Seminario de Geometría, Universidad de Granada

Granada, 1 Abril 2016

Uniqueness of immersed spheres in three-manifolds. Proof of a conjecture by Alexandrov

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Pablo Mira

Joint work with José A. Gálvez, available at arXiv.org

Objective 1: a very general Hopf-type theorem

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Constant mean curvature spheres in $\mathbb{R}^3$ *are round spheres.*

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2. CMC by an arbitrary **elliptic PDE** on each tangent plane.
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(3) $S \subset \mathbb{R}^3$ **strictly convex graph** over a bounded $D \subset \mathbb{R}^2$, that converges asymptotically to $\partial D \times \mathbb{R}$.

$S^* = \Phi(S)$ where Φ is a π-rotation around a horizontal line.

Then $\{S, S^*, \partial D \times \mathbb{R}\}$ and their translations is a candidate family.

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(4) Similar constructions when M is a **homogeneous** 3-manifold.

Rotational symmetry of immersed spheres

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$M =$ simply connected homogeneous 3-manifold, $n = \dim(Iso(M))$.

- $n = 6$: $\mathbb{R}^3$, $\mathbb{S}^3(c)$, $\mathbb{H}^3(c)$, or
- $n = 4$: rotational $\mathbb{E}^3(\kappa, \tau)$ space, or
- $n = 3$: general Lie group with a left invariant metric.

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Using our general theorem, we can extend the Abresch-Rosenberg theorem and prove that for very general elliptic equations

$$\mathcal{W}(H, K, K_{\text{ext}}) = 0, \tag{1}$$

any sphere in $\mathbb{E}^3(\kappa,\tau)$ satisfying (1) is a sphere of revolution.

- Gálvez, — , *Rotational symmetry of immersed spheres in homogeneous three-manifolds*, in preparation (2016).

Objective 2: Alexandrov's conjecture

A surface Σ in $\mathbb{R}^3$ satisfies a *prescribed curvature equation* if

$$W(\kappa_1, \kappa_2, \eta) = 0$$

holds on Σ, where:

- η is the unit normal of Σ,
- $\kappa_1 \geq \kappa_2$ are the principal curvatures,
- W is C^1 for k_1, k_2.

The ellipticity condition of the equation is given by

$$\frac{\partial W}{\partial k_1}\frac{\partial W}{\partial k_2} > 0.$$

Prescribed curvature equations: classical examples

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1. **Weingarten surfaces**: $W(\kappa_1, \kappa_2) = 0$.
2. **Christoffel problem (1844)**: $\frac{1}{\kappa_1} + \frac{1}{\kappa_2} = f(\eta)$.
3. **Minkowski problem (1903)**: $\kappa_1 \kappa_2 = f(\eta)$.
4. **Prescribed mean curvature (1950s)**: $\kappa_1 + \kappa_2 = f(\eta)$.

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The classification of spheres of prescribed curvature has been deeply studied:

(Alexandrov, Pogorelov, Minkowski, Nirenberg, Hopf, Hartman, Wintner, Lewy, Chern...)

The Alexandrov conjecture (1956)

Theorem (Gálvez, —)

Let $S \subset \mathbb{R}^3$ be closed, strictly convex, satisfying the prescribed curvature equation

$$W(\kappa_1, \kappa_2, \eta) = 0, \tag{2}$$

where W satisfies the ellipticity condition $W_{k_1} W_{k_2} > 0$ on

$$\Omega := \{(\lambda\kappa_1^0(p), \lambda\kappa_2^0(p), \eta^0(p) : p \in S, \lambda \in \mathbb{R}\}.$$

Then any other sphere Σ immersed in $\mathbb{R}^3$ that satisfies (2) is a translation of S.

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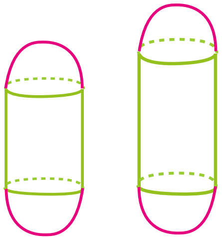
Then any other sphere Σ immersed in $\mathbb{R}^3$ that satisfies (2) is a translation of S.

(The theorem is known if Σ is also strictly convex).

The Alexandrov conjecture (1956)

Remark: The theorem is not true if S does not have positive curvature.

These two (non-strictly) convex spheres in $\mathbb{R}^3$ have the same principal curvatures at points with coinciding unit normals.



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Our proof of Alexandrov's conjecture settles another classical problem: the classification of elliptic Weingarten spheres in $\mathbb{R}^3$: *(Hopf, Chern, Alexandrov, Pogorelov, Hartman-Wintner, ...)*.

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Corollary

Let Σ be an immersed sphere in $\mathbb{R}^3$. Assume that around each umbilical point of Σ, the principal curvatures satisfy a C^1 relation $W(\kappa_1, \kappa_2) = 0$ with $W_{\kappa_1} W_{\kappa_2} > 0$.

Then Σ is a round sphere.

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3. Known if $W(\kappa_1, \kappa_2) = W(\kappa_2, \kappa_1)$ (Hartman-Wintner, 1954).

(Cases like $\kappa_1 + 2\kappa_2 = 3$ were not known beforehand).

A general Hopf uniqueness theorem

Surfaces modeled by elliptic PDEs

A *field of elliptic PDEs* in M assigns to each $(p, \Pi) \in G_2^+(M)$ a second order PDE

$$\Phi_{(p,\Pi)}(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

where $\Phi_{(p,\Pi)} = \Phi(x, y, z, p, q, r, s, t) \in C^{1,\alpha}(\mathcal{U})$ satisfies the ellipticity conditions on a convex $\mathcal{U} \subset \mathbb{R}^8$:

$$\Phi_r > 0, \qquad 4\Phi_r\Phi_t - \Phi_s^2 > 0.$$

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Definition

A class of surfaces $\mathcal{A}$ in M is modeled by an elliptic PDE if its elements are the solutions of a field of elliptic PDEs in M.

The general uniqueness theorem

Theorem (Gálvez, —)

Let $\mathcal{A}$ be a class of oriented surfaces in M. Assume:

1. *$\mathcal{A}$ is modeled by an elliptic PDE.*
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Sketch of Proof: Let $\Sigma \in \mathcal{A}$. We define the smooth tensor

$$\sigma := II - \Lambda : T\Sigma \times T\Sigma \to C^1(\Sigma),$$

where $\Lambda(p)$ is the 2nd fundamental form of $S(p, T_p\Sigma)$ at $p \in \Sigma$.

- $S = S(p, T_p\Sigma) \in \mathcal{S}$ is the tangent candidate of Σ at p.

Properties of $\sigma = II - \Lambda$

1. σ vanishes identically on every candidate surface.
2. $\sigma(q) = 0$ if and only if Σ has contact of order $k \geq 2$ with its tangent candidate at $q \in \Sigma$.

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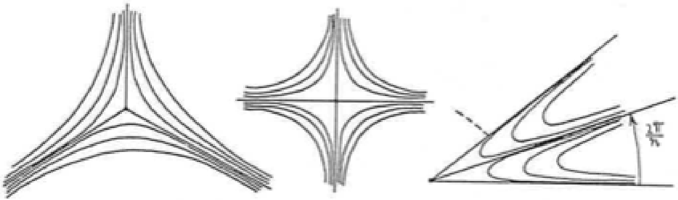
We want to prove that σ is a Lorentzian metric on Σ whose null directions only have isolated zeros of negative index.



9) $j = -\frac{1}{2}$ 10) $j = -1$ 11) $j = \frac{2-n}{2}$, $n \geq 1$

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We write

$$\sigma = II - \Lambda = (II - II_S) + (II_S - \Lambda),$$

where S is the tangent candidate to Σ at some fixed $q \in \Sigma$

We are going to study each term separately.

Expressions for $II - II_S$ and $II_S - \Lambda$

We can parametrize locally Σ, S in a neighborhood of q as immersions

$$\psi(u,v) = (u, v, h(u,v)), \qquad \psi^0(u,v) = (u, v, h^0(u,v)).$$

Let e, f, g and e^0, f^0, g^0 be the coefficients of II and II_S.

A computation shows that

$$e - e^0 = \Phi_1^e(h - h^0) + \Phi_2^e(h_u - h_u^0) + \Phi_3^e(h_v - h_v^0) + \Phi_4^e(h_{uu} - h_{uu}^0),$$

where the functions $\Phi_i^e(u,v)$ are smooth a neighborhood of $(0,0)$.

Similar formulas hold for the other coefficients f, g of the second fundamental form.

Expressions for $II - II_S$ and $II_S - \Lambda$

As both $h(u,v)$, $h^0(u,v)$ are solutions of the same $C^{1,\alpha}$ elliptic PDE, by Bers' theorem, we have $h = h^0$ or

$$h(u,v) - h^0(u,v) = p(u,v) + \cdots ,$$

where $p(u,v)$ is a homogeneous harmonic polynomial of degree $k \geq 2$. Thus

$$II - II_S = \begin{pmatrix} p_{uu} & p_{uv} \\ p_{uv} & p_{vv} \end{pmatrix} + o(\sqrt{u^2+v^2})^{k-2}.$$

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For the blue part in

$$\sigma = II - \Lambda = (II - II_S) + (II_S - \Lambda)$$

we can prove similarly that

$$II_S - \Lambda = o(\sqrt{u^2 + v^2})^{k-2}.$$

Completion of the proof

- If $\sigma \not\equiv 0$, we have proved that

$$\sigma = \begin{pmatrix} p_{uu} & p_{uv} \\ p_{uv} & p_{vv} \end{pmatrix} + o(\sqrt{u^2 + v^2})^{k-2}$$

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- If $\sigma \not\equiv 0$, σ is a Lorentzian metric with **isolated singularities** on $\Sigma - \{q : \sigma(q) = 0\}$, whose null lines coincide with the asymptotic directions in $\mathbb{R}^3$ of the graph of $p(u, v)$.

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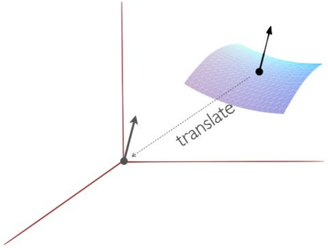
- If $\sigma \not\equiv 0$, σ is a Lorentzian metric with **isolated singularities** on $\Sigma - \{q : \sigma(q) = 0\}$, whose null lines coincide with the asymptotic directions in $\mathbb{R}^3$ of the graph of $p(u, v)$.

- By harmonicity of $p(u, v)$, the index of these line fields is **negative** at every singularity.
  (Impossible if Σ is diffeomorphic to $\mathbb{S}^2$, by **Poincaré-Hopf**).
  Thus, Σ is a **candidate sphere** S (since $\sigma \equiv 0$).

Any simply connected homogeneous three-manifold other than $\mathbb{S}^2(\kappa) \times \mathbb{R}$ is a **Lie group** M with a left invariant metric.

The Gauss map η of a surface in such an M is obtained by left-translating its unit normal to the Lie algebra at e.

It takes its values in $\mathbb{S}^2 \subset T_eM$.



A geometric application

M = three-dimensional Lie group with a left-invariant metric.

Corollary

Let Σ_1, Σ_2 be immersed spheres in M such that

$$\mathcal{W}(\kappa_1, \kappa_2, \eta) = 0$$

where $\mathcal{W}$ is $C^{1,\alpha}$, symmetric in κ_1, κ_2, and satisfies the ellipticity condition

$$\frac{\partial \mathcal{W}}{\partial k_1}\frac{\partial \mathcal{W}}{\partial k_2} > 0.$$

Then, if the Gauss map of Σ_1 is a diffeomorphism, Σ_2 is a left translation of Σ_1.

(Uniqueness of Christoffel-Minkowski problems in Lie groups).

Theorem (Guan-Guan, 2002)

Let $\mathcal{H} \in C^2(\mathbb{S}^2)$ satisfy $\mathcal{H}(-x) = \mathcal{H}(x) > 0$ for all $x \in \mathbb{S}^2$.

Then, there exists a closed strictly convex sphere $S_{\mathcal{H}} \subset \mathbb{R}^3$ of prescribed mean curvature $\mathcal{H}$, i.e.

$$H_\Sigma = \mathcal{H}(\eta),$$

where $\eta : \Sigma \to \mathbb{S}^2$ is the unit normal of Σ.

Prescribed mean curvature spheres in $\mathbb{R}^3$

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Theorem (Gálvez, —)

Let Σ be an immersed sphere in $\mathbb{R}^3$ with prescribed mean curvature $\mathcal{H} \in C^2(\mathbb{S}^2)$, where

$$\mathcal{H}(-x) = \mathcal{H}(x) > 0 \qquad \forall x \in \mathbb{S}^2.$$

Then Σ is the Guan-Guan sphere $S_{\mathcal{H}}$.

Proof of Alexandrov's conjecture

The Alexandrov conjecture (1956)

Theorem (Gálvez, —)

Let $S \subset \mathbb{R}^3$ be closed, strictly convex, satisfying the prescribed curvature equation

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where W satisfies the ellipticity condition $W_{k_1} W_{k_2} > 0$ on

$$\Omega := \{(\lambda\kappa_1^0(p), \lambda\kappa_2^0(p), \eta^0(p)) : p \in S, \lambda \in \mathbb{R}\}.$$

Then any other sphere Σ immersed in $\mathbb{R}^3$ that satisfies (3) is a translation of S.

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Then any other sphere Σ immersed in $\mathbb{R}^3$ that satisfies (3) is a translation of S.

- **Difficulty 1:** the ellipticity condition is not global.
- **Difficulty 2:** Equation (3) is not $C^{1,\alpha}$ (problem at umbilics).

Step 1: the Legendre transform

The Legendre transform of $\psi = (\psi_1, \psi_2, \psi_3)$ is:

$$L_\psi = \left(-\frac{\eta_1}{\eta_3}, -\frac{\eta_2}{\eta_3}, -\psi_1 \frac{\eta_1}{\eta_3} - \psi_2 \frac{\eta_2}{\eta_3} - \psi_3 \right).$$

If ψ is a graph of positive curvature, so is L_ψ: $(x, y, h(x, y))$.

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If ψ is a graph of positive curvature, so is L_ψ: $(x, y, h(x, y))$.

With respect to the parameters (x, y), we have for $\psi = \psi(x, y)$:

- The unit normal is
$$\eta = \frac{(-x, -y, 1)}{\sqrt{1 + x^2 + y^2}}.$$
- The second fundamental form is $II = \frac{1}{\sqrt{1+x^2+y^2}} \nabla^2 h$.
- The principal curvatures are $\kappa_i = \Phi^i(x, y, h_{xx}, h_{xy}, h_{yy})$.

Step 2: a discontinuous linear elliptic PDE

Take $p \in \Sigma$ with $\sigma(p) = 0$ and $\sigma \not\equiv 0$.

$(x, y, h(x, y))$ and $(x, y, h^0(x, y))$ = Legendre transforms of Σ, S.

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That both Σ, S satisfy $W(\kappa_1, \kappa_2, \eta) = 0$ is rewritten as

$$F(x, y, h_{xx}, h_{xy}, h_{yy}) = F(x, y, h^0_{xx}, h^0_{xy}, h^0_{yy}),$$

uniformly elliptic on compact sets with $0 < c \leq W_{k_i} \leq C$.

Step 2: a discontinuous linear elliptic PDE

Take $p \in \Sigma$ with $\sigma(p) = 0$ and $\sigma \not\equiv 0$.

$(x, y, h(x, y))$ and $(x, y, h^0(x, y))$ = Legendre transforms of Σ, S.

That both Σ, S satisfy $W(\kappa_1, \kappa_2, \eta) = 0$ is rewritten as

$$F(x, y, h_{xx}, h_{xy}, h_{yy}) = F(x, y, h^0_{xx}, h^0_{xy}, h^0_{yy}),$$

uniformly elliptic on compact sets with $0 < c \leq W_{k_i} \leq C$.

Lemma

The function $\varrho := h - h^0$ satisfies around the origin a uniformly elliptic **linear** *homogeneous PDE*

$$a_{11}\varrho_{xx} + 2a_{12}\varrho_{xy} + a_{22}\varrho_{yy} = 0,$$

where $a_{ij} = a_{ij}(x, y)$ are maybe not continuous.

The Bers-Nirenberg representation

Theorem (Bers-Nirenberg)

Let $u(x,y)$ be a non-linear solution on $\mathbb{D}$ to

$$L[u] \equiv a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0,$$

where $L[u]$ is uniformly elliptic (maybe not continuous). Then

$$u_x - iu_y = F(\varphi(x,y)),$$

where

1. *F is a holomorphic map.*
2. *φ is a C^α homeomorphism, whose inverse is also C^α.*

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Consequence 1: The zeros of u_x, u_y are of finite order.

Consequence 2: u_x, u_y have no interior extrema.

Step 3: σ has isolated zeros of non-positive index

Define again $\sigma := II - \Lambda$. Note that Λ is a Riemannian metric on Σ.

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Let $p \in \Sigma$ where $\sigma(p) = 0$ but $\sigma \not\equiv 0$ around p. Then:

$$\sigma = \frac{1}{\sqrt{1 + x^2 + y^2}} \begin{pmatrix} \varrho_{xx} & \varrho_{xy} \\ \varrho_{xy} & \varrho_{yy} \end{pmatrix}, \qquad \varrho := h - h^0.$$

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Thus $w_{xx}w_{yy} - w_{xy}^2 < 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$.

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Thus, $\varrho_{xx}\varrho_{yy} - \varrho_{xy}^2 < 0$ on $D^*(0,\delta)$, i.e.

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The index of the null lines of σ at the origin is ≤ 0 if and only if the index of $\nabla\varrho_x := (\varrho_{xx}, \varrho_{xy})$ is ≤ 0.

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Proof.

By Bers-Nirenberg, ϱ_x has no local extrema. So, the origin is a saddle point of $\varrho_x \Rightarrow \mathrm{Index}(\nabla\varrho_x) \leq 0$. $\square$

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The null lines of σ have isolated singularities of index ≤ 0.

Step 4: A global line field with isolated singularities

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$$\Lambda(T(X), Y) = \sigma(X, Y).$$

The **eigenspaces** $\mathcal{L}_1, \mathcal{L}_2$ of T define on $\Sigma - \{\sigma = 0\}$ two smooth line fields with isolated singularities at the points $\sigma = 0$.

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$\Longrightarrow$ the index of $\mathcal{L}_i$ at the isolated zeros of σ is **non-positive** (since the index of the null directions of σ is non-positive).

(Impossible if Σ diffeomorphic to $\mathbb{S}^2$) END.

Two related open problems

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(1) Consider a warped product 3-space $\mathbb{Q}^2(c) \times_f \mathbb{R}$ with metric

$$\langle , \rangle = f(t) \langle , \rangle_{\mathbb{Q}^2(c)} + dt^2, \qquad f > 0.$$

Problem

Are CMC spheres in $\mathbb{Q}^2(c) \times_f \mathbb{R}$ spheres of revolution?

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(2) A self-similar shrinker of the mean curvature flow is a surface $x : \Sigma \to \mathbb{R}^3$ satisfying $H_\Sigma = \frac{1}{2}\langle x, \eta \rangle$.

Conjecture

Self-similar shrinkers diffeomorphic to $\mathbb{S}^2$ are spheres of revolution.

(Proved by Brendle in the embedded case).