Higher genus helicoids

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Joint work with David Hoffman & Brian White

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Main theorem. For every positive integer k, there exists a genus-k helicoid in $\mathbb{R}^3$.

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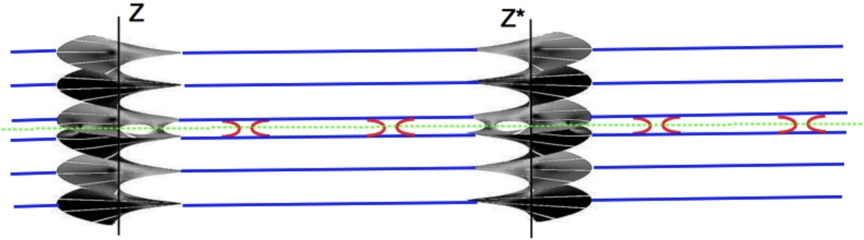
- Recall : g' is even if M_n is positive at O and odd if M_n is negative. We choose the sign of M_n at O so that g' and k have the same parity.

Catenoidal necks

For large values of R_n, M_n looks like this :



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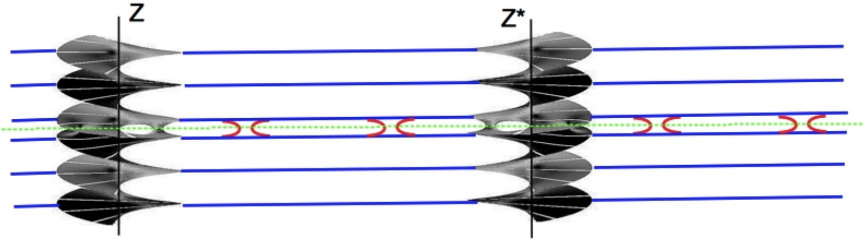

Let $2N$ be the number of handles that are escaping from both Z and Z^* (red guys). Then

$$g = 2k = 2g' + 2N \quad \Rightarrow \quad N = k - g'$$

Observe that N is even.

Goal : prove that $N \leq 1$

Change scale

- Let $\widetilde{M}_n = \frac{1}{R_n} M_n$. This is a genus-$g$ helicoid in $\mathbb{S}^2(1) \times \mathbb{R}$ asymptotic to the helicoid with pitch

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- Let $p_{i,n} \in Y$ be the "center" of the i-th catenoidal neck (midpoint of the two intersection points of the catenoidal neck with Y-axis).
  Label the necks so that $\mathrm{Im}(p_{i,n}) > 0$ for $1 \leq i \leq N$ and $p_{N+i,n} = -p_{i,n}$.

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- Passing to a subsequence :

$$p_i := \lim p_{i,n} \in Y$$

- First assume that all limit points p_i are distinct and $\neq 0, \infty$.

Write $\widetilde{M}_n$ as a graph.

Remove vertical cylinders of axis Z and Z^* and $2N$ small balls with centers $p_{i,n}$. This disconnects $\widetilde{M}_n$ into two components which are both vertical graphs over the helicoid and are exchanged by Y-symmetry.

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Consider the component which contains the positive X-axis. We can write it as the graph over the plane of a multivalued function f_n which has the form :

$$f_n = \frac{t_n}{2\pi} \arg(z) + u_n$$

with

- $u_n = 0$ on $\arg(z) = 0$
- $|u_n| < \frac{t_n}{2}$
- $u_n < 0$ on $\arg(z) > 0$
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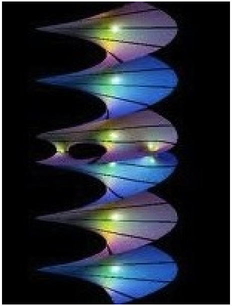
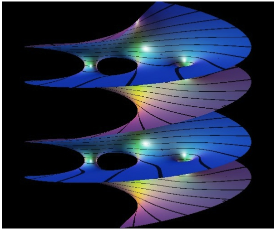
Observe : u_n is multivalued in the plane. It is well defined in the universal cover $\widetilde{\mathbb{C}^*}$ where $\arg(z)$ is well defined.

A simpler case

Assume for simplicity that u_n is a single-valued function of z.
Geometrically, this means we are considering periodic helicoidal surfaces invariant by a vertical translation :



non-periodic case



periodic case

Limit of u_n

Key Proposition. In the periodic case :

$$\widetilde{u} := \lim \frac{|\log t_n|}{t_n} u_n = \sum_{i=1}^{2N} c_i \log |z - p_i|$$

Moreover $c_i > 0$ for $1 \leq i \leq N$ and $c_{N+i} = -c_i < 0$.

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In the non-periodic case :

$$\lim \frac{|\log t_n|}{t_n} u_n = c_0 \arg z + \sum_{i=1}^{2N} c_i \log |\log z - \log p_i|$$

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- Bôcher Theorem implies it has log singularities at $p_1, \cdots, p_{2N}$.

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- If $M \subset \mathbb{S}^2(1) \times \mathbb{R}$ is the graph of a function f and χ is a horizontal Killing field

$$\text{Flux}_\chi(\gamma) = -\mathrm{Im} \int_\gamma 2(f_z)^2 \chi(z) dz + O(|f_z|^4)$$

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Claim : $F_i := \lim \dfrac{|\log t_n|^2}{t_n^2} F_{i,n} = -\mathrm{Re} \displaystyle\int_{C(p_i,\varepsilon)} (\widetilde{u}_z)^2(1 - z^2)dz$

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$$F_i = \pi \mathrm{Im} \sum_{j \neq i} c_i c_j \frac{1 - p_i p_j}{p_i - p_j}$$

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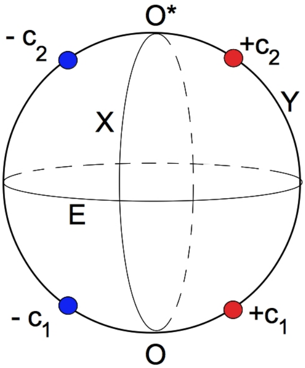
Write $p_j = i \tan \frac{\theta_j}{2}$:

$$F_i = \pi \sum_{j \neq i} c_i c_j \cot \frac{\theta_j - \theta_i}{2}$$

Physical model (periodic case)



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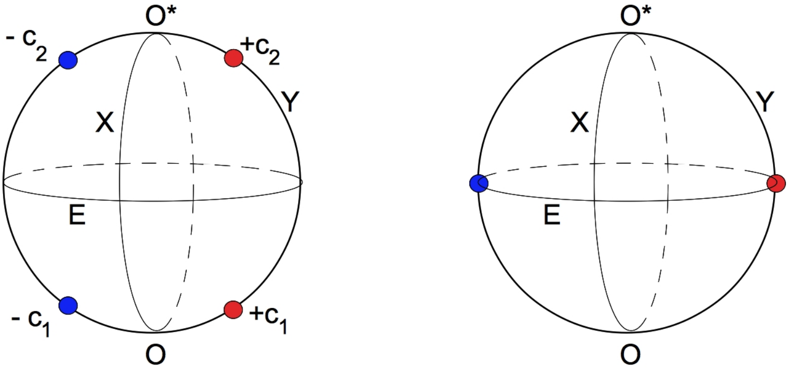

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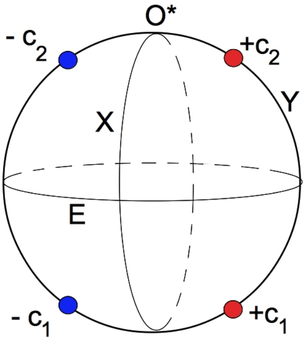
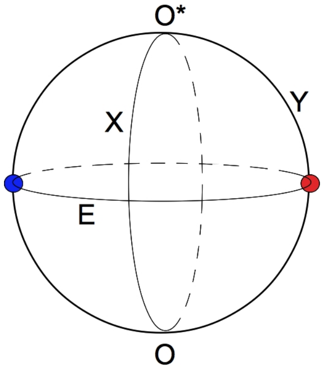
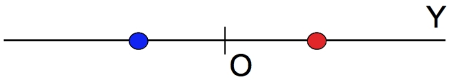




Conclusion : $N \leq 1$.

Case where some points $p_{i,n}$ converge to O (or $O*$)

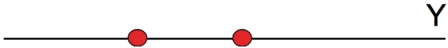
A blowup at O produces a configuration like



Cannot be balanced !

Case where all points $p_{j,n}$ for $1 \leq j \leq N$ converge to i

A blowup at i produces a configuration like



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Forces in the non-periodic case

$$F_i = \frac{y_i^2+1}{2y_i}\left[c_i^2\frac{1-y_i^2}{y_i^2+1} + \sum_{\substack{j\neq i\\ 1\leq j\leq N}} \frac{-2\pi^2\, c_i c_j}{(\log y_i - \log y_j)|\log y_i - \log y_j + i\pi|^2}\right].$$

where $y_i = \mathrm{Im}(p_i)$.