Inverse Spectral Positivity on Surfaces

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joint work with **Pierre Bérard** (Grenoble)

(M, g) an complete orientable Riemannian surface, K its curvature

For some $a > 0$ consider the operator $L = \Delta + aK + W$ where Δ is the (positive) Laplacian and $W \in L^1_{loc}(M)$.
The associated quadratic form is $q(u) = \int_M |du|^2 + aKu^2 + Wu^2$

This kind of operators naturally appear when studying the stability of minimal or CMC surfaces

- if $M \hookrightarrow \mathbb{R}^3$ is minimal, its Jacobi operator is $J = \Delta + 2K$
- if $M \hookrightarrow N$ is minimal, its Jacobi operator is $J = \Delta + K - (\mathrm{Scal}_N + \frac{|A|^2}{2})$

Stable minimal surfaces are those for which the (quadratic form associated to the) Jacobi operator is non-negative.

Inverse spectral problem

From the non-negativity of $\Delta + aK + W$ on a surface M, derive

- topological and/or geometrical properties of M
- properties of W

For this talk, we shall assume that $W \equiv 0$

Assume that M is a compact surface and that $\Delta + aK \geq 0$ on M.

Plugging the constant function $u = \mathbf{1}$ in the quadratic form we get

$$0 \leq \int_M |du|^2 + aKu^2 = a \int_M K = 2\pi a \chi(M)$$

and $M \sim \mathbb{S}^2$ or $M \sim \mathbb{T}^2$.

Moreover, if $M \sim \mathbb{T}^2$, since $\int_M K = 0$, we have

$$\frac{q(\mathbf{1})}{\|\mathbf{1}\|^2} = 0 = \inf \left\{ \frac{q(u)}{\|u\|^2} \;\middle|\; u \in C^\infty(M) \right\}$$

The constant function $\mathbf{1}$ is an eigenfunction associated to the eigenvalue 0 and

$$0 = \Delta \mathbf{1} + aK\mathbf{1} = aK$$

Therefore the surface is a flat torus.

If M is non-compact, the constant functions do not belong to $H^1(M)$.

A good property to check is the parabolicity of M : do we have $M \overset{conf}{\sim} \bar{M} \setminus \{p_1, \dots, p_n\}$ with $\bar{M}$ compact ?

On parabolic surfaces, there exist sequences of functions $u_k \in C_c^\infty(M)$ such that $u_k \to \mathbf{1}$ and $\int_M |du_k|^2 \to 0$.

This approach was used by D. Fischer-Colbrie and R. Schoen for the study of stable minimal surfaces.

Theorem (D. Fischer-Colbrie & R. Schoen)

If $\Delta + aK \geq 0$ on M with $a \geq 1$ then

i. $M \overset{conf}{\sim} \mathbb{C}$ or $M \overset{conf}{\sim} \mathbb{C}^*$ (in particular $\chi(M) \geq 0$)

ii. if $M \overset{conf}{\sim} \mathbb{C}^*$ and $\int_M |K| < \infty$ then $K \equiv 0$

They mention that the finiteness of the total curvature should not be essential in (ii.).

Question

Assume that $\Delta + aK \geq 0$ on M. For which values of a do we have that

i. $M \overset{conf}{\sim} \mathbb{C}$ or $M \overset{conf}{\sim} \mathbb{C}^*$

ii. $K \equiv 0$ if $M \overset{conf}{\sim} \mathbb{C}^*$

For the hyperbolic plane $\mathbb{H}^2(-1) = \left(\mathbb{D}, \left(\frac{2}{1-z^2}\right)^2 |dz|^2\right)$ we have

- $K \equiv -1$
- the spectrum of the Laplacian is $\sigma(\Delta) = [\frac{1}{4}, +\infty)$

Therefore

$$\Delta + \frac{1}{4}K = \Delta - \frac{1}{4} \geq 0$$

To extend the result of D. Fischer-Colbrie and R. Schoen we have to assume that $a > \frac{1}{4}$.

When $a \leq \frac{1}{4}$ we have to add some hypothesis on M to rule out the hyperbolic plane.

In the sequel, $V(R)$ is the volume of the ball of radius R centered in some fixed point p_0.

Assume that $\Delta + aK$ is non-negative on M. Then

- if $a > \frac{1}{4}$
  - $M \overset{conf}{\sim} \mathbb{C}$ or $M \overset{conf}{\sim} \mathbb{C}^*$ (S. Kawai, 1988 if $K \leq 0$, and –, 2006)
  - if M is a cylinder, then $K \equiv 0$ (M. Reiris 2010 for $a \geq 1$, and J. Espinar 2011)
- if $a \leq \frac{1}{4}$
  - if $V(R) = O(R^k)$ for some $k < 2$ then $M \overset{conf}{\sim} \mathbb{C}$ or $M \overset{conf}{\sim} \mathbb{C}^*$ (J. Espinar & H. Rosenberg, 2010)

Questions

When $a \leq \frac{1}{4}$,

- Can we relax the hypothesis on the volume growth ?
- Do we have a flatness result for cylinders ?

theorem (P. Bérard & —, 2011)

Let (M, g) be a complete non-compact Riemannian surface. Assume that the operator $\Delta + aK$ is non-negative on M, and that either

i. $a > \frac{1}{4}$, or
ii. $a = \frac{1}{4}$, and (M, g) has subexponential volume growth, or
iii. $a < \frac{1}{4}$, and (M, g) has k_a-subpolynomial volume growth, with $k_a = 2 + \frac{4a}{1-4a}$

Then,

a. The surface (M, g) is conformally equivalent to $\mathbb{C}$ or $\mathbb{C}^*$.
b. If M is a cylinder, then (M, g) is flat.

(M, g) has *subexponential volume growth* if $\limsup_{R\to\infty} \dfrac{\ln V(R)}{R} = 0.$

(M, g) has *k-subpolynomial volume growth* if $\limsup_{R\to\infty} \dfrac{V(R)}{R^k} = 0.$

i. Because of the hyperbolic space, assuming $a > \frac{1}{4}$ is necessary if there is no assumption on the volume growth of M.

ii. The hyperbolic spaces $\mathbb{H}(-b^2)$ satisfy $\Delta + \frac{1}{4}K \geq 0$ and have volume growth $V(R) \sim C\, e^{bR}$.
The hypothesis for $a = \frac{1}{4}$ must rule out the surfaces with arbitrary small exponential growth rate.
The hypothesis of subexponential volume growth is sharp.

iii. The surface $\left(\mathbb{D}, \left(\frac{2}{1-z^2}\right)^{2\alpha} |dz|^2\right)$, with $\alpha = \frac{1}{4a}$, has the following properties :

- completeness because $\alpha \geq 1$
- $\Delta + aK = \Delta + \frac{1}{4\alpha} K \geq 0$
- volume growth $V(R) \sim C\, R^{k_a}$, with $k_a = 2 + \frac{4a}{1-4a}$

The hypothesis for $a < \frac{1}{4}$ must rule out the surfaces with polynomial volume growth of degree k_a.
The hypothesis of k_a-subpolynomial volume growth is sharp.

Fix some point $p_0 \in M$ and consider r the distance function to p_0.

General method : plug some test functions $\xi(r)$ in the quadratic form and use the following estimate due to Pogorelov and Colding-Minicozzi

Technical lemma

If $\xi :]x,y[\to \mathbb{R}$ is C^1, piecewise C^2 with $\xi \geq 0$, $\xi' \leq 0$ and $\xi'' \geq 0$, then

$$\int_{C(x,y)} K\xi^2(r) \leq \xi^2(s)G(s)\Big|_x^y - 2\pi A\xi^2(s)\Big|_x^y + 2\xi(s)\xi'(s)L(s)\Big|_x^y - \int_{C(x,y)} (\xi^2)''(r)$$

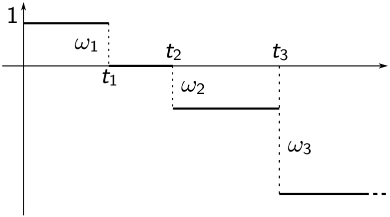
Where

- $G(s) = \int_{B(s)} K$ is the total curvature of the ball $B(s)$
- $L(s) = \mathrm{vol}(\partial B(s))$ is the length of the geodesic circle of radius s
- A is an upper bound of the $\chi(B(s))$, $s \in]x,y[$
- $C(x,y) = \{p \in M \mid x < r(p) < y\} = B(y) - \overline{B(x)}$

Consider the function $\widehat{\chi}(t) = \sup \left\{ \chi\left(B(s)\right) \mid s \in]t, \infty) \right\}$.

- non-increasing, with values in $\mathbb{Z}$
- discontinuities at the points $0 < t_1 < \cdots < t_n < \dots$
- at the discontinuity t_n, a jump $\omega_n = \widehat{\chi}(t_n^-) - \widehat{\chi}(t_n^+) \in \mathbb{N}^*$

Call $\overline{N} \in \mathbb{N} \cup \{\infty\}$ the number of discontinuities.



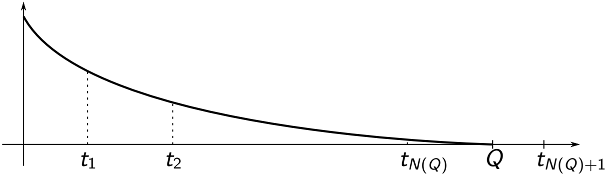
Topological Lemma

Let (M, g) be a complete Riemannian surface and $\chi(M)$ its Euler characteristic (with $\chi(M) = -\infty$ if M does not have finite topology)

$$1 - \sum_{n=1}^{\overline{N}} \omega_n \leq \chi(M)$$

$$\int_{C(x,y)} K\xi^2(r) \leq \xi^2(s)G(s)\Big|_x^y - 2\pi A\xi^2(s)\Big|_x^y + 2\xi(s)\xi'(s)L(s)\Big|_x^y - \int_{C(x,y)} (\xi^2)''(r)$$

For a function $\xi : [0, Q] \to \mathbb{R}$, apply Colding-Minicozzi inequality on each $]t_n, t_{n+1}[$ with $A = \widehat{\chi}(t_n)$.



Technical lemma revisited

If $\xi : [0, Q] \to \mathbb{R}$ is C^1, piecewise C^2, with $\xi \geq 0$, $\xi' \leq 0$, $\xi'' \geq 0$ and $\xi(Q) = 0$, then

$$\int_{B(Q)} K\xi^2(r) \leq 2\pi\left(\xi^2(0) - \sum_1^{N(Q)} \omega_n \xi^2(t_n)\right) - \int_{B(Q)} (\xi^2)''(r)$$

Consider a function $\xi : [0, Q] \to \mathbb{R}$ which is C^1, piecewise C^2 with $\xi \geq 0$, $\xi' \leq 0$, $\xi'' \geq 0$ and $\xi(Q) = 0$. By the technical lemma we have

$$\int_{B(Q)} K\xi^2(r) \leq 2\pi\left(\xi^2(0) - \sum_1^{N(Q)} \omega_n \xi^2(t_n)\right) - \int_{B(Q)} (\xi^2)''(r)$$

The non-negativity of $\Delta + aK$ gives

$$\begin{aligned} 0 &\leq \int_{B(Q)} (\xi')^2(r) + aK\xi^2(r) \\ &\leq 2\pi a\left(\xi^2(0) - \sum_1^{N(Q)} \omega_n \xi^2(t_n)\right) + \int_{B(Q)} (1-2a)(\xi')^2(r) - 2a(\xi\xi'')(r) \end{aligned}$$

In the sequel we give the test functions for the case $a = \frac{1}{4}$.

For $a = \frac{1}{4}$ we have

$$0 \leq \frac{\pi}{2}\left(\xi^2(0) - \sum_{1}^{N(Q)} \omega_n \xi^2(t_n)\right) + \frac{1}{2}\int_{B(Q)} (\xi')^2(r) - (\xi\xi'')(r)$$

Choosing $\xi(r) = e^{-\alpha r} - e^{-\alpha Q}$ gives

$$0 \leq \frac{\pi}{2}\left(\xi^2(0) - \sum_{1}^{N(Q)} \omega_n \xi^2(t_n)\right) + \frac{\alpha^2}{2} e^{-\alpha Q} \int_{B(Q)} e^{-\alpha r}$$

Because M has sub-exponential volume growth, $\int_M e^{-\alpha r} < \infty$, and letting $Q \to \infty$ gives

$$0 \leq \frac{\pi}{2}\left(1 - \sum_{1}^{\overline{N}} \omega_n e^{-2\alpha t_n}\right)$$

with $\alpha \to 0$, using the topological lemma we get

$$0 \leq \frac{\pi}{2}\left(1 - \sum_{1}^{\overline{N}} \omega_n\right) \leq \frac{\pi}{2}\chi(M)$$

To prove that the surface is parabolic it is sufficient to prove that $V(R) \leq C\,R^2$ for some constant C.

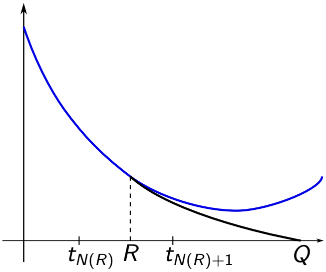
$$0 \leq \frac{\pi}{2}\left(\xi^2(0) - \sum_{1}^{N(Q)} \omega_n \xi^2(t_n)\right) + \frac{1}{2}\int_{B(Q)} (\xi')^2(r) - (\xi\xi'')(r)$$

We want to choose ξ such that the integral term is negative and related to the volume of the ball.

Choose

$$\xi(r) = \begin{cases} e^{\left(1-\frac{r}{2R}\right)^2} & \text{if } r \in [0,R] \\ \beta(e^{-\alpha r} - e^{-\alpha Q}) & \text{if } r \in [R,Q] \end{cases}$$

with α and β such that ξ is C^1.



With this choice of ξ we get

$$\frac{1}{4R^2}\int_{B(R)} e^{2(1-\frac{r}{2R})^2} \leq \frac{\pi}{2}\left(\xi^2(0)-\sum_{1}^{N(Q)}\omega_n\xi^2(t_n)\right)+\frac{\alpha^2\beta^2}{2}e^{-\alpha Q}\int_{C(R,Q)} e^{-\alpha r}$$

and with $Q \to +\infty$

$$\frac{V(R)}{R^2} \leq 2\pi\left(\xi^2(0)-\sum_{1}^{N(R)}\omega_n\xi^2(t_n)\right) \leq C$$

The surface has quadratic volume growth and is parabolic. Moreover, if $M \sim \mathbb{C}^*$, then, for $R > t_1$ we have

$$\frac{V(R)}{R^2} \leq 2\pi\left(e^2 - e^{2(1-\frac{t_1}{2R})^2}\right) \sim_\infty \frac{C}{R}$$

The cylinders have linear volume growth.

The flatness of cylinders is a consequence of the following proposition

proposition

If M is a complete cylinder such that

- $\Delta + aK \geq 0$ for some $a > 0$
- $\limsup_{R \to \infty} \frac{V(R)}{R^2} = 0$

then $K \geq 0$.

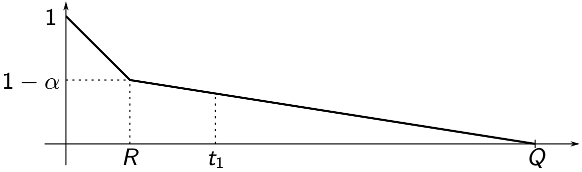
Using Cohn-Vossen inequality we get

$$0 \leq \int_M K \leq 2\pi\chi(M) = 0$$

and $K \equiv 0$

$$\int_{C(x,y)} K\xi^2(r) \leq \xi^2(s)G(s)\Big|_x^y - 2\pi A\xi^2(s)\Big|_x^y + 2\xi(s)\xi'(s)L(s)\Big|_x^y - \int_{C(x,y)} (\xi^2)''(r)$$

To prove the proposition we use the boundary terms with the following test function



$$\begin{aligned} 0 &\leq \int_{B(Q)} (\xi')^2(r) + aK\xi^2(r) \leq \text{Boundary terms } + (1-2a)\int_{B(Q)} (\xi')^2(r) \\ &\leq 2\pi a\Big(1-(1-\alpha)^2\big(\frac{Q-t_1}{Q-R}\big)^2\Big) + 2a(1-\alpha)\frac{R-\alpha Q}{Q-R}\frac{L(R)}{R} \\ &\quad + (1-2a)\alpha^2\frac{V(R)}{R^2} + (1-2a)\big(\frac{1-\alpha}{Q-R}\big)^2\Big(V(Q)-V(R)\Big) \end{aligned}$$

Letting $Q \to +\infty$ gives

$$0 \leq 2\pi a\alpha(2-\alpha) - 2a\alpha(1-\alpha)\frac{L(R)}{R} + (1-2a)\alpha^2\frac{V(R)}{R^2}$$

The functions $L(R)$ and $V(R)$ have the following expansions at 0 :

- $L(R) = 2\pi R\left(1 - \frac{K(p_0)}{6}R^2 + R^2\varepsilon_1(R)\right)$
- $V(R) = \pi R^2\left(1 - \frac{K(p_0)}{12}R^2 + R^2\varepsilon_2(R)\right)$

where $K(p_0)$ is the curvature at p_0. Using these expansions we get

$$0 \leq \alpha^2 + \frac{K(p_0)R^2}{12}\alpha\Big(8a-(1+6a)\alpha\Big) + \alpha R^2\Big((1-2a)\alpha\varepsilon_2(R) - 4a(1-\alpha)\varepsilon_1(R)\Big)$$

dividing by α and letting $\alpha \to 0$ gives

$$0 \leq \frac{2K(p_0)}{3}R^2 - 4aR^2\varepsilon_1(R)$$

which implies that $K(p_0) \geq 0$

Thank you for your attention