Surfaces immersed in a half-space with the same Gaussian curvature induced by the Euclidean and Hyperbolic metrics

Pedro Roitman (Univ. de Brasília)
joint with Nilton Barroso

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Terminology

Surfaces such that $K_h = K_e$ will be called *isocurved surfaces*.

Some simple examples

- horizontal planes (Horospheres)

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Question

Are there other examples? How to construct them?

The associated PDE

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and for an isocurved graph we have the PDE:

$$A(\varphi_{uu}\varphi_{vv} - \varphi_{uv}^2) + B\varphi_{uu} + D\varphi_{vv} + 2C\varphi_{uv} + E = 0, \qquad (2)$$

where $A = (1-\varphi^2)$, $B = -\varphi(1+\varphi_v^2)$, $C = \varphi\varphi_v\varphi_u$, $D = -\varphi(1+\varphi_u^2)$, and $E = (1+\varphi_u^2+\varphi_v^2)(\varphi_u^2+\varphi_v^2)$.

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Note that it seems difficult to find explicit solutions without symmetry assumptions.

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- eliptic if $\Delta < 0$,
- hiperbolic if $\Delta > 0$,
- parabolic if $\Delta = 0$.

Remark

The PDE for isocurved graphs is a mixed type equation

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Intuition

Since isocurved surfaces are defined using both metrics, maybe we should think of a geometric construction that involves both geometries.

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This procedure defines a congruence of geodesics: C_Σ.

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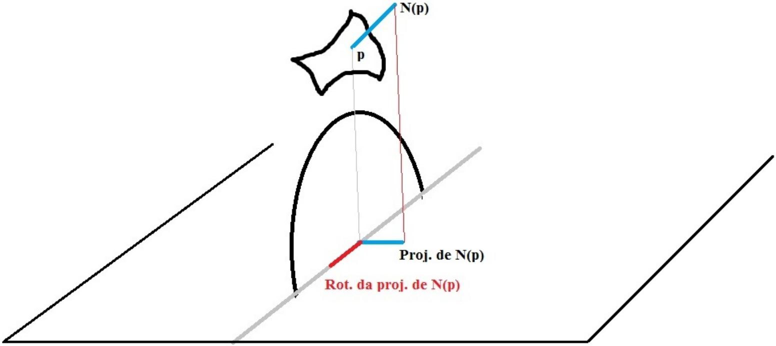

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Intriguing Property

If Σ is minimal, then the orthogonal surfaces of C_Σ are isocurved (at smooth points).

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A local parametrization of such surfaces would look like:

$$\mathbf{Y} = \boldsymbol{\sigma} + R\left(\cos\theta \mathbf{e_1} + \sin\theta \mathbf{e_3}\right), \tag{4}$$

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where $\boldsymbol{\sigma}$ is the center of the circle and so on...
The orthogonality condition is equivalent to

$$\langle d\mathbf{Y}, -\sin\theta \mathbf{e_1} + \cos\theta \mathbf{e_3}\rangle = 0,$$

where $\langle\,,\rangle$ stands for the euclidean inner product.

the integrable system

After some computations...we arrive at this system

$$\theta_u = \frac{\sin\theta}{R}\langle \boldsymbol{\sigma}_u, \mathbf{e_1}\rangle, \quad \theta_v = \frac{\sin\theta}{R}\langle \boldsymbol{\sigma}_v, \mathbf{e_1}\rangle \tag{5}$$

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It is better to make the following change of variables:
$\sin\theta = 1/\cosh\beta$ e $\cos\theta = \tanh\beta$, and work with the equations:

$$\beta_u = -\frac{\langle \boldsymbol{\sigma}_u, \mathbf{e_1}\rangle}{R}, \quad \beta_v = -\frac{\langle \boldsymbol{\sigma}_v, \mathbf{e_1}\rangle}{R}, \tag{6}$$

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Frobenius' theorem gives us the integrability condition:

$$\left(\frac{\langle \sigma_u, \mathbf{e_1} \rangle}{R}\right)_v = \left(\frac{\langle \sigma_v, \mathbf{e_1} \rangle}{R}\right)_u. \tag{7}$$

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Since

$$\boldsymbol{\sigma}(u,v) = (u,v,0),$$

$$R = \frac{\sqrt{1+\psi_u^2+\psi_v^2}}{\sqrt{\psi_u^2+\psi_v^2}},$$

$$\mathbf{e_1} = \frac{(-\psi_v, \psi_u)}{\sqrt{\psi_u^2+\psi_v^2}},$$

it follows from simple computations that (7) is equivalent to

$$(1+\psi_v^2)\psi_{uu} - 2\psi_u\psi_v\psi_{uv} + (1+\psi_u^2)\psi_{vv} = 0,$$

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- Integration can be difficult.

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- If we start with an immersion Y into $\mathbb{R}^3_+$ and ask if Y comes from a surface by the inverse process then Y must be isocurved.
- Stricktly speaking, we actually don't need a surface to start our geometric construction, all we need is a smooth two parameter family of contact elements of $\mathbb{R}^3$.

Geometric Remark

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$$\mathbf{X}(x, y) = (\alpha(x) \sin y, \alpha(x) \cos y, \gamma(x)),$$

where

$$\alpha(x) = \frac{1}{4} \frac{\left(-\mathrm{e}^{5x} + 8\,\mathrm{e}^{3x} + 45\,\mathrm{e}^{x} - 8\,\mathrm{e}^{-x} + 16\,\mathrm{e}^{-3x}\right)}{\left(4\,\mathrm{e}^{-2x} + 1\right)\left(\mathrm{e}^{4x} + 1\right)},$$

and

$$\gamma(x) = \frac{1}{2} \frac{\mathrm{e}^{3x}\sqrt{1 + 8\,\mathrm{e}^{-2x} + 16\,\mathrm{e}^{-4x}}}{\mathrm{e}^{4x} + 1}.$$

Surface of Revolution

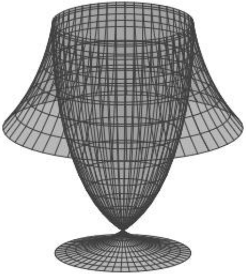

Figure: Isocurved of revolution

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From our method we get the following isocurved parametrized surface:

$$\mathbf{X}(x, y) = (x - \Lambda_1 \sin y \cos x, y - \Lambda_1 \sin x \cos y, \Lambda_2),$$

where

$$\Lambda_1 = \frac{\sqrt{\cos^2 x + \cos^2 y - \cos^2 x \cos^2 y}\tanh(\arcsin(\sin x \sin y))}{\sin^2 x \cos^2 y + \sin^2 y \cos^2 x},$$

and

$$\Lambda_2 = \frac{\sqrt{\cos^2 x + \cos^2 y - \cos^2 x \cos^2 y}}{\cosh(\arcsin(\sin x \sin y))\sqrt{\cos^2 x + \cos^2 y - 2\cos^2 x \cos^2 y}}.$$

Scherk type

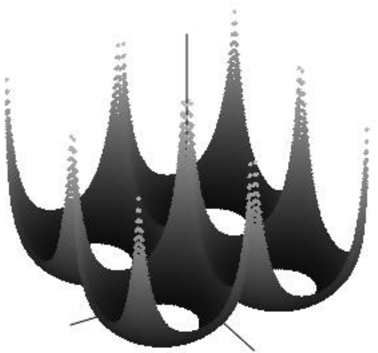

Figure: 2-periodic Scherk type Isocurved

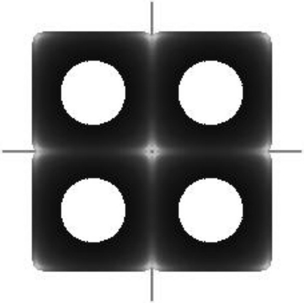

Figure: Top view

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- However, locally we can generate all the elliptic isocurved surfaces.
- The hyperbolic isocurved surfaces are generated in a similar way using timelike minimal surfaces in $\mathbb{L}^3$.
- There are also parabolic examples and probably a degenerate type of minimal surface can be associated to it.

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Muchas Gracias!!