Higher genus helicoids

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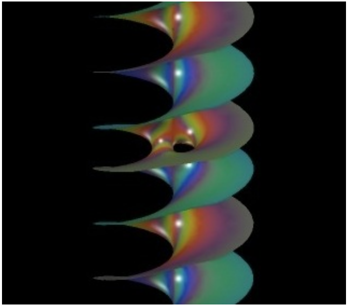
Main result (joint work with D. Hoffman and B. White)

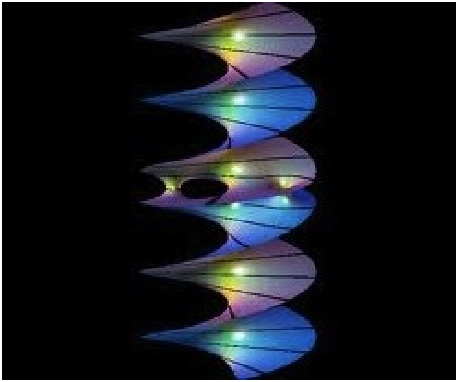
Theorem For every $g \geq 1$, there exists a properly embedded, genus g minimal surface in $\mathbb{R}^3$ that is asymptotic to the helicoid.

Previously known : the genus 1 helicoid

- Discovered by Hoffman Karcher Wei using Weierstrass Representation (1993)
- Proven to be embedded by Hoffman Weber Wolf (2009)



Picture of a genus 2 helicoid



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2. Prove that for even g, these surfaces converge to a genus $g/2$ helicoid in $\mathbb{R}^3$ as $r \to \infty$.

Minimal surfaces in $\mathbb{S}^2(r) \times \mathbb{R}$

- Model for $\mathbb{S}^2(r)$: $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with conformal metric

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- The standard helicoid H :

$$z = \pm e^{ih}$$

is minimal in $\mathbb{S}^2(r) \times \mathbb{R}$ for all r. It has two axes : $0 \times \mathbb{R}$ and $\infty \times \mathbb{R}$

Main result follows from three theorems

Theorem 1 [Hoffman White] For each g there exists two helicoidal, genus g minimal surfaces $M^{+}_{g,r}$ and $M^{-}_{g,r}$ in $\mathbb{S}^2(r) \times \mathbb{R}$. Both have a top end and a bottom end asymptotic to a vertical translate of H.

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Theorem 2 [Hoffman White] Given a sequence $r_n \to \infty$, there exists a subsequence such that $M^\pm_{g,r_n}$ converges to a minimal surface $M^\pm_g$ in $\mathbb{R}^3$ which is asymptotic to the standard helicoid H. Moreover M^+_g has even genus and M^-_g has odd genus.

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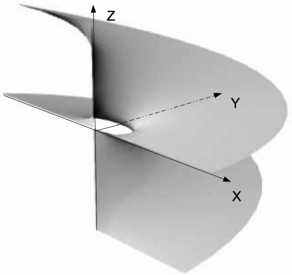
Theorem 3 [Hoffman Traizet White] Assume g is even. If $g/2$ is even then the limit M^+_g has genus $g/2$. If $g/2$ is odd then M^-_g has genus $g/2$.

Notations

- $M \subset \mathbb{S}^2(r) \times \mathbb{R}$ denotes $M^+_{g,r}$ or $M^-_{g,r}$
- The vertical axes $0 \times \mathbb{R}$ and $\infty \times \mathbb{R}$ are denoted Z and Z^*
- The horizontal axes $\mathbb{R} \times 0$ and $i\,\mathbb{R} \times 0$ are denoted X and Y
- The standard helicoid H divides $\mathbb{S}^2 \times \mathbb{R}$ in two components denoted H^+ and H^- so that H^+ contains the positive Y axis
- $S = M \cap H^+$

Symmetries of higher genus helicoids in $\mathbb{S}^2 \times \mathbb{R}$

1. $M \cap H = Z \cup Z^* \cup X$
   So the boundary of S is $Z \cup Z^* \cup X$



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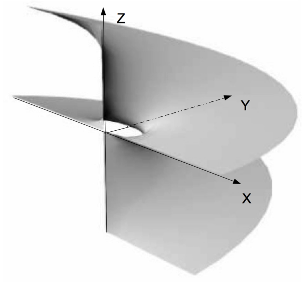

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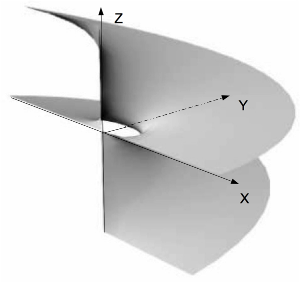

2. M intersects Y in $2g + 2$ points. M is invariant by 180^o rotation about the axis Y and the quotient is a disk.
3. if genus g is even : M is symmetric with respect to the vertical cylinder $\mathbb{S}^1(r) \times \mathbb{R}$.

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Goal : prove that $g - 2g' \leq 2$ so at most two handles escape.

Then $\frac{g}{2} - g' \leq 1$ so if $\frac{g}{2}$ and g' have the same parity, $g' = \frac{g}{2}$.

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Idea : handles which escape interact via some forces which can be explicitely computed in the limit.

Proof of Theorem 3 in 8 slides

1. Change scale : work in $\mathbb{S}^2(1) \times \mathbb{R}$

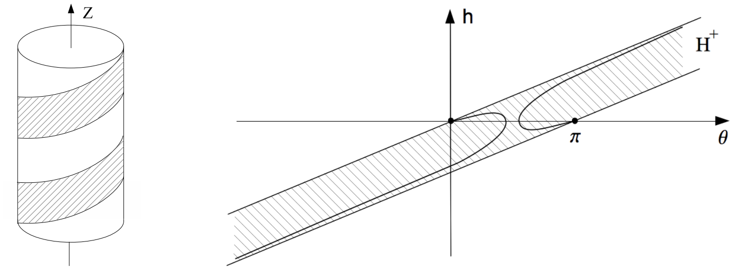
- $t_n = \frac{1}{r_n} \to 0$
- $M_n = t_n M^{\pm}(g, r_n) \subset \mathbb{S}^2(1) \times \mathbb{R}$
- $H_n = t_n H$ is a helicoid of vertical period $2\pi t_n$
- $S_n = M_n \cap H_n^+$

2. Work in universal cover of $\mathbb{R}^3 \setminus Z$

Work in cylindrical coordinates (r, θ, h)

Metric is still the metric of $\mathbb{S}^2 \times \mathbb{R}$

H_n^+ is the domain $t_n(\theta - \pi) < h < t_n\theta$



3. Limit as $t_n \to 0$

Proposition S_n converges to the plane $h = 0$. Convergence is smooth outside of a discrete singular set $\mathcal{S}$. Multiplicity of the limit is

$$\begin{cases} 1 & \text{on } \theta < 0 \text{ and } \theta > \pi \\ 2 & \text{on } 0 \leq \theta \leq \pi \end{cases}$$

4. Formation of catenoidal necks

Proposition Let $p \in \mathcal{S}$, $p \neq 0, \infty$. There exists $p_n \in S_n$, $p_n \to p$, and $\lambda_n \to \infty$ such that $\lambda_n(S_n - p_n)$ converges to the standard catenoid in $\mathbb{R}^3$. Moreover, $p \in Y$.

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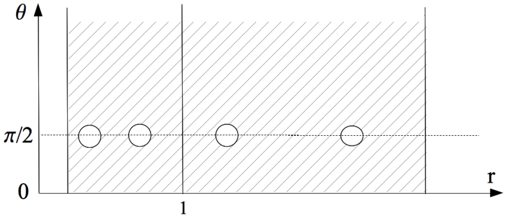
Goal : prove that $N \leq 1$.

Remark : there could be several handles converging to the same singular point. There could also be handles converging to 0 or ∞ at this scale but not fast enough to be captured by the limit helicoids.

5. Write S_n as a graph

Remove vertical cylinders of small radius about Z and Z^*
Remove small balls centered at $p_1, \cdots, p_N$
This disconnects S_n in two components. The top one is the graph of a function f_n on a domain $\Omega_n \subset \widetilde{\mathbb{C}^*}$



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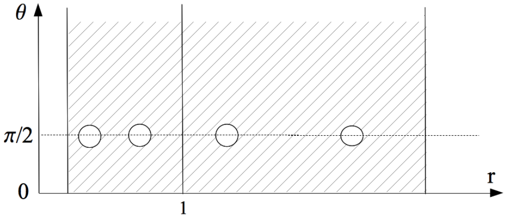

$$f_n(z) = t_n \arg z - u_n(z)$$

- $u_n = 0$ on $\theta = 0$
- $0 < u_n < t_n\pi$ on Ω_n

6. Limit of u_n

Proposition

$$\lim \frac{|\log t_n|}{t_n} u_n(z) = c_0 \arg z - \sum_{i=1}^{N} c_i \log \left| \frac{\log z - \log p_i}{\log z - \log \overline{p_i}} \right|$$

Moreover $c_i > 0$ for $1 \leq i \leq N$.

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- If $M \subset \mathbb{S}^2(1) \times \mathbb{R}$ is the graph of a function f_n and χ is a horizontal Killing field

$$\mathrm{Flux}_\chi(M,\gamma) = -\Im \int_\gamma 2(f_z)^2 \chi(z) dz + O(|f_z|^4)$$

8. Conclude

$$F_{i,n} = \mathrm{Flux}_{\chi_Y}(S_n, C(p_i, \varepsilon))$$

$$F_i = \lim \left(\frac{\log t_n}{t_n}\right)^2 F_{i,n}$$

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$$F_i = -c_1^2 \sinh x_i - \sum_{j \neq i} \frac{2\pi^2 c_i c_j}{(x_i - x_j)((x_i - x_j)^2 + \pi^2)}$$

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- If $N \geq 2$ then $F_1 > 0$, contradiction.
- If $N = 1$ then $p_1 = i$.