# Sobre la clasificación de los espacios lorentzianos r-ésimo simétricos 

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## Introduction

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■ To classify the 2nd-symmetric Lorentzian manifolds, i.e.:

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- To provide properties and open questions on the $r$ th-symmetric case $\nabla^{r} R=0$ and, in general on the implications of

$$
\nabla^{r} T=0
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for any tensor field.

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- Penrose limit type constructions

■ "Super-energy" tensor

- Higher order Lagrangian theories, supergravity, vanishing of quantum fluctuations...


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\begin{aligned}
\nabla^{2} R(X, Y ; \ldots) & -\nabla^{2} R(Y, X ; \ldots)= \\
& =\nabla_{X}\left(\nabla_{Y} R\right)-\nabla_{Y}\left(\nabla_{X} R\right)-\nabla_{[X, Y]} R \\
& =: R(X, Y) \cdot R=0
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- Lorentzian and higher signatures: $\nabla^{r} R=0 \nRightarrow \nabla R=0$


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■ So, a ladder of conditions appear in the Lorentzian case: Locally symmetric $\subset \mathbf{2 n d}$-symmetric $\subset$ semi-symmetric How hadn't 2nd-symmetry been studied before?

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Main result to be proven:
Theorem (Blanco, Senovilla, - )
Let $(M, g)$ be a properly 2nd-symmetric Lorentzian n-manifold:

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Let $(M, g)$ be a properly 2nd-symmetric Lorentzian n-manifold:

- (Local classification). $(M, g)$ is locally isometric to a product
- a (non-flat) symmetric Riemannian space ( $N, g_{N}$ )
- a proper 2 nd-order Cahen-Wallach space $\left(\mathbb{R}^{d+2}, g_{A}\right)$, $g_{A}=-2 d u\left(d v+\left(\mathbf{a}_{\mathrm{ij}} \mathbf{u}+\mathbf{b}_{\mathrm{ij}}\right) x^{i} x^{j} d u\right)+\delta_{i j} d x^{i} d x^{j}$ with some $a_{i j} \neq 0$.


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- (Global classification). Moreover, if $(M, g)$ is 1-connected and geodesically complete, then it is globally isometric to $\left(\mathbb{R}^{d+2} \times N, g_{A} \oplus g_{N}\right)$.

Local symmetry vs. 2nd-symmetry

## Characterizations of local symmetry vs 2nd-symmetry

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## Local symmetry

## Proposition

For a (connected) semi-Riemannian manifold ( $N, h$ ), they are equivalent:
(i) $(N, h)$ is locally symmetric, i.e. $\nabla R=0$.
(ii) If $X, Y$ and $Z$ are parallel vector fields along a curve $\gamma$, then so is $R(X, Y) Z$.
(iii) The sectional curvature of non-degenerate planes is invariant under parallel transport
(iv) The local geodesic symmetry $s_{p}$ is an isometry at any $p \in N$.
(v) $(N, h)$ is locally isometric to a symmetric space.

## Characterizations of local symmetry vs 2nd-symmetry

## Remark

" $(N, h)$ is locally isometric to a symmetric space"
$\rightsquigarrow$ as a difference with the locally homogeneous case, as there exists even Riemannian non-regular ones (Kowalski'97)

## Characterizations of local symmetry vs 2nd-symmetry

## 2nd symmetry

## Lemma

For a semi-Riemannian ( $N, h$ ), they are equivalent:

- Skew symmetry of $\nabla^{2} R$ in the derivatives slots.
- For any non-degenerate tangent plane $\Pi_{p} \subset T_{p} N$, its parallel transport $\Pi_{\gamma}$ along any geodesic $\gamma$, the derivative of its sectional curvature $\frac{d}{d \tau}\left(K\left(\Pi_{\gamma}\right)\right)$ is a constant along $\gamma$.
- For any parallelly propagated vector fields $X, Y, Z$ along any geodesic $\gamma$, the vector field $\left(\nabla_{\gamma^{\prime}} R\right)(X, Y) Z$ is itself parallelly propagated along $\gamma$.


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## Proposition

For a semi-Riemannian ( $N, h$ ), they are equivalent:
(i) $(N, h)$ is 2nd-symmetric, $\nabla \nabla R=0$
(ii) $(N, h)$ is semi-symmetric $(R(X, Y) R=0)$ and satisfies any of the equivalent conditions to skew-symmetry in the lemma .
(iii) If $V, X, Y, Z$ are parallelly propagated vector fields along any curve, then so is $\left(\nabla_{V} R\right)(X, Y) Z$.

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## Remark

Characterizations in terms of an analog of the geodesic symmetry or local isometries to a model space are conspicuously absent.

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Proof. Use de Rham decomposition

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- $g_{R}$ es reducible: $g_{R}=g_{R}^{(1)} \oplus g_{R}^{(2)} \oplus \ldots \oplus g_{R}^{(s)}$.
- $L=\sum_{m=1}^{s} \lambda_{m} g_{R}^{(m)}$ for some $\lambda_{m} \in \mathbb{R}$.


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-and locally homogeneous with Ric $\leq 0$ are regular (Spiro '93)

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Lorentzian symmetric spaces
Theorem (Cahen, Wallach '70)
A complete 1-connected Lorentzian symmetric space $(M, g)$ is isometric to the product of a simply-connected Riemannian symmetric space and one of the following Lorentzian manifolds:

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## Lorentzian symmetric spaces

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A complete 1-connected Lorentzian symmetric space $(M, g)$ is isometric to the product of a simply-connected Riemannian symmetric space and one of the following Lorentzian manifolds:
$1\left(\mathbb{R},-d t^{2}\right)$
2 The universal cover of de Sitter or anti-de Sitter d-spaces, $d \geq 2$,
3 A Cahen-Wallach space $C^{d}(A)=\left(\mathbb{R}^{d}, g_{A}\right), d \geq 2$, where $A \equiv\left(A_{i j}\right)$ is a $(d-2) \times(d-2)$ matrix and $g_{A}=-2 d u\left(d v+A_{i j} x^{i} x^{j} d u\right)+\sum_{i j} \delta_{i j} d x^{i} d x^{j}$

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## Remark

Choosing $A$ with $\operatorname{trace}(A)=0$ : there are Ricci flat non-flat Lorentzian symmetric spaces.

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## Remark

Lorentzian symmetric space with a parallel lightlike v.f. $K \Rightarrow$ : Locally isometric to the product of a $C W^{d}(A), d>2$ and Riemannian symmetric space.

## Classification locally symmetric vs 2 nd-symmetric

## 2nd-symmetric:

The theorem to be proven shows:
proper 2nd-symmetric spaces only appear generalizing the family of Cahen-Wallach spaces $\operatorname{CW}^{d}(A), d>2$ :

■ $\rightsquigarrow$ allow an affine dependence of the matrix $A$ in $u$

## Generalization of Cahen-Wallach family

Generalized Cahen-Wallach $d$-space of order $r$, $C W_{r}^{d}(A)=\left(\mathbb{R}^{d}, g_{A}\right), d \geq 2$ : metric:

$$
g_{A}=-2 d u\left(d v+\sum_{i j} A_{i j}(u) x^{i} x^{j} d u\right)+\sum_{i j} \delta_{i j} d x^{i} d x^{j}
$$

where $A \equiv\left(A_{i j}(u)\right)$ is a $(d-2) \times(d-2)$ matrix:

$$
A_{i j}(u)=A_{i j}^{(r-1)} u^{r-1}+\cdots+A_{i j}^{(1)} u+A_{i j}^{0}
$$

for symmetric (constant) matrixes $A_{i j}^{k}$.

## Generalization of Cahen-Wallach family

## Proposition

Any generalized Cahen-Wallach space $C W_{r}^{d}(A)$ satisfies:
1 If $A_{i j}^{(r-1)} \neq 0\left(C W_{r}^{d}(A)\right.$ is proper) then it is proper rth-symmetric

1. Direct computation: in an appropriate basis
$\left\{E_{\alpha}\right\}=\left\{E_{0}=\partial_{u}-\sum A_{i j} x^{i} x^{j} \partial_{v}, E_{1}=\partial_{v}, \partial_{i}\right\}$ the only non-vanishing components of $\nabla^{\prime} R, I \in\{0, \ldots r-1\}$ are:
$\nabla_{0} \stackrel{(I)}{!} . \nabla_{0} R_{i 0 j}^{1}=\frac{d^{\prime} A_{i j}}{d u}=\sum_{k=1}^{r-1} \frac{k!}{(k-l)!} A_{i j}^{(k)} u^{k-I} \square$

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$2 K=\partial_{v}$ is a lightlike parallel vector field
3 It is analytic
4 it is geodesically complete
Proof. 2,3: Trivial

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Proof. 2,3: Trivial
4. Direct computation or general results (Candela, Romero, - '13)

## Generalization of Cahen-Wallach family

## Corollary

A complete 1-connected Lorentzian manifold locally isometric to some $C W_{r}^{d}(A)$ is globally isometric too.

This will allow to go from the local to the global result.

## Must rth-symmetry imply local symmetry ?

This is a particular case of:
■ When $\nabla^{r} T=0 \Rightarrow \nabla T=0$ ?

## Riemannian case

## Theorem

Let $(M, g)$ be Riemannian and $T$ a tensor field such that $\nabla^{r} T=0$. Then $\nabla T=0$ if either
(a) (Nomizu-Ozeki '62) $g$ is complete and irreducible, or
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## Remark

In particular, from (b), Riemmannian $r$-th symmetric implies locally symmetric.

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2. Put $f:=g(T, T) / 2$. Using $\nabla^{2} T=0$ :
$\operatorname{Hess} f(X, Y)=g\left(\nabla_{X} T, \nabla_{Y} T\right) \quad$ and $\quad \nabla$ Hess $f=0$

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3. By Eisenhart thm: Hess $f=c g, c \in \mathbb{R}$. Thus $Z:=\operatorname{grad}(f)$
satisfies $\nabla_{X} Z=c X$ (in particular, is homothetic)
4. Under irreducibility + completeness homothetic vectors are Killing: $c=0 g\left(\nabla_{X} T, \nabla_{Y} T\right)=0$. As $g$ is Riemannian $\nabla T_{\equiv}=0$.

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Proof (b) 1. Irreducibility can be assumed: $T=0$ on the flat part of (local) de Rham decomposition (as well as on mixed elements)
2. As before, one has $\nabla_{X} Z=c X$ and needs $c=0$.
3. As $Z$ is homothetic, it is affine. Thus $L_{z} \nabla=0=L_{Z} T$ and:

$$
0=L_{Z} \nabla T=\nabla_{Z}(\nabla T)+(s+1) c \nabla T=(s+1) c \nabla T
$$

( $s$ : covar minus contrav slots for $T$ ). That is, if $c \neq 0$ directly $\nabla T=0 . \square$

Local symmetry vs. 2nd-symmetry

Sketch of proof

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- The manifold is incomplete with a proper (non-Killing) homothetic vector field (necessarily without zeroes)

In the latter case the metric can be written locally as a cone: $M=I \times S, I \subset(0, \infty),\left(S, g_{S}\right)$ Riemannian

$$
g=d t^{2}+t^{2} \pi_{S}^{*} g_{S}
$$

being $Z=t \partial_{t}$ proper homothetic. In particular:

$$
\nabla Z=2 \cdot \operatorname{ld}(\neq 0) \quad \nabla^{2} Z=0
$$

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## Difficulties for the semi-Riemannian extension

1 The (full, local) de Rham decomposition cannot be carried out when the subspaces invariant by local holonomy are degenerate
2 The conclusion $c=0$ only means $g(T, T)$ constant and $g(\nabla T, \nabla T)=0$ i.e. $\nabla T$ is a lightlike tensor
3 Even in the non-degenerate irreducible case, to apply Eisenhart one needs : if the restricted homogeneous holonomy group is irreducible and a symm. 2-cov tensor $h$ is invariant by the group, then $h=c g$ for some function $c$, which is constant if $h$ is parallel
However, this holds in Lorentzian signature and others (Tanno'67, $n=2$ or non-neutral signature)

## Further properties: $\nabla^{r} T=0$ in generic points

## Definition

A point $p$ is generic if the curvature endomorphism:

$$
R: \Lambda^{2}(M) \rightarrow \Lambda^{2}(M) \quad v^{b} \wedge w^{b} \mapsto 2 R(v, w)
$$

is an isomorphism when restricted to $p$.

## Theorem <br> If there exists a generic point, $\nabla^{r} T=0$ implies $\nabla T=0$, for any semi-Riemannian metric.

## $\nabla^{r} T=0$ in generic points

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Proofs of increasing generality:
1 Simply, no conic metric (nor flat one) is generic.

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Proofs of increasing generality:
1 Simply, no conic metric (nor flat one) is generic. Remarks

- Valid only for the Riemannian case
- Extensible to generic (non-degenerate) Ric, as $\operatorname{Ric}\left(\partial_{t}, X\right)=0$ in the conic metric


## $\nabla^{r} T=0$ in generic points

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Proofs of increasing generality:
2 (Tanno '72) As we had $Z$ with $\nabla_{X} Z=c X$ : $0=L_{Z} \nabla=\nabla^{2} Z+R(Z, \cdot)=R(Z, \cdot)$ So $R$ is not invertible except if $Z=0$.

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So $R$ is not invertible except if $Z=0$.
Remarks:

- Also valid for Riemannian and extensible to generic Ric

■ For Lorentz and non-neutral sign. + irreducibility, it applies, but then implies only $g(\nabla T, \nabla T)=0$ and $g(T, T)=$ const.

## $\nabla^{r} T=0$ in generic points

## Theorem

(Senovilla '08) If there exists a generic point, $\nabla^{r} T=0$ implies $\nabla T=0$ on all $M$, for any semi-Riemannian metric.

Proofs of increasing generality:
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Proofs of increasing generality:
3 (Senovilla '08) Apply the Ricci identities to $T$ and $\nabla T$ : The invertibility of $R$ allows to clear $\nabla T=0$.
Remarks:

- Independent of both, signature or previous computations
- Extensible to: all semi-symmetric spaces have constant curvature around generic points


## Limits of old techniques

A computation in the spirit of old papers:

## Proposition

Let $(M, g)$ be semi-Riemannian and $r$-th symmetric. If there exists a vector field $Z$ :

$$
\nabla_{X} Z=c X \quad c \in \mathbb{R} \quad \forall X \in \mathfrak{X}(M)
$$

then either $Z$ is parallel or $R=0$.

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$$

then either $Z$ is parallel or $R=0$.
Proof. As $Z$ is homothetic, $L_{z} \nabla=0, L_{z} \nabla^{k} R_{i j k}^{\prime}=0$ and:
$0=L_{Z}\left(\nabla^{r-1} R\right)=\nabla_{Z}\left(\nabla^{r-1} R\right)+(1+r) c \nabla^{r-1} R=(1+r) c \nabla^{r-1} R$
So, if $c \neq 0$, use induction. $\square$

## Limits of old techniques

## Corollary

A proper rth-symmetric Lorentzian $(M, g)$ either admits a parallel lightlike direction or satisfies that $\nabla^{r-1} R$ is (parallel and) null and $g\left(\nabla^{r-2} R, \nabla^{r-2} R\right)$ is a constant.

Proof. The first possibility occurs either when degenerately reducible or when admits a lightlike parallel v.f.

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Proof. The first possibility occurs either when degenerately reducible or when admits a lightlike parallel v.f.
Otherwise, in each irreducible part, put again $T=\nabla^{r-2} R$, $f=g(T, T)$, $\operatorname{Hess} f(X, Y)=g\left(\nabla_{X} T, \nabla_{Y} T\right)$ and $Z=\operatorname{grad} f$ By previous Prop., necessarily $Z \equiv 0$. $\square$

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## Remark

Limit of "old" results: this suggests that at least 2nd-symmetric Lorentzian spaces must admit a parallel lightlike v.f. K.

## Existence of a lightlike parallel vector field

## Theorem

(Senovilla '08). Any proper 2nd-symmetric Lorentzian space admits a unique lightlike parallel vector field K.
(Alternative proof by Aleeksevski \& Galaev, '11.)

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- Previous result for $\exists$ parallel light. vector, not only a line: $\exists$ Parallel $L \neq c g$ plus no decomposable (non-degenerately reducible) $\Rightarrow \exists$ ! independent parallel lightlike vector $K$. (proof by discussing possible Segre types )


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Uniqueness: a linear combination of $K_{1} \pm K_{2}$ would be (parallel and) timelike in contradiction with no-decompsability/properness.

## Existence of a lightlike parallel vector field

## Theorem

(Senovilla '08). Any proper 2nd-symmetric Lorentzian space admits a unique independent lightlike parallel vector field K.

- Analyze the curvature concomitants showing that, either such a $K$ exists, or they vanish:
(a) 1-form concomitants of order $m$ and degree up to $m+1$
(b) scalar or 2-cov. concomitants of equal order and degree.

■ Using Ricci identity, such restrictions force the existence of $K$

## Brinkmann spaces

## Definition

A Brinkmann space is any Lorentzian n-manifold endowed with a complete lightlike parallel vector field $K$.

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Brinkmann decomposition $\{u, v\}$ :
$1 K$ parallel: fix $u$ (up to a constant) s.t.: $K=\operatorname{grad} u$
$2 K$ lightlike: $K^{\perp}$ degenerate totally geodesic integrable foliation with leaves $\Sigma_{u}$

3 Choose a hypersup. $\Omega$ transverse to $K$ so that $\bar{M}:=\Sigma_{u=0} \cap \Omega$ is spacelike and transverse
4 Let $\phi$ the flow of $K$, define $v$ so that $\phi_{-v(p)}(p) \in \Omega$

## Construction of the Brinkmann decomposition



## Construction of a Brinkmann chart

■ Brinkmann chart $\left\{u, v, x^{i}\right\}$ : complete $u, v$ to a chart by choosing $n-2$ coordinates $x^{i}$ independent of $u$ in $\Omega$.

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- Expression of $g$ in a Brinkmann chart:

$$
g=-2 d u\left(d v+H\left(u, x^{k}\right) d u+W_{i}\left(u, x^{k}\right) d x^{i}\right)+g_{i j}\left(u, x^{k}\right) d x^{i} d x^{j}
$$

(natural sum in repeated indexes, $K \equiv-\partial_{v}$ )

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## Remark

Being more careful, one could get $H=0$ and $W_{i}=0$ !
But it is preferred as above, as we wish to remove the $u$-dependence of $g_{i j}\left(u, x^{i}\right)$.

## Geometric developments

- In general:

Study of degenerate hypersurfaces
$\rightsquigarrow$ Transverse vector field $\xi$
Non-unique $\xi$ : wise choice when possible.

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■ In general:
Study of degenerate hypersurfaces
$\rightsquigarrow$ Transverse vector field $\xi$
Non-unique $\xi$ : wise choice when possible.

- This happens in Brinkmann spaces too:
degenerate hypersurfaces $\Sigma_{u}$ with transverse $\partial_{u}$ (non-univocally determined)
- Issues on Brinkmann spaces:
- Relations between different choices of $\partial_{u}$ (and $\Omega$ )
- To introduce associated geometric objects with nice properties
- Study potentially extensible to other degenerate cases


## Geometric developments

## - Foliations

1 Spacelike ( $n-2$ )-foliation $\mathcal{M}$ : $\left\{u=u_{0}, v=v_{0}\right\}$
2 Timelike 2 foliation: $\mathcal{U}:\left\{x^{i}=x_{0}^{i}\right\}$

## Geometric developments

- Foliations

1 Spacelike ( $n-2$ )-foliation $\mathcal{M}:\left\{u=u_{0}, v=v_{0}\right\}$
2 Timelike 2 foliation: $\mathcal{U}:\left\{x^{i}=x_{0}^{i}\right\}$

- Tangent bundle decompositions:

1 Non-orthogonal: $T M=T \mathcal{M} \oplus T \mathcal{U}$
2 Orthogonal: $T M=T \mathcal{U} \oplus(T \mathcal{U})^{\perp}$

- Natural bases:
$1 \mathrm{TU}=\operatorname{span}\left\{E_{0}:=\partial_{u}-H \partial_{v}, E_{1}:=\partial_{\nu}\right\}$
$2(T \mathcal{U})^{\perp}=\operatorname{span}\left\{E_{i}:=\partial_{i}-W_{i} \partial_{\nu}\right\}$
$3 T \mathcal{M}=\operatorname{span}\left\{\partial_{i}\right\}$


## The spacelike foliation $\mathcal{M}$

Foliation $\mathcal{M}:\left\{u=u_{0}, v=v_{0}\right\}$
Metric induced bundle by the foliation:

$$
\bar{g}=g_{i j} \overline{d x^{i}} \overline{d x^{j}}
$$

(Notation: if $d x^{i}, \alpha$ on $M$, then $\overline{d x^{i}}, \bar{\alpha}$ on the foliation)

## Exterior derivative $\bar{d}$

For any 1-form $\alpha$ on $M$ :

$$
\bar{d} \bar{\alpha}=\overline{d \alpha}
$$

Satisfies the properties of a derivation for $\omega, \tau \in \Lambda^{q} \mathcal{M}$ :
1 Linearity plus $\bar{d}(\omega \wedge \tau)=\bar{d} \omega \wedge \tau+(-1)^{s} \omega \wedge \bar{d} \tau$.
$2 \bar{d}(\bar{d} \omega)=0$.
3 If $\omega=\frac{1}{s!} \omega_{i_{1} \ldots i_{s}} \overline{\bar{d}} x^{i_{1}} \wedge \ldots \overline{\bar{d}} x^{i_{s}}$, then $\bar{d} \omega=\frac{1}{s!} \partial_{k}\left(\omega_{i_{1} \ldots i_{s}}\right) \bar{d} x^{k} \wedge \bar{d} x^{i_{1}} \wedge \ldots \bar{d} x^{i_{s}}$
4 Poincaré Lemma: $\bar{d}$-closed implies $\bar{d}$-exact.

## Covariant derivative $\bar{\nabla}$ for $\mathcal{M}$

- Vector fields on $\mathcal{M}$ are naturally on $M$
- $\mathcal{M}$ is endowed with a Riemannian metric and then a natural $\bar{\nabla}$

$$
\bar{\nabla}_{X} Y(\in \mathfrak{X}(\mathcal{M})) \quad \forall X, Y \in \mathfrak{X}(\mathcal{M})
$$

Extended to tensor fields on $\mathcal{M}$ satisfies

$$
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Defines a foliation curvature $\overline{\mathcal{R}}$ :
$\overline{\mathcal{R}}(X, Y) Z=\left(\bar{\nabla}_{X} \bar{\nabla}_{Y}-\bar{\nabla}_{Y} \bar{\nabla}_{X}-\bar{\nabla}_{[X, Y]}\right) Z \in \mathfrak{X}(\mathcal{M}), \forall X, Y, Z \in \mathfrak{X}(\mathcal{M})$ plus Ricci tensor $\overline{\mathcal{R} i c}$ and scalar curvature $\overline{\mathcal{S}}$.

## Covariant derivative $\bar{\nabla}$ for $\mathcal{M}$

## Definition

- $\mathcal{M}$ is flat (resp. locally symmetric) if $\overline{\mathcal{R}}=0$ (resp. $\bar{\nabla} \overline{\mathcal{R}}=0)$

■ $u$-Einstein if $\overline{\mathcal{R i c}}=\mu \overline{\mathrm{g}}$ for some $\mu$ s.t. $d \mu \wedge d u=0$ (Schur lemma Ric $=f g \Rightarrow f \equiv c$ does not apply to foliations) and:
$1 \mathcal{M}$ is Einstein if $\mu=$ const.,
$2 \mathcal{M}$ is Ricci-flat if $\mu \equiv 0$.

## Covariant derivative $\bar{\nabla}$ for $\mathcal{M}$

From Riemannian results:

## Proposition

Let $(M, g)$ be a Brinkmann space:
$1 \bar{\nabla}^{r} \overline{\mathcal{R}}=0$ (rth-symmetric) $\Longrightarrow \bar{\nabla} \overline{\mathcal{R}}=0$ (locally symmetric).
$2 \bar{\nabla} \overline{\mathcal{R}}=0$ (locally symmetric) and $\overline{\mathcal{R i c}}=0$ (Ricci-flat) $\Longrightarrow \overline{\mathcal{R}}=0$ (flat)
3 If $\mathcal{M}$ is flat, there exists a chart $\left\{u, v, y^{i}\right\}$ s.t.: $g=-2 d u\left(d v+H d u+W_{i} d y^{i}\right)+\delta_{i j} d y^{i} d y^{j}$. ( $g_{i j}=\delta_{i j}$ independent of $u$ )

## Transverse operators for $\mathcal{M}$ : dot derivative

For $T \in \Gamma\left(T_{s}^{r} \mathcal{M}\right)$ :

$$
\dot{T}=\overline{\mathcal{L}_{\partial_{u}} \stackrel{\circ}{T}} \in \Gamma\left(T_{s}^{r} \mathcal{M}\right)
$$

That is, in the base $\left\{\partial_{i}\right\}$ :

$$
\dot{T}_{j_{1} \ldots j_{s}}^{i_{1} \ldots i_{r}}=\partial_{u}\left(T_{j_{1} \ldots j_{s}}^{i_{1} \ldots i_{r}}\right)
$$

## Transverse operators for $\mathcal{M}: D_{0}$ derivative

Recall $E_{0}=\partial_{u}-H \partial_{v}$

$$
\begin{array}{ccc}
D_{0}: \Gamma\left(T_{s}^{r} \mathcal{M}\right) & \longrightarrow & \Gamma\left(T_{s}^{r} \mathcal{M}\right) \\
T & \rightarrow & D_{0} T=\overline{\left(\nabla_{E_{0}} \stackrel{\circ}{T}\right)}
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\end{array}
$$

Properties:
1 Algebraic properties of a tensor derivation
$2 D_{0} \bar{g}=0$

## Lemma

Each vector field on a leave of $\mathcal{M}$ can be extended to a unique $K\left(=-\partial_{v}\right)$-invariant $D_{0}$-parallel vector field in $\mathfrak{X}(\mathcal{M})$.

## Reducibility in $\mathcal{M}$

$T \in \Gamma\left(T_{s}^{k} \mathcal{M}\right)$ is reducible if, there are foliations $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ s.t., in a natural sense:

$$
T \mathcal{M}=T \mathcal{M}^{(1)} \oplus T \mathcal{M}^{(2)} \quad T=T^{(1)} \oplus T^{(2)}
$$

i.e. there exists a Brinkmann chart $\left\{u, v, x^{i}\right\}$ and a partition of the indexes $I_{1}=\{2, \ldots, d+1\}, I_{2}=\{d+2, \ldots, n-1\}$ s.t.

$$
T_{a a^{\prime}}=0 \quad \text { y } \quad \partial_{a^{\prime}} T_{a b}=0,
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where $a, b$ belong to some $I_{m}$ and $a^{\prime}, b^{\prime}$ to the other one.

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where $a, b$ belong to some $I_{m}$ and $a^{\prime}, b^{\prime}$ to the other one. In particular, when $\bar{g} \in \Gamma\left(T_{2} \mathcal{M}\right)$ is reducible the sum is orthogonal and we write $\mathcal{M}=\mathcal{M}^{(1)} \times \mathcal{M}^{(2)}$,

$$
g=-2 d u(d v+H d u+\dot{W})+\stackrel{\circ}{g}^{(1)} \oplus \stackrel{\circ}{g}^{(2)}
$$

## Extended Eisenhart theorem

## Theorem

Let $(M, g)$ be a Brinkmann space and $\left\{u, v, x^{i}\right\}$ a Brinkmann chart. If there exist a symmetric $\bar{L} \in \Gamma\left(T_{2}^{0} \mathcal{M}\right), \bar{L} \neq c \bar{g}$, which is v-invariant, $\bar{\nabla}$-parallel and $D_{0}$-parallel.

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Then there exists a Brinkmann chart $\left\{u, v, y^{i}\right\}$ in the Brinkmann decomposition $\{u, v\}$ such that:
$1 \bar{g}$ is reducible: $\bar{g}=\bar{g}^{(1)} \oplus \ldots \oplus \bar{g}^{(s)}, s \geq 2$ (u-dependent)
$2 \bar{L}=\sum_{m=1}^{s} \lambda_{m} \bar{g}^{(m)}$ for some $\lambda_{m} \in \mathbb{R}$ (u-independent, $\dot{\lambda}_{m}=0$ ).

## Local version of the theorem

Aim:

## Theorem

A properly 2nd-symmetric Brinkmann space is locally isometric to a product of:

- a proper 2 nd-order Cahen-Wallach space $\left(\mathbb{R}^{d+2}, g_{A}\right)$, $g_{A}=-2 d u\left(d v+\left(\mathbf{a}_{\mathrm{ij}} \mathbf{u}+\mathbf{b}_{\mathrm{ij}}\right) x^{i} x^{j} d u\right)+\delta_{i j} d x^{i} d x^{j}$ with some $a_{i j} \neq 0$, and
- symmetric Riemannian space ( $N, g_{N}$ ).


## Step 1: define appropriate elements on $\mathcal{M}$

Express the non-trivial parts of $R, \nabla R$ in terms of tensors on $\mathcal{M}$

- Tensors for $R: A \in T_{2} \mathcal{M}, B \in T_{3} \mathcal{M}, \bar{R} \in T_{3}^{1} \mathcal{M}$

$$
\begin{aligned}
& A(X, Y)=\theta^{1}\left(R\left(E_{0}, \dot{Y}\right) \dot{X}\right) \text {, i.e. } A_{i j}=R^{1}{ }_{i 0 j} \\
& \square B(X, Y, Z)=\theta^{1}(R(\dot{Y}, \dot{Z}) \dot{X}) \text {, i.e., } B_{i j k}=R^{1}{ }_{i j k} \\
& \text { } \bar{R}(X, Y) Z=\bar{R}(\dot{X}, \dot{Y}) \dot{Z}, \text { i.e., } \bar{R}^{i}{ }_{j k l}=R^{i}{ }_{j k l}
\end{aligned}
$$

- Tensors for $\nabla R: \widetilde{A} \in T_{2} \mathcal{M}, \widehat{A}, \widetilde{B} \in T_{3} \mathcal{M}, \widehat{B}, \widetilde{R} \in T_{3}^{1} \mathcal{M}$

$$
\begin{array}{cl}
\widetilde{A}(X, Y)=\theta^{1}\left(\left(\nabla_{E_{0}} R\right)\left(E_{0}, \stackrel{\circ}{Y}\right) \dot{X}\right), & \widehat{A}(X, Y, Z)=\theta^{1}\left(\left(\nabla_{\dot{X}} R\right)\left(E_{0}, \dot{Z}\right) \dot{Y}\right), \\
\widetilde{B}(X, Y, Z)=\theta^{1}\left(\left(\nabla_{E_{0}} R\right)(\stackrel{\circ}{Y}, \dot{Z}) \dot{X}\right), & \widehat{B}(X, Y, Z, V)=\theta^{1}\left(\left(\nabla_{\dot{X}} R\right)(\dot{Z}, \dot{V}) \dot{\mathscr{Y}}\right), \\
\widetilde{R}(X, Y) Z=\overline{\nabla_{E_{0}} R(\dot{X}, \stackrel{\circ}{Y}) \dot{Z}} . \\
\widetilde{\widetilde{A}_{i j}=\nabla_{0} R^{1}{ }_{i 0 j} ; \widehat{A}_{s i j}=\nabla_{s} R^{1}{ }_{i 0 j}} \\
\widetilde{B}_{i j k}=\nabla_{0} R^{1}{ }_{i j k} ; \widehat{B}_{s i j k}=\nabla_{s} R^{1}{ }_{i j k} ; \widetilde{R}^{i}{ }_{j k l}=\nabla_{0} R^{i}{ }_{j k l}
\end{array}
$$

## Step 2: simplification of chart-dependent elements

## Proposition

For any 2nd-symmetric Brinkmann decomposition $\{u, v\}$ :
(a) All the (chart-dependent) elements for $\nabla R$ vanish but $\tilde{A}$, i.e.

$$
\widehat{B}=\widetilde{R}=\widehat{A}=\widetilde{B}=0
$$

(b) $\widetilde{A}$ is independent of the chosen chart
(c) The equations of 2 nd symmetry reduce to:

$$
\begin{array}{ll}
\bar{\nabla} \widetilde{A}=0, & D_{0} \widetilde{A}=0 \\
\bar{\nabla} \bar{R}=0, & D_{0} \bar{R}=0
\end{array}
$$

with $\widehat{B}=0, \widetilde{B}=0, \widehat{A}=0$.

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Ingredients of the proof. A first simplification comes from $\bar{\nabla}^{2} R=0 \Rightarrow \bar{\nabla} R=0$.

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■ Use the conditions of integrability of 2nd symmetry equations

$$
\begin{aligned}
& \left(\bar{\nabla}_{k} D_{0}-D_{0} \bar{\nabla}_{k}\right) F^{i}{ }_{j}=\left(H_{, k}\right)\left(\partial_{v} F^{i}{ }_{j}\right)+F^{i}{ }_{m} B_{k j}{ }^{m}-F^{m}{ }_{j} B_{k m}{ }^{i}-t^{m}{ }_{k} \bar{\nabla}_{m} F^{i}{ }_{j} \\
& \left(\bar{\nabla}_{n} \bar{\nabla}_{m}-\bar{\nabla}_{m} \bar{\nabla}_{n}\right) T_{j_{1} \ldots j_{s}}^{i_{1} \ldots j_{k}}=\sum_{b=1}^{s} \bar{R}^{\prime}{ }_{j_{b} n m} T_{j_{1} \ldots j_{b-1} j_{j+1} \ldots i_{k}}^{i_{s}}-\sum_{a=1}^{k} \bar{R}^{i_{a}}{ }_{l n m} T_{j_{1} \ldots j_{s}}^{i_{1} \ldots i_{a}-1 i_{a+1} \ldots i_{k}}
\end{aligned}
$$

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& \left(\bar{\nabla}_{n} \bar{\nabla}_{m}-\bar{\nabla}_{m} \bar{\nabla}_{n}\right) T_{j_{1} \ldots j_{s}}^{i_{1} \ldots i_{k}}=\sum_{b=1}^{s} \bar{R}^{\prime}{ }_{j_{b} n m} T_{j_{1} \ldots j_{b-1} j_{b+1} \ldots j_{s}}^{i_{1} \ldots i_{k}}-\sum_{a=1}^{k} \bar{R}^{i_{a}}{ }_{{ }^{\prime} m m} T_{j_{1} \ldots j_{s}}^{i_{1} \ldots i_{a-1} i_{a+1} \ldots i_{k}}
\end{aligned}
$$

- Use the equations derived from 2nd Bianchi identity

$$
\nabla_{[\alpha} R_{\beta \lambda] \nu \mu}=0 \Longrightarrow \widetilde{R}_{i j k l}=-2 \widehat{B}_{[i j] k l}, \quad \widetilde{B}_{k i j}=2 \widehat{A}_{[i j] k}
$$

Technical point: algebraic criteria for the vanishing of tensor fields are also introduced, as:
In an Euclidean vector space, $T_{i j k}$ vanishes if
$T_{i[j k]}=T_{i j k}, T_{i j k}+T_{j k i}+T_{k i j}=0$ and $T_{(i j)}^{r} T_{r n m}=0$

## Step 2: simplification of chart-dependent elements

## Remark

- $\nabla R \neq 0$ iff $\widetilde{A} \neq 0$.
- The scalar curvature $S$ (not only of $\mathcal{M}$ but also ) of $M$ is constant.


## Step 3: Reducibility of $\tilde{A}$ and $\overline{\operatorname{Ric}}$

From the equations of 2nd-symmetry:

$$
\begin{array}{ll}
\bar{\nabla} \widetilde{A}=0, & D_{0} \widetilde{A}=0 \\
\bar{\nabla} \bar{R}=0, & D_{0} \bar{R}=0
\end{array}
$$

$\widetilde{A}$ and $\overline{\operatorname{Ric}}$ (and also $\bar{g}$ ) are $D_{0}$ - $\bar{\nabla}$-invariant so that Extended Eisenhart theorem applies and:

## Step 3: Reducibility of $\tilde{A}$ and $\overline{\text { Ric }}$

- $\mathcal{M}=\mathcal{M}^{(1)} \times \mathcal{M}^{(2)}$ with $\mathcal{M}^{(1)}$ flat and $\mathcal{M}^{(2)}$ locally symmetric non Ricci-flat.


## Step 3: Reducibility of $\tilde{A}$ and $\overline{\text { Ric }}$

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## Remark

For any Brinkmann decomposition $\{u, v\}$ :

- $\widetilde{A}, \overline{\mathrm{Ric}}$ and $\bar{g}$ are simultaneously reducible


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## Remark

For any Brinkmann decomposition $\{u, v\}$ :

- $\widetilde{A}, \overline{\text { Ric }}$ and $\bar{g}$ are simultaneously reducible
- The non-trivial part of $\widetilde{A}$ lies in $\mathcal{M}^{(1)}$ and the non-trivial one of Ricci on $\mathcal{M}^{(2)}$


## Step 4: reduction to two independent Lorentzian problems

From previous result in a Brinkmann chart:

$$
g=-2 d u(d v+H d u+W ْ W)+\dot{\bar{g}}^{(1)} \oplus \stackrel{\circ}{g}^{(2)}
$$

and one can check that $H, W$ are also simultaneously reducible, so that in some new chart:

$$
g=-2 d u\left(d v+\left(H^{(1)}+H^{(2)}\right) d u+\grave{W}^{(1)}+\dot{W}^{(2)}\right)+\stackrel{\circ}{g}^{(1)} \oplus \stackrel{\circ}{\bar{g}}(2)
$$

## Step 4: reduction to two independent Lorentzian problems

Now, define two lower dimensional Lorentzian spaces
$M^{[m]}=\mathbb{R}^{2} \times \bar{M}^{(m)}, m=1,2:$

$$
g^{[m]}=-2 d u\left(d v+H^{(m)} d u+W^{(m)}\right)+\stackrel{\circ}{g}^{(m)}
$$

## Remark

- These two Lorentzian spaces are 2 nd symmetric as so was the original one.
■ So, the problem is reduced to the 2nd symmetry of two simple spaces


## Step 4: reduction to two independent Lorentzian problems

- $\left(M^{[2]}, g{ }^{[2]}\right) 2$ nd symmetric with $\widetilde{A}^{[2]}=0$ :
- Locally symmetric
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- ( $\left.M^{[1]}, g^{[1]}\right) 2$ nd-symmetric with flat $\mathcal{M}^{[1]}\left(\widetilde{A}^{[1]} \neq 0\right)$ : 2nd-symmetric plane wave: directly computable obtaining a generalized Cahen-Wallach of orden 2 :

$$
g_{A}=-2 d u\left(d v+\left(a_{i j} u+b_{i j}\right) x^{i} x^{j} d u\right)+\delta_{i j} d x^{i} d x^{j}
$$

## Further open questions

Modest:
1 Characterize accurately when $\nabla^{2} T=0 \nRightarrow \nabla T=0$ in the Lorentzian case.
2 Classify 3rd symmetric Lorentzian spaces.

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Senovilla's:
1 Solve all the linear conditions for curvature:

$$
\nabla^{r} R+t_{1} \otimes \nabla^{r-1} R+t_{2} \otimes \nabla^{r-2} R+\cdots+t_{r-1} \otimes \nabla R+t_{r} \otimes R=0
$$

for some $m$ - covariant tensors $t_{m}$.

