y} = |\cos(y)| & x > 0 \\ \psi(0, y) = 0, \quad y \in \mathbb{R}. \end{cases}$$

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But : there in some cases there are Schauder estimates with unbounded coefficients.

Total curvature entropy of cmc hypersurfaces

Recapitulation (stated for cmc in $\mathbb{R}^n$)

- Complete noncompact stable ends of positive cmc are of exponential volume growth $(nH^2 \leq \frac{\mu_M^2}{4})$.
- Properly embedded complete noncompact stable ends of bounded curvature are minimal $(nH^2 \leq \frac{\mu_{\mathcal{N}}^2}{4} = 0)$
- What are cmc with $\mu_{\mathcal{T}_p} = 0$?

Total curvature entropy cmc hypersurfaces

Caccioppoli's inequality of type III - $H \neq 0$ **[IliasNelliSoret12]**

Theorem

Let M be a complete cmc hypersurface immersed in a manifold of cnst curvature c. Assume M has finite index and $n \leq 5$. Then there exist a compact subset K in M and a constant γ such that, for any $\phi \in C_0^\infty(M \setminus K)$

$$\gamma \int_{M\setminus K} |A_0|^{2x}\phi^2 \leq \mathcal{D} \int_{M\setminus K} |A_0|^{2x}|\nabla\phi|^2$$

provided either (1) $c = 0$ or 1, $x \in [1, \frac{2\sqrt{n-1}}{n}\left(1 + \sqrt{1 - \frac{n-2}{2\sqrt{n-1}}}\right))$ or (2) $c = -1$, $\varepsilon > 0$, $x \in [1, x_2 - \varepsilon]$, $H^2 \geq g_n(x)$.

Total curvature entropy of cmc hypersurfaces

Proposition

Let $w \in L^1_{loc}(M;\mathbb{R}^+)$ and $W(R) := \int_{B_R} w$. Assume that the entropy of W is zero. If for some positive constant C, and compact $K \subset M$, w satisfies

$$\int_{M\setminus K} w\psi^2 \leq C \int_{M\setminus K} w|\nabla\psi|^2, \qquad \forall \psi \in W_0^{1,2}(M \setminus K),$$

then $\int w < \infty$

With the same hypothesis as in Caccioppoli inequality
$\int |A_0|^{2x} < \infty \implies |A_0| = 0$ [IliasNelliSoret12]

Total curvature entropy of cmc hypersurfaces

Theorem

There is no complete noncompact stable hypersurface M with constant positive mean curvature H in $\mathbb{R}^{n+1}$, $n \leq 5$, with $\mu_{\mathcal{T}_p} = 0$ for some $p \in [1, \frac{2\sqrt{n-1}}{n}\left(1 + \sqrt{1 - \frac{n-2}{2\sqrt{n-1}}}\right))$

Stable CMC *vs* Stable minimal

Let M be a stable minimal hypersurface of $\mathbb{R}^{n+1}$

- $|B(r)|_M \leq r^{5+\epsilon} \implies M$ is a hyperplane
- But is a prop. emb. minimal hypersurface with bounded curvature and $|B(r)|_M \sim r^{10}$ $(n = 8)$.
- There is no stable minimal hypersurface with bounded curvature and embedded tube in $\mathbb{R}^{n+1}$, $n \leq 5$

Question

Is a noncompact complete stable constant mean curvature hypersurface properly embedded in $\mathbb{R}^{n+1}$, for $7 \geq n \geq 3$, necessarily a hyperplane ?

reference2 reference1

References I

L. Barbosa, P. Bérard: *Eigenvalue and "twisted" eigenvalue problems, applications to cmc surfaces,* J. Math. Pure Appl. 79, 5 (2000) 427-450.

L. Barbosa, M. do Carmo: *Stability of hypersurfaces with constant mean curvature,* Math. Z., 185 (1984) 339-353.

L. Barbosa, M. do Carmo, J. Eschenburg: *Stablity of hypersurfaces with constant mean curvature in a Riemannian manifold,* Math. Z., 197 (1988) 123-128.

P. Bérard, M. do Carmo, W. Santos: *Complete hypersurfaces with constant mean curvature and finite total curvature*, Annals Glob. Anal. Geom. 16 (1998) 273-290.

References II

- V. G. Bondarenko: *Covariant derivatives of Jacobi fields on a manifold of nonpositive curvature* , Ukrainian Mathematical Journal., Volume 50, Number 6, 185 (1998) 857-869.
- R. Brooks: *Exponential Growth and the Spectrum of the Laplacian* Proceedings of the American Mathematical Society, Vol. 82, No. 3 (Jul., 1981) 473-477.

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References III

- R. Brooks: *On the spectrum of non-compact manifold with finite volume,* Math. Zeit. 187 (1984) 425-432.
- G. Carron: *Inégalités isopérimétriques et inégalités de Faber-Krahn,* Séminaire de géométrie spectrale et géométrie, tome 13 (1994-1995) 63-66.
- J. Cheeger: *A lower bound for the smallest eigenvalue of the Laplacian,* in Problems in analysis (ed. R.C. Gunning). A symposium in honor os Salomon Bochner (Princeton 1969), Princeton University Press (1970) 195-199.
- J. Cheeger, D. Ebin : *Comparison Theorems in Riemannian Geometry,*, American Elsevier, New-York, 1975

References IV

- X. Cheng: *On Constant Mean Curvature Hypersurfaces with Finite Index*, Arch. Math. 86 (2006) 365-374.
- A. da Silveira: *Stability of Complete Noncompact Surfaces with Constant Mean Curvature*, Math. Ann. 277 (1987) 629-638.
- M. do Carmo , *Hypersurfaces of constant mean curvature*,Lecture Notes in Mathematics, Vol 1410, p. 133.
- M. do Carmo, D. Zhou: *Eigenvalue estimate on complete noncompact Riemannian manifolds and applications*, Trans. Amer. Math. Soc. 351 (4) (1999) 1391-1401.

References V

- M. F. Elbert, B. Nelli, H. Rosenberg: *Stable constant mean curvature hypersurfaces*, Proc. Amer. Math. Soc. 135,10 (2007) 3359-3366.
- D. Fischer-Colbrie: *On complete minimal surfaces with finite Morse index in three manifolds*, Invent. Math. 82 (1985). 121-132
- D. Fischer-Colbrie, R. Schoen: *The structure of complete stable minimal surfaces in 3-manifolds of non negative scalar curvature*, Comm. Pure Appl. Math. 33 (1980) 199-211.

References VI

- K. Frensel: *Stable complete surfaces with constant mean curvature,* Bull. of the Brazilian Math. Soc.Volume 27, Number 2 (1996) 129-144.
- D. Gilbarg, N.S. Trudinger: *Elliptic Partial Differential Equations of Second order,* Springer Verlag, 1983.
- V. Gimeno, V. Palmer: *Volume growth of submanifolds and the Cheeger isoperimetric constant,* arXiv:1104.5625v2 [math.DG].
- A. Gray: *Tubes,* Addison-Wesley, 1990.

References VII

- E. Hebey: *Sobolev Spaces on riemannian manifolds*, Lecture Notes in Maths 1635.
- W.Y. Hsiang: *Generalized rotational hypersurfaces of constant mean curvature in the euclidean spaces*, J. Diff. Geom. 17 (1982) 337-356.
- W.Y. Hsiang: *On Generalization of theorems of A.D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature*, Duke Math. Journal 49 (3) (1982) 485-496.
- Y. Higuchi: *A remark on exponential growth and the spectrum of the Laplacian*, Kodai math. J. 24 (2001) 42-47.

References VIII

- S. Ilias, B. Nelli, M. Soret: *Caccioppoli's inequality on constant mean curvature hypersurfaces in Riemannian manifolds,* , arXiv:1107.3653v2 , to appear in Annals Glob. Anal. Geom.
- H. Karcher : *Manifolds of nonpositive curvature* MAA studies in mathematics v. 27 (1989) Studies in mathematics (Mathematical Association of America)
- L. Karp *Differnetial inequalities on complete Riemannian manifolds and applications,* Math. Ann. 272 (1985) 449-459.

References IX

- F. Lopez, A. Ros: *Complete minimal surfaces with index one and stable constant mean curvature surfaces,* Comment. Math. Helvetici 64 (1989) 34-43.
- E. L. Lima: *The Jordan-Brouwer Separation Theorem for Smooth Hypersurfaces* The American Mathematical Monthly, Vol. 95, No. 1 (Jan., 1988), pp. 39-42.
- B. P. Lima, J. F. Montenegro, N. L. Santos: *Eigenvalues Estimates for the p-Laplace Operator on Manifolds,* arXiv:0808.2028v1 [math.DG].
- W. F. Moss, J. Piepenbrink: *Positive solutions of elliptic equations* Pacific J. Math., 75, 1 (1978), 219-226.

References X

- W. Meeks III, J. Perez, A. Ros: *Stable constant mean curvature surfaces,* Handbook of Geometric Analysis n 1, International Press of Boston (2008).
- F. Morgan: *Geometric Measure Theory, a beginner's guide,* Academic Press (1988).
- M. Nakai, L. Sario: *Quasi-harmonic classification*, Proceedings of the AMS, 31, 1 (1972) 165-169.
- B. Nelli, M. Soret: *Stably embedded minimal hypersurfaces,* Math. Z. 255, 3 (2007) 493-514.

References XI

H. Okumura: *Infimum of the exponential volume growth and the bottom of the essential spectrum of the Laplacian,* arXiv:0707.0185v3 [math.DG].

B. Palmer: PhD thesis, Stanford (1986).

R. Reilly: *Variational properties of functions of the mean curvatures for hypersurfaces in space forms,* J. Diff. Geom. 8 (1973) 465-477.

R. Schoen, L. Simon, S.T. Yau: *Curvature Estimates for Minimal Hypersurfaces*, Acta Math. 134 (1975) 275-288.

References XII

- R. Schoen, S. T. Yau: *Lectures on Differential Geometry*, International Press (1994).
- M. Soret: *Minimal surfaces with bounded curvature*, Comm. in Geom. and Analysis 9 (2001) 921-950.
- J. Spruck: *Remarks on the stability of minimal hypersurfaces of* $\mathbb{R}^n$, Math. Zeit. 144 (1975) 169-174.
- Y. B. Shen, X. H. Zhu: *Rigidity of stable minimal hypersurfaces*, Math. Ann. 309 (1997) 107- 116.

References XIII

J. TYSK: *Finiteness of index and total scalar curvature for minimal hypersurfaces:* Proc. of the AMS, 105, 2, (1989) 429-435.

Entropies

Example

- $\mu_v = 0$

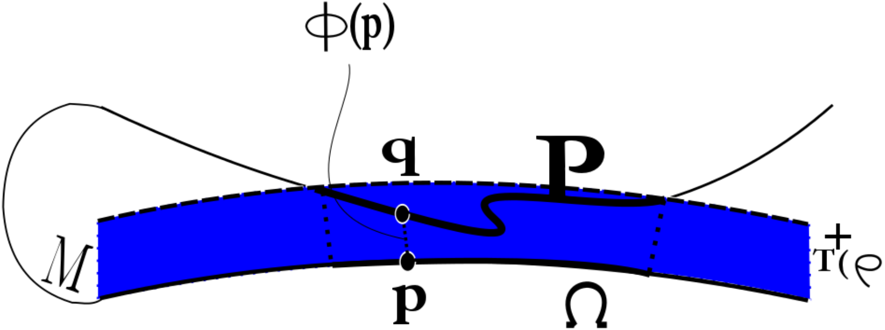
$$v(r) = \sum_{p \leq N} a_p r^p, v(r) = e^{r^\alpha}, \alpha < 1$$

- $\mu_w = \alpha$

$$w(r) = v(r) e^{\alpha r}$$

retour

$M \cap T^+(M) \neq \emptyset$



◂ 1 ◂ 2

The cheesebox property : Bounds on the first derivatives of curvature are necessary I

Consider the general case, where the ambient space $\mathcal{N}$ is not necessarily Euclidean. There exists a radius ρ_0 such that for each point $p \in \mathcal{N}$, the exponential chart is a diffeomorphism $f_p : U\,(:= B_{\mathbb{R}^{n+1}}\,(0, \rho_0)) \subset \mathbb{R}^{n+1} \longrightarrow V := f_p\,(U) \subset \mathcal{N}$. Furthermore, since $\mathcal{N}$ has bounded curvature and bounded first derivatives of curvature, the diffeomorphism f_p is a C^1-norm which is uniformly bounded with respect to p.

The cheesebox property : Bounds on the first derivatives of curvature are necessary II

The previous result of this paragraph concerning Euclidean cheeseboxes applies to $f_p^{-1}(M) \cap V$ and $f_p^{-1}(P) \cap V$ to prove the C^1-uniformly boundedness $\phi \circ f_p$ with respect to $p \in M$. Since f_p has a C^1-norm uniformly bounded with respect to p, this is also the case for ϕ. 1

Sketch of proof: barrier functions on B_R I

Construction of functions ψ_R such that $\Delta\psi_R \geq \varepsilon$ on $B_R \cap \Omega$.

Let r be the distance function from a fixed point of M, and consider the radial test function $\psi_R(x) = f_R \circ r(x)$ where

$$f_R(r) := \begin{cases} \beta\left(1 - \left(\frac{r}{R}\right)^2\right) & \forall\ r \leq R, \\ 0 & \forall\ r \geq R. \end{cases}$$

with $\beta = \rho - \delta$, for small positive δ. Notice that ψ_R vanishes on $\partial B_R \cap \Omega$ and $\psi_R \leq \phi$ on $B_R \cap \partial\Omega$, since $\phi|_{\partial\Omega} = \rho$. Therefore $\psi_R \leq \phi$ on $\partial(\Omega \cap B_R)$.

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Sketch of proof: barrier functions on B_R II

Construction of functions ψ_R such that $\Delta\psi_R \geq \varepsilon$ on $B_R \cap \Omega$.

M has bounded curvature $\Longrightarrow \exists k > 0 : Ric_M \geq -(n-1)k^2$.
Standard comparison theorems $\Longrightarrow \Delta r \leq \frac{n-1}{r}(1 + kr)$
Remark inequality holds outside the cut-locus of M and holds in the weak sense at any point of M.
Computation gives $\Delta\psi_R \geq -\frac{2\beta}{R^2}(n + (n-1)kR) \geq -\varepsilon$, for R large.
$\Delta\phi \leq \Delta\psi_R$ on $B_R \cap \Omega$, for R large. Then, by Corollary **??**, $\phi \geq \psi_R$ on $B_R \cap \Omega$, for R large. Letting $R \to \infty$ we obtain $\phi \geq \beta$ on Ω. Therefore $\phi \geq \rho - \delta$ in Ω for any $\delta > 0$. Thus $\phi \geq \rho$ in Ω.

Jacobi fields expansion I

$J_1 = T$ on γ. The Jacobi fields $\{J_i\}_{i\neq 1}$ are defined according to the evolution equation

$$J_i''(y) + R_M(y)(J_i, T)T(y) = 0, \quad i \neq 1 \quad \forall y \in \gamma \subset exp(D(p,\rho))$$

(where $J_i''(y) := \nabla_T \nabla_T J_i(y)$ and R_M is the curvature tensor of M), with the second order initial conditions at p :

$$\begin{cases} J_i(p) & = e_i \\ \nabla_T J_i(p) & = 0. \end{cases}$$

Jacobi fields expansion II

Since the curvature of M is bounded, the classical Rauch Comparison Theorem and equation (62) yield the following expansion of $J_i(x)$ in terms of the distance ρ:

$$J_i(x) = J_i(p) + O(\rho^2), \qquad x \in exp(D(p, \rho)),$$

$$g_{ij}(x) := g_M\left(J_i(x), J_j(x)\right) = \delta_{ij} + 0(\rho^2) \qquad x \in \gamma \subset exp\left(D\left(p, \rho\right)\right)$$

where $O(\rho^2)$ depends on bounds of the curvature tensor R_M of M.