New conformal methods in Riemannian and Lorentzian geometry

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Plan of the talk

1. Short introduction to globally hyperbolic manifolds
2. Conformal extendibility and its physical consequences
3. The flatzoomer technique and bounded geometry
4. Consequences for the Yamabe flow

Causality properties

Definition

A spacetime ($=$ time oriented Lorentzian manifold) (M, g) is called

- **causal** $:\Leftrightarrow$ it contains no closed future causal curve,
- **diamond-compact** $:\Leftrightarrow$ $J_g^+(p) \cap J_g^-(q)$ compact for all $p, q \in M$,
- **globally hyperbolic** $:\Leftrightarrow$ it is causal and diamond-compact.

Examples of g.h. spacetimes

- $(M, g) = (\mathbb{R} \times N, -dt^2 \oplus g_N)$ for a complete Riemannian metric g_n on N (in particular $\mathbb{R}^{1,n}$)
- If (M, g) g.h. and (M, h) spacetime with $J_h \subset J_g$, then (M, h) g.h. (in particular: conformally invariant notion)
- If (M, g) g.h. and $A \subset M$ is **causally convex** Lorentzian submanifold (that is, no causal curve leaves and re-enters A), then A is g.h. as well
- To be g.h. is a fine-C^0-stable property. Together with the third property: Every Lorentzian manifold is locally g.h.

Consequences of global hyperbolicity

- Geodesic connectedness of causally related points
- With K.O. Friedrich's classical results and the structural result of Bernal/Sánchez 2005-2006 (cf next slide): Well-posedness of Cauchy problems for **symmetric-hyperbolic** 1st order linear operators A (i.e such that $\mathrm{symb}(A)$ is symmetric and is positive-definite on $\mathrm{int}(J_g)$), and even of appropriate semilinear symmetric-hyperbolic operators as Yang-Mills (Chrusciel-Grant 2000)

Structural result

Theorem (OM - Miguel Sánchez 2009)

Any globally hyperbolic manifold is isometric to $(\mathbb{R} \times N, -fdt^2 + g_t)$ *where* g_t *is some one-parameter family of Riemannian metrics on* N *and* $f \in C^\infty(\mathbb{R} \times N)$ *that can be chosen bounded.*

Consequence: Lorentzian Nash's theorem: Every *stably causal* manifold can be embedded *conformally* into a Minkowski space; every g.h. even *isometrically*

$\rightsquigarrow$ **analytical advantages**: Morse Theory etc

Can we ask for more?

Question: Can we get additional bounds on $g(\mathrm{grad}^g t, \mathrm{grad}^g t)$ and $\dot{g}_t$? Completeness of level sets? Would be important e.g. for

- Well-posedness for open Lorentzian minimal surfaces,
- Decay estimates for Dirac, Laplace, Yang-Mills...

In general: No!

But...

Conformal equivalence and conformal maps

Isometries $D : (M, g) \to (N, h)$ between Riemannian manifolds: condition $D^*(h) = g$. Now relax condition a bit:

Definition

Two Riemannian metrics g, h on a manifold M are called **conformally equivalent** if there is a smooth positive function f on M such that $h = f \cdot g$. A map $D : (M, g) \to (N, h)$ is called **conformal** iff D^*h is conformally equivalent to g.

- Conformal equivalence preserves angles (e.g. orthogonality).
- Many important equations possess some covariance under conformal transformations. Examples: Conformal Laplace (Yamabe) operators, Dirac operators...

Conformal compactifications

Definition

Let (M, g) be a g.h. spacetime. An open conformal embedding F of (M, g) into a g.h. spacetime (N, h) is called **conformal compactification of** (M, g) iff $\overline{F(M)}$ is compact and causally convex.

A Riemannian ingredient

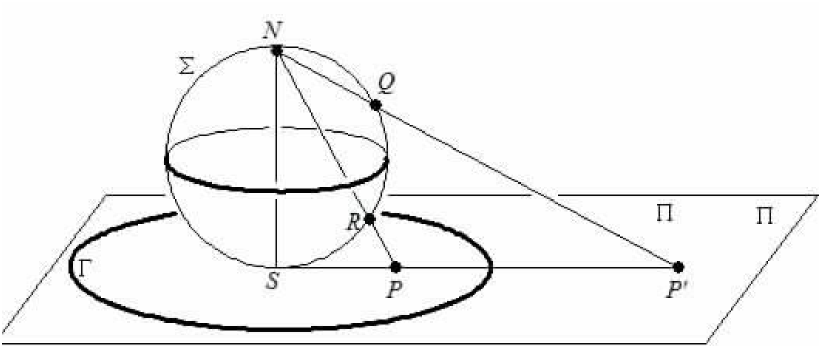

Figure: The stereographic projection

A Lorentzian example

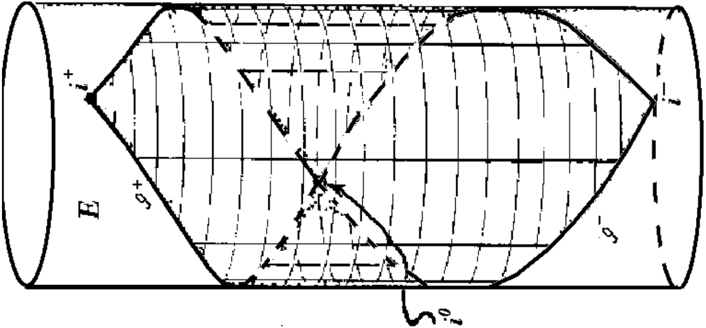

Figure: A conformal compactification of $\mathbb{R}^{1,n}$, used for solving small initial value problem for neutral Dirac-YangMills-Higgs systems in $\mathbb{R}^{1,3}$

Conformal extensions

Definition

Let (M, g) be a g.h. spacetime and S_0 be a Cauchy surface of (M, g). An open conformal embedding F of $I^+(S_0)$ into a g.h. spacetime (N, h) is called **conformal future compactification** or **(future) conformal extension of** (M, g) iff $\overline{F(M)}$ is future compact and causally convex.

Anderson-Chrusciel: Each element of a weighted Sobolev neighborhood of the Minkowski metric in the space of Lorentzian metrics admits a sufficiently regular conformal extension.

Nicolas Ginoux-OM 2014: Conformal extendibility $\Rightarrow$ well-posedness of neutral small Dirac-Higgs-YangMills systems

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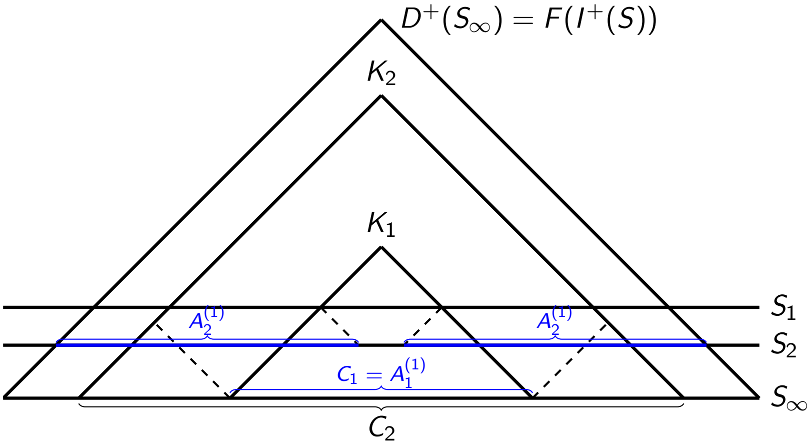
Scheme of the global existence proof



Bounded geometry

For a (k, m)-tensor A with $m \in \{0, 1\}$ and a Riemannian metric g:

$$|A|_g := \sup\{|A(V_1, \dots V_k)|_g : \quad g(V_i, V_i) = 1 \quad \forall i\}$$

$\mathrm{inj}_g : M \to (0, \infty]$: injectivity radius of g

Definition

A Riemannian metric g on a manifold M is called **of bounded geometry** iff inj_g^{-1} and $|\nabla^i \, \mathrm{Riem}_g|_g$ are bounded, for all $i \in \mathbb{N}$.

Examples: compact manifolds, homogeneous spaces (e.g. $\mathbb{R}^n$), ...

Consequences of bounded geometry:

- Sobolev and Morrey embeddings
- Uniformly good charts

(Roe, Atiyah & Bott & Patodi, Eichhorn...)

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An Unexpected Link, Riemannian version

Theorem (OM - Marc Nardmann 2013)

Every Riemannian metric is conformally equivalent to a metric of bounded geometry.

An Unexpected Link, Lorentzian version

Theorem (OM - Marc Nardmann 2013)

For every g.h. mf (M, g) and every Cauchy temporal function t on M, we can find a causally complete metric $h \in [g]$ s.t.

- *t satisfies $h(\mathrm{grad}^h t, \mathrm{grad}^h t) > -1$,*
- *The 2nd fundamental form of all level sets is C^k-bounded, $\forall k$.*
- *All t-level sets (metric induced by h) are of bounded geometry.*

Main result, general version for foliated mfs

Theorem (OM - Marc Nardmann 2013)

Let F be a foliation on a manifold M, let g_0, h_0 be semi-Riemannian metrics on M which induce nondegenerate semi-Riemannian metrics on the leaves of F. Let $K = (K_i)_{i \in \mathbb{N}}$ be a smooth compact exhaustion of M, let $b \in C^0(M, \mathbb{R}_{>0})$. Then there exists a real-analytic function $u : M \to \mathbb{R}$ with $u > b^{-1}$ such that, for $g := e^{2u} g_0$ and $h := e^{2u} h_0$:

1. *For every $i \in \mathbb{N}$, $\left| {}^g\nabla^i \mathrm{Riem}_g \right|_h < b$ holds on $M \setminus K_i$.*
2. *If g_0 Riemannian, then g complete with $\mathrm{conv}_g > b^{-1}$.*
3. *For every $i \in \mathbb{N}$, $\left| {}^{g_F}\nabla^i \mathrm{Riem}_{g_F} \right|_{h_F} < b$ holds on $M \setminus K_i$.*
4. *If $(g_0)_F$ is Riemannian, then for each F-leaf L, g_L is complete with $\mathrm{conv}_{g_L} > b^{-1}\big|_L$.*
5. *For every $i \in \mathbb{N}$, $\left| {}^g\nabla^i II^F_g \right|_h < b$ holds on $M \setminus K_i$.*

Plan of the proof of the main result

Notation: $g[u] := e^{2u} \cdot g$.

- **Step 1**: Define *flatzoomers*.
- **Step 2**: Show: $u \mapsto \left| \nabla^{(i)}_{g[u]} \mathrm{Riem}_{g[u]} \right|_{g[u]}$ and $u \mapsto \left| \nabla^{(i)}_{g[u]} II^{g[u]}_{t^{-1}(a)} \right|_{g[u]}$ are flatzoomers, $\forall i \in \mathbb{N}, a \in \mathbb{R}$.
- **Step 3**: Define *quasi-flatzoomers*.
- **Step 4**: Show that $u \mapsto \mathrm{inj}^{-1}_{g[u]}$ is a quasi-flatzoomer.
- **Step 5**: Consider an appropriate sum of the QFZs above, which is a QFZ again. Show that every quasi-flatzoomer Q 'is surjective on rapid falloff classes', i.e. if ϵ is a continuous function (a 'falloff profile') then there is a smooth function u such that $Q(u) < \epsilon$ pointwise.

Step 1: Define flatzoomers

Definition

Let M be a manifold. A functional $\Phi\colon C^\infty(M,\mathbb{R}) \to C^0(M,\mathbb{R}_{\geq 0})$ is a **flatzoomer** iff for some — and hence every — Riemannian metric η on M, there exist $k, d \in \mathbb{N}$, $\alpha \in \mathbb{R}_{>0}$, $u_0 \in C^0(M,\mathbb{R})$ and a polynomial-valued map $P \in C^0\left(M, \mathbb{R}\mathrm{Poly}^d_{k+1}\right)$ such that

$$\left|\Phi(u)(x)\right| \;\leq\; \mathrm{e}^{-\alpha u(x)} P(x) \left(u(x),\, \left|\nabla^1_\eta u\right|_\eta(x),\, \ldots,\, \left|\nabla^k_\eta u\right|_\eta(x) \right)$$

holds for all $x \in M$ and all $u \in C^\infty(M,\mathbb{R})$ with $u(x) > u_0(x)$.

Step 2: R and its derivatives are flatzoomers

Theorem (covariant derivatives of the Riemann tensor)

Let (M, g) be a semi-Riemannian manifold, let $k \in \mathbb{N}$. Then $\Phi\colon C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}_{\geq 0})$ defined by

$$\Phi(u) := \left|\nabla^k_{g[u]} \mathrm{Riem}_{g[u]}\right|^2_{g[u]}$$

is a flatzoomer.

Proof: Riemann curvature $(4,0)$-tensor Riem_g under conformal change (with $\owedge$ being the Kulkarni-Nomizu product)

$$\mathrm{Riem}_{g[u]} = \mathrm{e}^{2u}\Big(\mathrm{Riem}_g - g \owedge \Big(\mathrm{Hess}_g\, u - \mathrm{d}u \otimes \mathrm{d}u + \tfrac{1}{2}\,|\mathrm{d}u|^2_g\, g\Big)\Big).$$

Higher derivatives: sophisticated induction.

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Higher derivatives: sophisticated induction.

Step 3: Define quasi-flatzoomers

Beware: $\Phi : u \mapsto \mathrm{inj}_{g[u]}^{-1}$ is *not* a flatzoomer, because $\Phi(u)(x)$ cannot be bounded in terms of a k-jet $j_x^k u$ of u *at* x.

Definition

Let $\mathcal{K} = (K_i)_{i \in \mathbb{N}}$ be a compact exhaustion of a manifold M.
A functional $\Phi : C^\infty(M, \mathbb{R}) \to \mathrm{Fct}(M, \mathbb{R}_{\geq 0})$ is a **quasi-flatzoomer for $\mathcal{K}$** iff for some — and hence every — Riemannian metric η on M, there exist $k, d \in \mathbb{N}$, $\alpha \in \mathbb{R}_{>0}$, $u_0 \in C^0(M, \mathbb{R})$ and $P \in C^0\left(M, \mathbb{R}\mathrm{Poly}_{k+1}^d\right)$ such that

$$|\Phi(u)(x)| \leq \sup\left\{ \mathrm{e}^{-\alpha u(y)} P(y) \left(\left|\nabla_\eta^0 u\right|_\eta(y), \ldots, \left|\nabla_\eta^k u\right|_\eta(y) \right) \,\middle|\, y \in K_{i+1} \setminus K_{i-2} \right\}$$

$\forall i \in \mathbb{N}, x \in K_i \setminus K_{i-1}, \ \forall u \in C^\infty(M, \mathbb{R})$ with $u > u_0$ on $K_{i+1} \setminus K_{i-2}$.

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$\forall i \in \mathbb{N}, x \in K_i \setminus K_{i-1}$, $\forall u \in C^\infty(M,\mathbb{R})$ with $u > u_0$ on $K_{i+1} \setminus K_{i-2}$.

Step 4: Show that inj^{-1} is a quasi-flatzoomer

Theorem

Let (M, g) be a Riemannian manifold M,
let $\mathcal{K} = (K_i)_{i \in \mathbb{N}}$ be a compact exhaustion of M.
Then $\Phi\colon C^\infty(M, \mathbb{R}) \to \mathrm{Fct}(M, \mathbb{R}_{\geq 0})$ given by

$$\Phi(u) := 1/\,\mathrm{inj}_{g[u]}$$

is a quasi-flatzoomer for $\mathcal{K}$.

Step 4, ctd.

Theorem (Generalized Klingenberg's lemma)

Let (M,g) be a Riemannian manifold, let $x \in M$, let $r, \delta, \ell \in \mathbb{R}_{>0}$. Assume

- *the ball $B_r^g(x) := \{z \in M \mid \mathrm{dist}_g(x,z) \leq r\}$ (which is closed in M) is compact;*
- *$\sec_g \leq \delta$ holds on $B_r^g(x)$;*
- *every self-intersecting geodesic in (M,g) contained in $B_r^g(x)$ has length $\geq \ell$.*

Then $\mathrm{inj}_g(x) \geq \min\left\{r, \frac{\pi}{\sqrt{\delta}}, \frac{\ell}{2}\right\}$.

$\rightsquigarrow$ Main task: bound ℓ from 0!

Step 4, ctd.

$\forall$ chart U_i of a locally finite chart cover $\mathcal{U} = \{U_i | i \in I\}$,
$\exists A_i, C_i \in \mathbb{R}$:

$$\left| {}^{g[u]}\Gamma^c_{ab} \right| \leq A_i \left(1 + \left| \mathrm{d}u \right|_g \right),$$

$$C_i \left| v \right|_{\mathrm{eucl}_i} \geq \left| v \right|_g \geq C_i^{-1} \left| v \right|_{\mathrm{eucl}_i}.$$

$\mathcal{U}$ locally finite, thus $\exists\ H \in C^0(M, \mathbb{R}_{>0})$ with $\forall x \in M \colon \forall i \in \mathbb{N}$:
$x \in U_i \Rightarrow H(x) \geq 4n^2 A_i C_i^3$.

Define the flatzoomer $\Phi_1 \colon u \mapsto \mathrm{e}^{-u} H \cdot \left(1 + \left| \mathrm{d}u \right|_g \right)$.

Lemma

$\sup \left\{ 4/\,\mathrm{length}(\gamma) \,\middle|\, \gamma \subset B_r^{g[u]}(x) \textit{ is a self-intersecting } g[u]\textit{-geodesic} \right\}$
$\leq \sup \left\{ \Phi_1(u)(y) \,\middle|\, y \in K_{i+1} \setminus K_{i-2} \right\}$

Step 4: Proof of Lemma

Let $\gamma\colon [0,\ell] \to B := B_r^{g[u]}(x)$ be a self-intersecting unit speed $g[u]$-geodesic.
Compactness $\Rightarrow$ There is s_0 with s.t. $u(\gamma(s_0)) = \min_{s\in[0,\ell]} u(\gamma(s))$.
Since $B \subset U_j$, we can regard B as subset of Euclidean $\mathbb{R}^n$.
Self-intersecting $\rightsquigarrow$ $\exists s_1 \in [0,\ell]$ with $\langle \gamma'(s_1), \gamma'(s_0)\rangle_{\mathrm{eucl}_j} < 0$.
In particular, $|\gamma'(s_0)|_{\mathrm{eucl}_j} \leq |\gamma'(s_1) - \gamma'(s_0)|_{\mathrm{eucl}_j}$ (*).

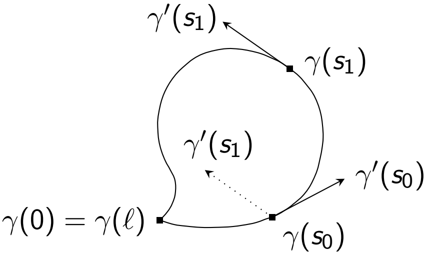

Figure: A self-intersecting $g[u]$-geodesic γ in $B \subseteq \mathbb{R}^n$.

Step 4: Proof of the lemma, ctd.

Choose reparametrization s.t. $s_0 = 0$ and define $\tau := s_1$.

$$
\begin{aligned}
1 &=_{\text{(unit speed)}} \left|\gamma'(0)\right|_{g[u]} =_{(\text{Def.}g[u])} \mathrm{e}^{u(\gamma(0))} \left|\gamma'(0)\right|_g \\
&\leq_{(\text{Def.}C_j)} C_j \mathrm{e}^{u(\gamma(0))} \left|\gamma'(0)\right|_{\mathrm{eucl}_j} \\
&\leq_{(*)} C_j \mathrm{e}^{u(\gamma(0))} \left|\gamma'(\tau) - \gamma'(0)\right|_{\mathrm{eucl}_j} \\
&=_{\text{(fund.th.calc.)}} C_j \mathrm{e}^{u(\gamma(0))} \left|\int_0^\tau \gamma''(s)\,\mathrm{d}s\right|_{\mathrm{eucl}_j} \\
&\leq_{\text{(geod.eq.)}} C_j \mathrm{e}^{u(\gamma(0))} \sum_{a,b,c=1}^{n} \int_0^\tau \left|{}^{g[u]}\Gamma^c_{ab}(\gamma(s))\right| \cdot \left|\gamma_a'(s)\right| \cdot \left|\gamma_b'(s)\right| \,\mathrm{d}s
\end{aligned}
$$

Step 4: Proof of the lemma, ctd.

Use definition of A_j and then $n^{1/2} |v|_{\mathrm{eucl}_j} \geq \sum_{a=1}^n |v_a|$ (**):

$$
\begin{aligned}
\ldots \;\leq\; & C_j \mathrm{e}^{u(\gamma(0))} \sum_{a,b,c=1}^{n} \int_0^\tau A_j \cdot \left(1 + |\mathrm{d}u|_g(\gamma(s))\right) \cdot |\gamma_a'(s)| \cdot |\gamma_b'(s)| \, \mathrm{d}s \\
=_{(**)} \; & n A_j C_j \mathrm{e}^{u(\gamma(0))} \int_0^\tau \left(1 + |\mathrm{d}u|_g(\gamma(s))\right) \cdot \left(\sum_{a=1}^n |\gamma_a'(s)| \right)^2 \mathrm{d}s \\
\leq_{(\mathrm{Def.} C_j)} \; & n^2 A_j C_j^3 \int_0^\tau \left(1 + |\mathrm{d}u|_g(\gamma(s))\right) \cdot \mathrm{e}^{u(\gamma(0))} \, \mathrm{e}^{-2u(\gamma(s))} \mathrm{d}s \\
\leq_{(u \text{ min. at } \gamma(0))} \; & n^2 A_j C_j^3 \int_0^\tau \left(1 + |\mathrm{d}u|_g(\gamma(s))\right) \cdot \mathrm{e}^{-u(\gamma(s))} \mathrm{d}s \\
\leq_{(\mathrm{int.est.})} \; & \ell n^2 A_j C_j^3 \left\| \mathrm{e}^{-u} \left(1 + |\mathrm{d}u|_g\right) \right\|_{C^0(U_j \cap (K_{i+1} \setminus K_{i-2}))} .
\end{aligned}
$$

Step 4: Proof of Lemma, last part

Consequently,

$$\begin{aligned} 4/\ell &\leq n^2 A_j C_j^3 \left\| \mathrm{e}^{-u}\left(1 + \left|\mathrm{d}u\right|_g\right) \right\|_{C^0(U_j \cap (K_{i+1} \setminus K_{i-2}))} \\ &\leq \left\| H\mathrm{e}^{-u}\left(1 + \left|\mathrm{d}u\right|_g\right) \right\|_{C^0(K_{i+1} \setminus K_{i-2})} \\ &= \sup\left\{ \Phi_1(u)(y) \,\middle|\, y \in K_{i+1} \setminus K_{i-2} \right\}. \end{aligned}$$

□

Step 5: Main result for general quasi-flatzoomers

Theorem (OM-Marc Nardmann 2013)

Let $\mathcal{K} = (K_i)_{i \in \mathbb{N}}$ *be a smooth compact exhaustion of* M*,*
let $(\Phi_i)_{i \in \mathbb{N}}$ *be a sequence of quasi-flatzoomers for* $\mathcal{K}$*,*
let $(\varepsilon_i)_{i \in \mathbb{N}}$ *be a sequence in* $C^0(M, \mathbb{R}_{>0})$*, let* $w \in C^0(M, \mathbb{R})$*.*

Then there exists a real-analytic $u \colon M \to \mathbb{R}$ *with* $u > w$ *such that*

$$\forall i \in \mathbb{N}: \ \left|\Phi_i(u)\right| < \varepsilon_i \ \text{ holds on } M \setminus K_i \, .$$

The Yamabe flow: Motivation

- Basic problem: Look for natural best metric in a given conformal class.
- Candidate: Metric of constant scalar curvature (Yamabe problem).
- On compact manifolds: Such a metric exists. It is unique if there is a solution of *negative* constant scalar curvature (Yamabe, Trudinger, Aubin, Schoen)
- To do Morse theory of the underlying functional: Ensure that the space of solutions is compact.
- Critical tool: Yamabe flow
- In the noncompact case, even existence is in general not true (counterexamples by Jin (1988) on $\mathbb{R}^n$, $n \geq 3$)

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- To do Morse theory of the underlying functional: Ensure that the space of solutions is compact.
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The Yamabe flow: Motivation

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The Yamabe flow: Definition and properties

$g_t = -\mathrm{scal}^g \cdot g, \ g(0) = g_0$
For $g = u^{4/n-2} g_0$ we have $\mathrm{scal}^g = -u^N L u$,
where $N = \frac{n+2}{n-2}$ and $L = \frac{4(n-1)}{n-2} \Delta^{g_0}$.
Equation for u after rescaling of the time:
$\partial(u^N) = Lu - \mathrm{scal}^g \cdot u, \ u(0,x) = 1$.

On compact manifolds: Eternal existence of an equivalent normalised version (R.S. Hamilton) and convergence to a csc metric for $dim \leq 5$ or M spin (Brendle)

On a noncompact manifold, little is known except for a lifetime estimate $T \sim |scal|_{C^0}^{-1}$ (*) and eternal existence if Ric^{g_0} bounded from below and $\mathrm{scal}^{g_0} \leq 0$. This can be proven via elementary parabolic methods (cf. Ma-An 1999).
Consequence of (*) and of the main theorem: In every conformal class there are representatives of arbitrarily large lifetime.

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The Yamabe flow: A recent result and a conjecture

Shao-Simonett (J.Evol.Eq. 2014): Using the flatzoomer result, it can be shown that there is a lifetime estimate uniform for the entire class of metrics ug_0 for u uniformly positive and of bounded C^1 norm, for appropriate g_0.

Conjecture: In every conformal class there is a metric of infinite lifetime for the Yamabe flow. (Work in Progress)
Envisaged joint work with Alexander Grigoryan (Bielefeld): Flatzoomability of pseudo-isometry invariant properties from **stochastic geometry** (heat kernels, p-parabolicity...)
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