Area Preserving Transformations and Classical Differential Geometry

P.Roitman, joint work with W.Ferreira

GRANADA, 2012

Outline

Small Gallery

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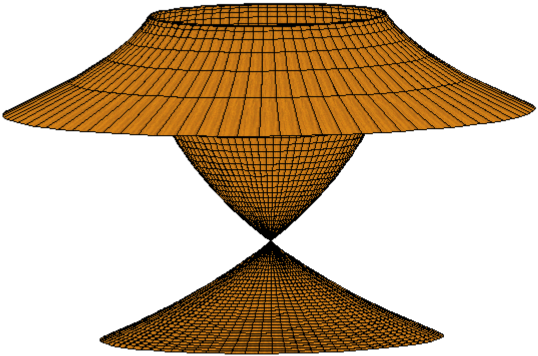
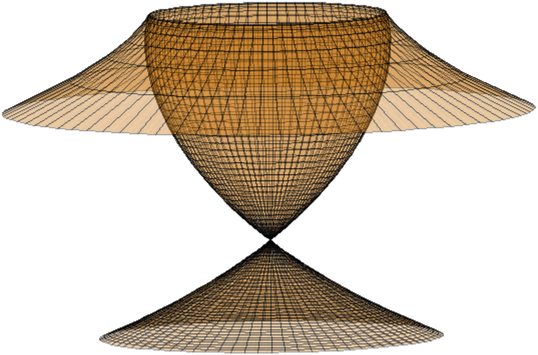
It's a pitty that they're not here to see some pictures... .

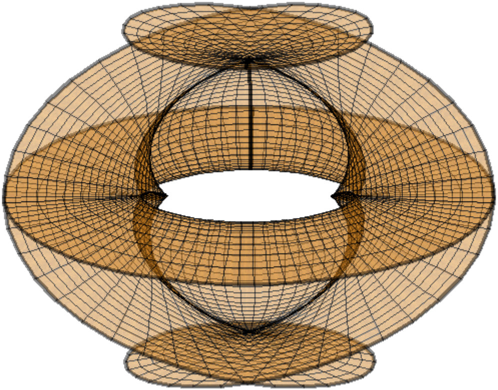
Rotational Appel-Bonnet Surface



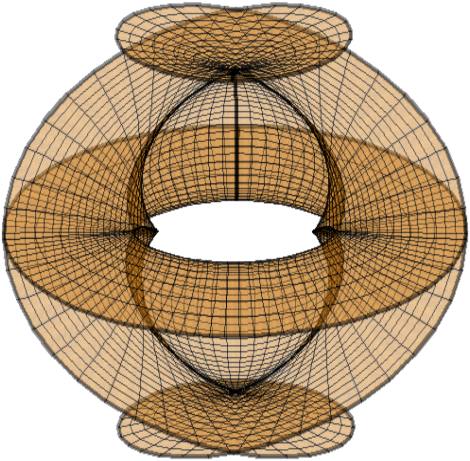
Rotational Bonnet Surface



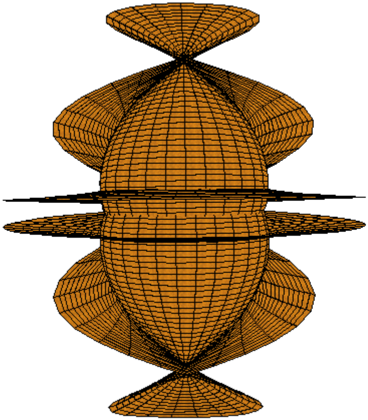
Non-trivial example of a Bonnet Surface



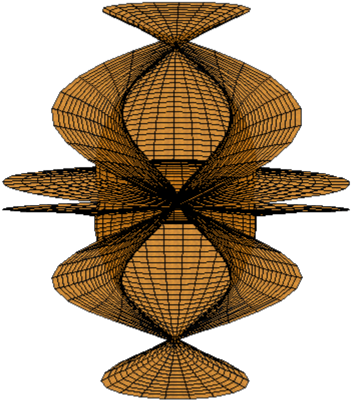
in motion...



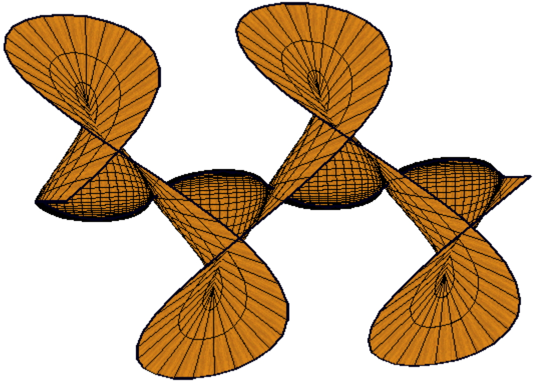
Appell Surface (Pollo Asado, according to my daughter)



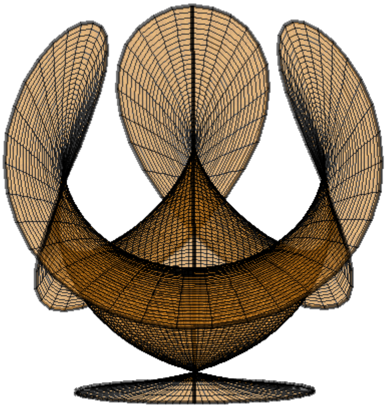
Rotated...



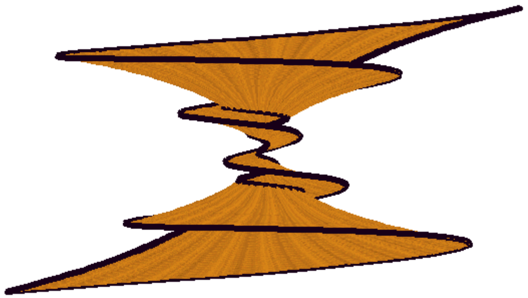
Another Bonnet (1-periodic)



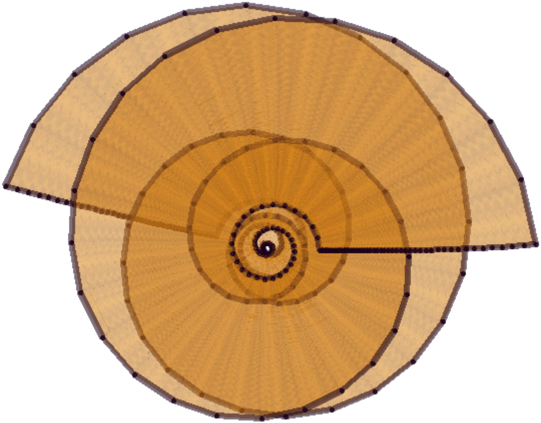
Another Appell



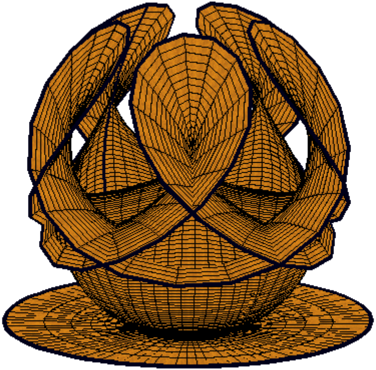
Appell Spiral



Top View



One more Appell



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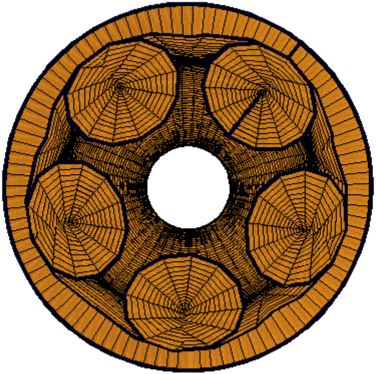
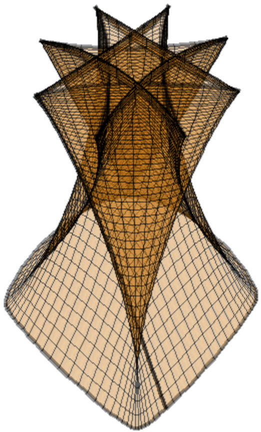


A "Bonnet Star" (Tarea: find it in Alhambra)



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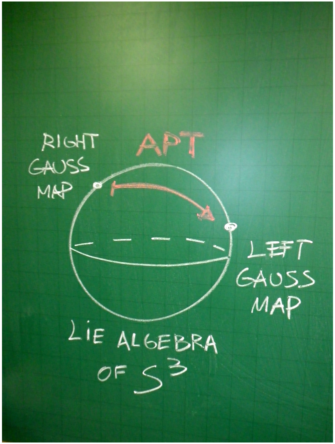

APT and Gauss Maps in $\mathbb{S}^3$

Nice Fact

The correspondence defined by the right and left Gauss maps is an APT.

Scheffers' method to generate APT in the plane

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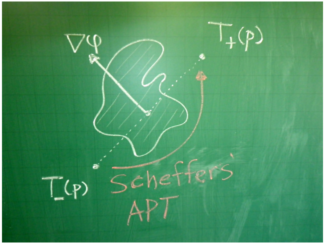
Let $\varphi : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a differentiable function.
Consider the following maps

$$T_{-}(x, y) = (x - \varphi_y, y + \varphi_x)$$

and

$$T_{+}(x, y) = (x + \varphi_y, y - \varphi_x)$$

Scheffers APT



Sheffers' APT in the plane

The map $T_+ \circ T_-^{-1}$ is an APT (up to degenerate cases).

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Let M denote either $\mathbb{S}^2$ or $\mathbb{H}^2$, the hyperbolic plane, and $\varphi : \Omega \subset M \longrightarrow \mathbb{R}$ be a differentiable function.
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and

$$T_+(p) = exp_p(J\nabla\varphi(p)))$$

After some computation we obtain the following:

Generalization of Scheffers' method

The map $T_+ \circ T_-^{-1}$ is an APT (up to degenerate cases).

Some concepts from classical differential geometry

To appreciate the connections of the APT generated a la Scheffers with some special surfaces we need to recall some classical concepts.

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The Middle Surface

Let Σ be an oriented surface in $\mathbb{R}^3$ with H and K the mean and Gaussian curvatures. For $p \in \Sigma$ consider the point $M(p)$ along the normal line to Σ at p that we get by moving from p a signed distance $\frac{H}{K}$ along the normal line.

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In other words:

$$M(p) = X(p) + \frac{H}{K} N(p)$$

This defines a surface called the Middle Surface of the congruence of lines normal to Σ.

Middle Plane and Middle Spheres

The plane passing through $M(p)$ and orthogonal to $N(p)$ is called the Middle Plane at p.

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The congruence of spheres centered at M and with radius $\left|\frac{H}{K}\right|$ is called the Middle Sphere Congruence.

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Bonnet Surfaces

Bonnet surfaces are defined as surfaces for which the middle surface is contained in a plane.

Appell surfaces and APT

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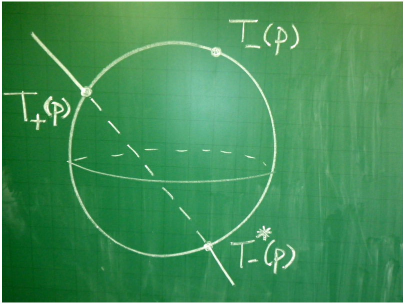

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So, we may generate examples of Appell surfaces using harmonic functions.

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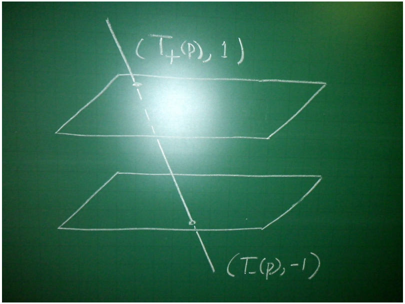

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So, we can generate examples of Bonnet surfaces with minimal surfaces.

And what about the $\mathbb{H}^2$ case?

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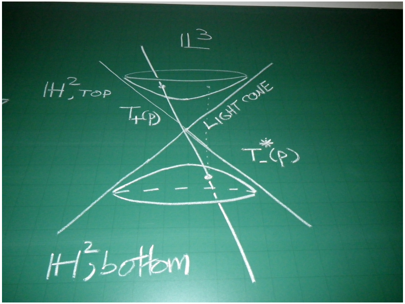

You may guess what happens...

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- Study global aspects of Appell and Bonnet surfaces.
- Study their singularities.
- Generalization to other spaces...maybe CP^n with the Fubini-Study metric.
- Use APT's a la Schefers to produce Lagrangian immersions in product spaces...

REFERENCES

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