Discrete Differential Geometry

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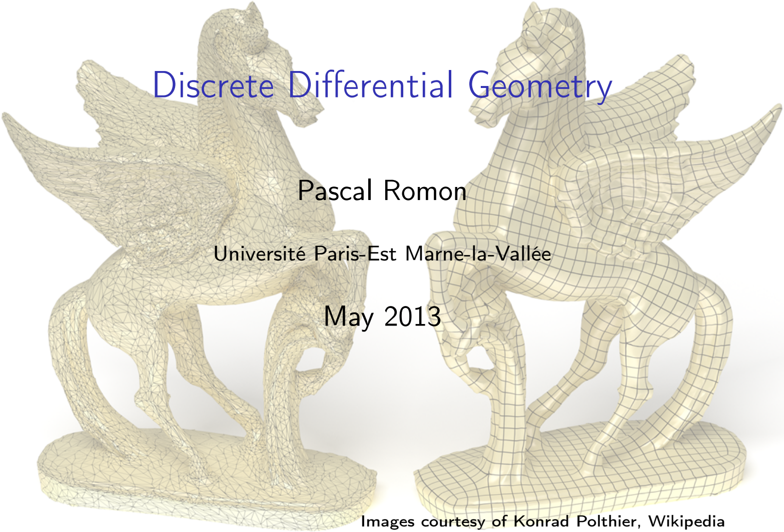

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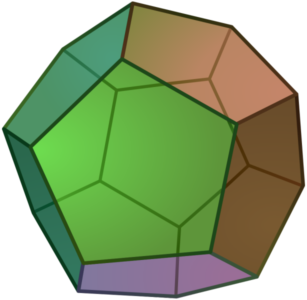
Introduction

Discrete differential geometry

- Deals with discrete geometric objects: points (vertices), segments (edges), faces, polygons, polyhedra.
- Finds properties and invariants (affine or metric)
- Focuses on differential quantities (such as curvature)



How do we discretize derivatives?

- Tangent vectors (s is arclength)

$$t(s) = \frac{d\gamma}{ds} = \lim_{h\to 0} \frac{\gamma(s+h) - \gamma(s)}{h} \longleftrightarrow t_i = \frac{p_{i+1} - p_i}{\ell_i}$$

- Curvature in $\mathbb{R}^2$ (change in the tangent direction)

$$\kappa(s)n(s) = \frac{dt}{ds} \overset{?}{\longleftrightarrow} \kappa_i n_i = \frac{t_{i+1} - t_i}{\ell_i}$$

  (note the lack of symmetry!)
- Curvature of surfaces in $\mathbb{R}^3$?

Smooth gaussian curvature

Gaussian curvature K at p measures

- Length of infinitesimal circle at p:

$$K = \lim_{r \to 0} \frac{3}{\pi r^3}(2\pi r - L(C(r)))$$

- variation of the normal vector w.r.t. the metric

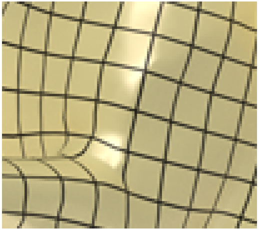
$$K = \frac{\det II}{\det I} = \frac{\begin{vmatrix} -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2} & \frac{E_u}{2} & F_u - \frac{E_v}{2} \\ F_v - \frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}}{(EG - F^2)^2}$$

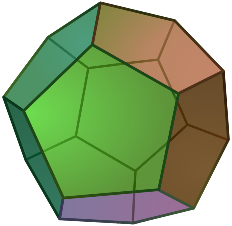
- spherical area of the normal

Curvature and derivatives in multiple variables

Problems:

- what are I and II?
- how can one take derivatives when there are no notion of "two variables"?



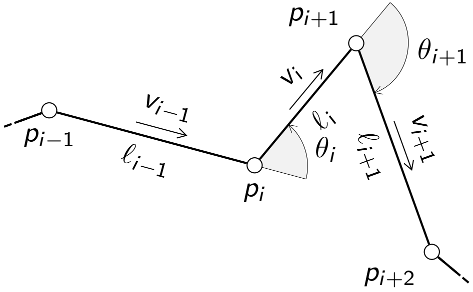


Conclusion: describe geometric invariants by their properties.

Discrete curves in the plane

A (connected) curve is set of vertices $p_1, \ldots, p_n$ linked by edges $(p_i p_{i+1})$.
Its (euclidean) invariants are:

- lengths $\ell_i = \|p_{i+1} - p_i\|$, defined on edges
- tangent vectors $v_i = \frac{p_{i+1} - p_i}{\ell_i}$
- rotation angles $\theta_i = (v_{i-1}, v_i)$



Bonnet theorem

Theorem (Bonnet theorem, smooth case)

Given a function κ, there exists a curve γ whose curvature is exactly κ. Such a curve is unique up to congruence.

Solve in arclength coordinate s, using Cauchy–Lipschitz theorem:

$$\frac{d^2\gamma}{ds^2} = \kappa J \frac{d\gamma}{ds},$$

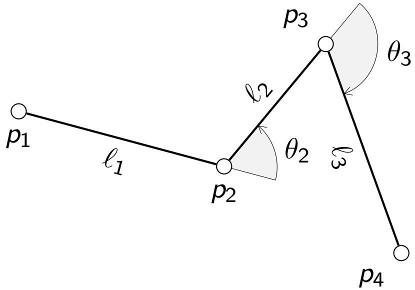
where J is the rotation by $+\pi/2$. The (local) solution is determined by initial data $\gamma(0)$, $\gamma'(0)$, hence up to congruence (equivariance of the equation).

Global existence is more complicated.

Bonnet theorem (discrete case)

Theorem

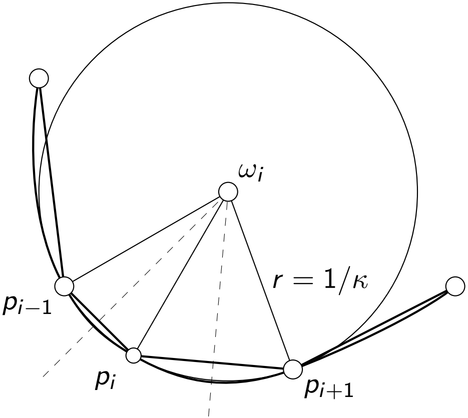
Given angles θ_i and lengths ℓ_i, there exists a discrete curve (p_i) corresponding to these. Such a curve is unique up to congruence.



Curvature as closest fitting circle

Menger curvature:

$$\kappa_i^M = \frac{2\sin\theta_i}{\|p_{i+1} - p_{i-1}\|}$$



Variation of tangent or normal vectors

In the smooth case, $\kappa = \langle dt/ds, Jt \rangle = \frac{d\phi}{ds}$, where ϕ is the angle of the tangent t (or normal n). For discrete curves, since $\phi_i - \phi_{i-1}$,

$$\kappa_i^\theta = \frac{2\theta_i}{\ell_{i-1} + \ell_i} \quad \text{or} \quad \kappa_i^n = \frac{2\langle t_i - t_{i-1}, n_i \rangle}{\ell_{i-1} + \ell_i} = \frac{4 \sin \frac{\theta_i}{2}}{\ell_{i-1} + \ell_i}$$

As a consequence, we recover Gauss-Bonnet

$$\theta(b) - \theta(a) = \int_a^b \kappa(s)\, ds$$

$$\text{total angular variation} = \sum_i \theta_i = \sum_i \kappa_i^\theta \, \frac{\ell_{i-1} + \ell_i}{2}$$

Variation of length

Consider the length $L = \sum_i \ell_i = \sum_i \|p_i - p_{i-1}\|$.

$$\nabla_{p_i} L = \nabla_{p_i} \|p_i - p_{i-1}\| + \|p_{i+1} - p_i\| = v_{i-1} - v_i$$

Mean curvature vector $\vec{H}_i := \frac{2(v_i - v_{i-1})}{\ell_{i-1}+\ell_i} = \kappa_i^L n_i$ with

$$\kappa_i^L = \kappa_i^n = \frac{4 \sin \frac{\theta_i}{2}}{\ell_{i-1} + \ell_i}$$

First variation of area, when $p_i \to p_i' = p_i + \epsilon u_i$ is

$$L' = L - \epsilon \sum_i \kappa_i^L \langle u_i, n_i \rangle + o(\epsilon)$$

Approximation properties

Convergence when points are close enough?

- γ is the graph of $x \mapsto y(x) = \frac{c}{2}x^2 + o(x^2)$
- $p_{i-1} = (-\tau_{i-1}, \frac{c}{2}\tau_{i-1}^2 + o(\tau_{i-1}^2))$, $p_{i+1} = (\tau_i, \frac{c}{2}\tau_i^2 + o(\tau_i^2))$ with positive $\tau_i, \tau_{i-1} \in O(\tau)$, $\tau \to 0$.

$$\ell_i = \tau_i \left(1 + \frac{c^2}{8}\tau_i^2 + o(\tau_i^2)\right), \quad \ell_{i-1} = \tau_{i-1}\left(1 + \frac{c^2}{8}\tau_{i-1}^2 + o(\tau_{i-1}^2)\right)$$

$$\overrightarrow{p_{i-1}p_i} \cdot \overrightarrow{p_i p_{i+1}} = \tau_{i-1}\tau_i \left(1 - \frac{c^2}{4}\tau_i\tau_{i-1} + o(\tau_{i-1}\tau_i)\right)$$

$$\begin{aligned}
\cos\theta_i &= \frac{\overrightarrow{p_{i-1}p_i} \cdot \overrightarrow{p_i p_{i+1}}}{\ell_i \ell_{i-1}} = \frac{\left(1 - \frac{c^2}{4}\tau_i\tau_{i-1} + o(\tau_{i-1}\tau_i)\right)}{\left(1 + \frac{c^2}{8}\tau_{i-1}^2 + o(\tau_{i-1}^2)\right)\left(1 + \frac{c^2}{8}\tau_i^2 + o(\tau_i^2)\right)} \\
&= 1 - \frac{c^2}{8}(\tau_i + \tau_{i-1})^2 + o(\tau^2) = 1 - \frac{\theta_i^2}{2} + o(\tau^2)
\end{aligned}$$

$$\theta_i \sim \frac{c}{2}(\tau_{i-1} + \tau_i)$$

Discrete surfaces

An (abstract) discrete (polyhedral) surface S is

- a set V of vertices
- a set E of (unordered) edges $e = (pq)$, $p, q \in V$, $p \neq q$
- a set F of faces $f = (p_1, \ldots, p_n = p_1)$, where $(p_i p_{i+1}) \in E$

such that

1. every vertex p belongs to 3 edges (2 on the boundary)
2. every edge e belongs to 2 faces (1 on the boundary)
3. every face f has at least 3 vertices (edges)
4. neighbors of p form a closed curve (except on the boundary)
5. boundary curves are closed

Metric structure

A metric structure on S is given by

- identification of each face f with a (non-unique) planar polygon (chart), with compatible edge length
- by the lengths $\ell(e)$ of each edge e, and angles between adjacent edges, denoted $\alpha_f(p)$.

Remarks:

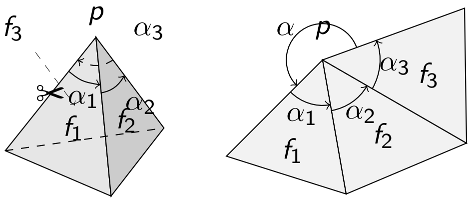
- for a triangle, lengths suffice
- for general polygons, lengths and angles are needed, and they have to satisfy a integrability relation

$$\sum_{j=1}^{n} \ell_j e^{i \sum_{k=1}^{j} \theta_k} = 0$$

where $\theta_k = \pi - \beta_k$ is the rotation angle.

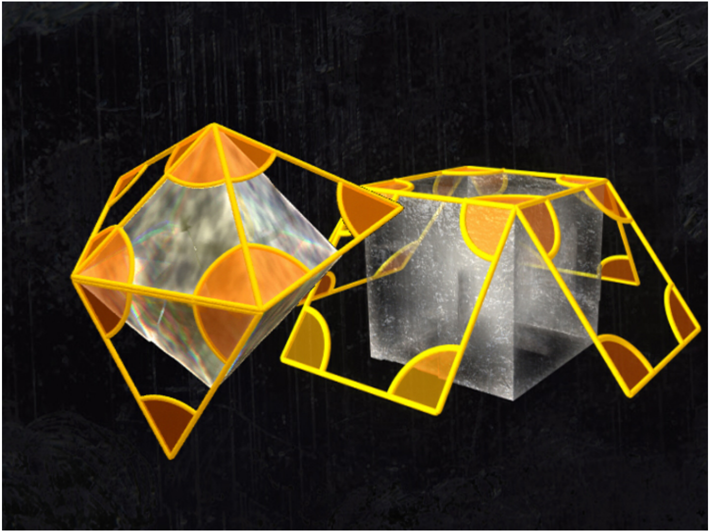
Intrinsic distance and unfolding

Unfolding = constructing isometric faces with connected edges.



The intrinsic metric inside S is flat with conical singularities ($\Rightarrow$ straight lines across edges).
The conical singularity is measured by the angle defect (impossibility to close the unfolded faces) $\alpha = 2\pi - \sum \alpha_k$. Also recover gaussian curvature via : length of circle around p.

Unfolding preserves intrinsic data

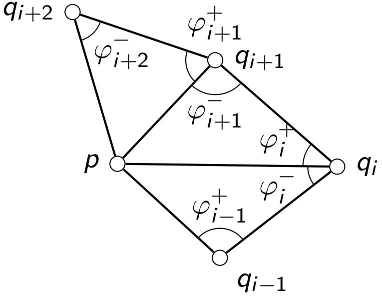


Gauss–Bonnet around p

Apply the Gauss–Bonnet formula to faces touching p (aka star(p)):

$$\int K\, dA + \int \kappa_g\, d\ell = 2\pi$$

$$\begin{aligned}\int \kappa_g\, d\ell &= \sum_i \theta_i \\ &= \sum_i (\pi - \phi_i^+ - \phi_i^-) \\ &= \sum_i (\pi - \phi_i^+ - \phi_{i-1}^-) \\ &= \sum_i \alpha_i\end{aligned}$$



We define the discrete gaussian curvature by $K(p) = \frac{3\,\alpha}{\text{area(star}(p))}$

Global discrete Gauss–Bonnet

We can directly prove

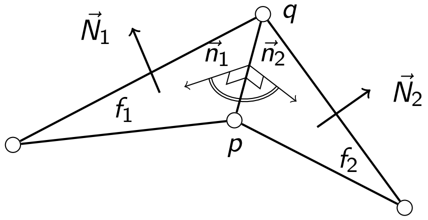
$$\sum_p K(p) = 2\pi\chi = 2\pi(|V| - |E| + |F|)$$

For a surface without boundary,

$$\begin{aligned}
\sum_p \alpha(p) &= \sum_p (2\pi - \sum_{f \ni p} \alpha_f(p)) = 2\pi|V| - \sum_f \sum_{p \in f} \alpha_f(p) \\
&= 2\pi|V| - \sum_f (|f| - 2)\pi = 2\pi|V| + 2\pi|F| - \pi \sum_f |f| \\
&= 2\pi(|V| - |E| + |F|)
\end{aligned}$$

Dihedral angles

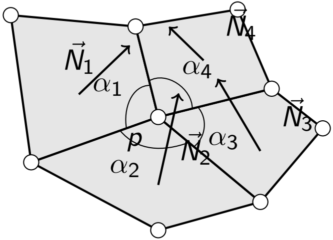
Gaussian curvature relies on intrinsic distances, but extrinsic position is determined by dihedral angles $\theta = (\vec{N}_1, \vec{N}_2)$.

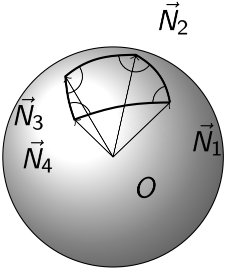


Theorem (Reconstruction)

A surface in space is uniquely determined by its lengths, vertex angles and diedral angles, up to congruence.

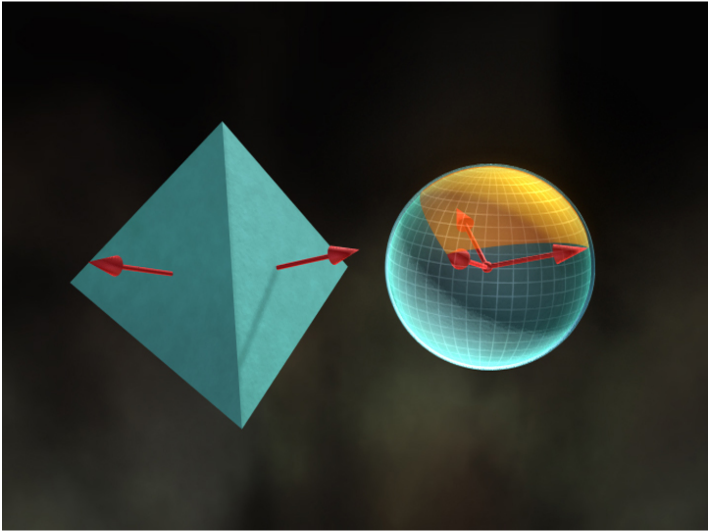
Spherical area





- spherical distance is equal to the dihedral angle
- spherical vertex angle at $\vec{N}_i$ is $\pi - \alpha_i$

Gauss image of the top vertex



Variation of area

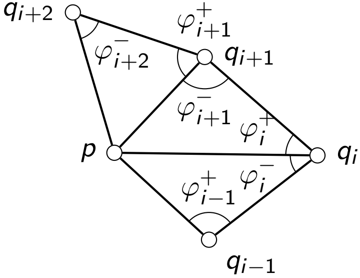
The area of triangle $f = (pqr)$ is $A(f) = \frac{1}{2}\overrightarrow{qp} \cdot J\overrightarrow{qr}$. So, the partial gradient of area of a triangulated polyhedral surface S is

$$\nabla_p A = \frac{1}{2} \sum_{(pqr)\in F} J\,\overrightarrow{qr} = \frac{1}{2} \sum_i J_i\,\overrightarrow{q_i q_{i+1}}$$

Cotangent formula

$$J\overrightarrow{q_i q_{i+1}} = \cot\varphi^-_{i+1}\overrightarrow{q_i p} + \cot\varphi^+_i\overrightarrow{q_{i+1} p}$$

$$\nabla_p A = \frac{1}{2} \sum_{i=1}^{n} (\cot\varphi^-_{i+1} + \cot\varphi^+_{i-1})\,\overrightarrow{q_i p}$$



Mean curvature vector

$$
\begin{aligned}
\vec{H}(p) &= -\frac{1}{2}\nabla_p A = -\frac{1}{4}\sum_{e\sim p} J\,\vec{e} = -\frac{1}{4}\sum_i J_i\,\overrightarrow{q_i q_{i+1}} \\
&= -\frac{1}{4}\sum_i J_i(\overrightarrow{q_i p} - \overrightarrow{q_{i+1} p}) = -\frac{1}{4}\sum_i (J_i - J_{i-1})\overrightarrow{q_i p} \\
&= \frac{1}{4}\left(\sum_i \vec{N}_i \times \overrightarrow{pq_i} + \sum_i \vec{N}_{i-1} \times \overrightarrow{q_i p}\right) = \frac{1}{4}\sum_i (\vec{N}_{i-1} - \vec{N}_i) \times \overrightarrow{pq_i} \\
&= \frac{1}{2}\sum_i |e_i| \sin\frac{\theta_i}{2}\vec{N}_{e_i} = \frac{1}{2}\sum_{e\ni p} |e| \sin\frac{\theta_e}{2}\vec{N}_e
\end{aligned}
$$

- $\vec{H}(p)$ defines a normal vector at p
- $\vec{H}(p) = \sum_{e\ni p}\vec{H}(e)$ is carried by the edges

Mean curvature flow

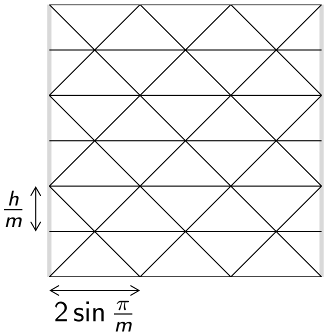
Solve the smoothing problem by moving the points to minimize area (hence bumps):

$$\frac{dp_i}{dt} = \vec{H}(p_i)$$

Schwarz lantern (1890)

Discretize the cylinder of radius r and height h with integers n, m

- $2nm$ triangles
- distance to cylinder bounded by $r(1-\cos\frac{\pi}{n}) \xrightarrow[n\to\infty]{} 0$
- area of a single triangle : $\frac{rh}{m}\sin\frac{\pi}{n}\sqrt{1+\frac{4r^2}{h^2}m^2\sin^4\frac{2\pi}{n}}$
- total area $2rh\left(n\sin\frac{\pi}{n}\right)\sqrt{1+\frac{4r^2}{h^2}m^2\sin^4\frac{2\pi}{n}}$
- limit when $n\to\infty$, $\frac{m}{n^2}\to\alpha\leq\infty$: $2\pi rh\sqrt{1+\frac{16\pi^4r^2}{h^2}\alpha^2}$
- vertical slope of triangles of angle approx $\frac{2h}{\alpha\pi^2 r}$



Discrete curvature

Xu, Xu & Sun (2005) prove that there is no discrete expression that will converge to curvature for all discretization

Proof :

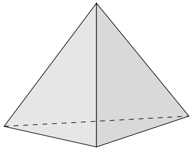
- fix points $p = (0,0)$, $q_i = (\epsilon x_i, \epsilon y_i, \epsilon z_i)$, $i = 1, \dots, 4$
- consider a surface S given by $z = \frac{1}{2}(ax^2 + 2bxy + cy^2)$, then the gaussian curvature is $K = ac - b^2$ and obviously $p \in S$
- solve $q_i \in S$ in terms of a, b, c, i.e. $ax_i^2 + 2bx_iy_i + c^2y_i^2 = 2z_i/\epsilon$ the rank may lower than 3, e.g. $(x_1, y_1) = (1,0) = -(x_3, y_3)$ and $(x_2, y_2) = (0,1) = -(x_4, y_4)$; then b is arbitrary.

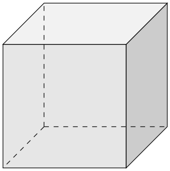
$$\begin{pmatrix} x_1^2 & 2x_1y_1 & y_1^2 \\ x_2^2 & 2x_2y_2 & y_2^2 \\ x_1^3 & 2x_3y_3 & y_3^2 \\ x_4^2 & 2x_4y_4 & y_4^2 \end{pmatrix}$$

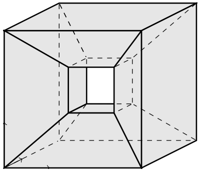
Why is the *discrete* Euler characteristic a topological invariant?

$$\chi = |V| - |E| + |F|$$

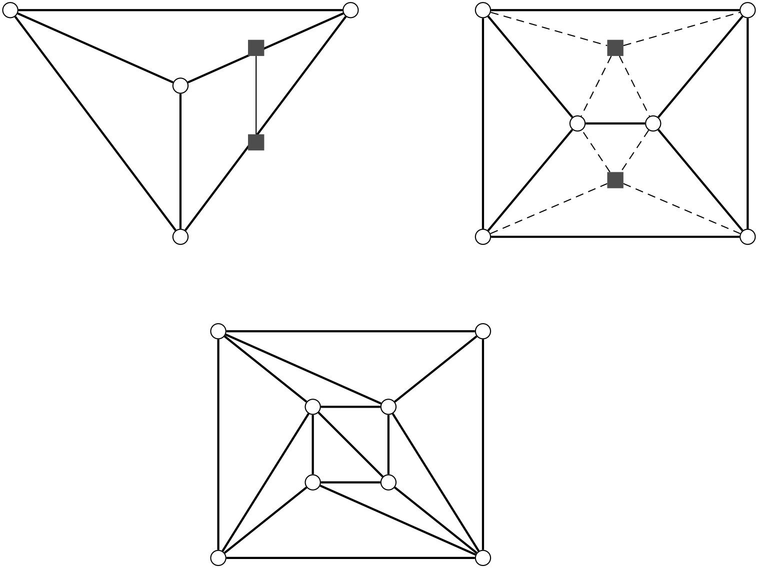
- add/remove vertex on edge, edge between vertices, vertex in face
- $\Rightarrow$ always assume triangulation if needed
- define an equivalence relation : $S \sim S'$ if they can be refined up to the same combinatorial structure. Then $\chi(S) = \chi(S')$.
- Is is trivial?







From the tetrahedron to the cube



Smooth Euler characteristic

Define χ on smooth S by picking a triangulation (or tiling).
Why is it well-defined?

- invariant under homeomorphisms (bicontinuous bijections)
- move vertices and edges
- refining to prove that χ is independent of the tiling

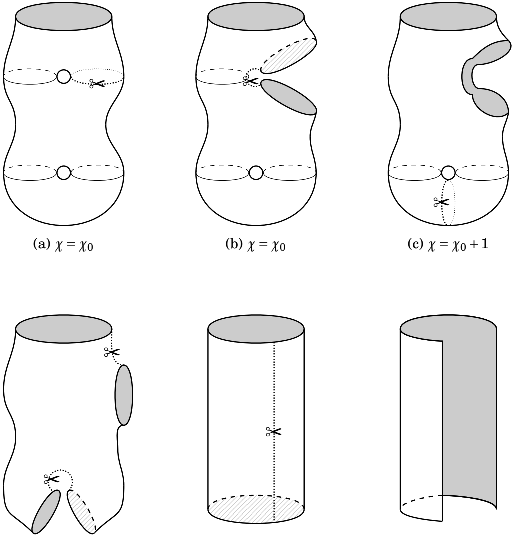
Notes:

- for extrinsic surfaces, one can deform from smooth surfaces to discrete polyhedra
- there are other meanings of χ ...

Cuts and handles

Theorem

For S with
g handles and
r boundary curves,
$\chi = 2 - 2g - r$



(a) $\chi = \chi_0$

(b) $\chi = \chi_0$

(c) $\chi = \chi_0 + 1$

(d) $\chi = \chi_0 + 1$

(e) $\chi = \chi_0 + 3$

(f) $\chi = \chi_0 + 4 = 1$

Total curvature

Theorem (discrete Gauss–Bonnet)

$\sum \alpha(p) = 2\pi\chi$ *(where $\alpha(p)$ is the angle defect)*

Apply to the equilateral metric:

- faces are regular n-gons
- edge length $= 1$
- inside angle is $(1 - 2/n)\pi$ (Gauss–Bonnet again)
- angle defect is $\alpha(p) = 2\pi K_c(p)$ (combinatorial curvature)

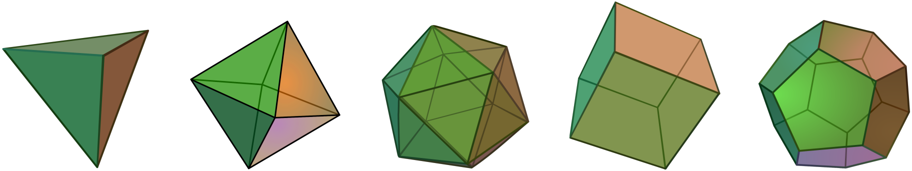
$$K_c(p) = 1 - \frac{d(p)}{2} + \sum_{f \ni p} \frac{1}{|f|}$$

Regular tilings of $\mathbb{S}^2$ and $\mathbb{T}^2$

A tiling is *regular* if degree d and number $n = |f|$ of edges are constant.

$$\chi = \sum_{p \in V} K_c(p) = |V| \left(1 - d\,\frac{n-2}{2n}\right)$$

$\mathbb{S}^2$	K	$d=3$	$d=4$	$d=5$	$d=6$
$n=3$	$1-d/6$	$1/2$	$1/3$	$1/6$	0
$n=4$	$1-d/4$	$1/4$	0		
$n=5$	$1-3d/10$	$1/10$			
$n=6$	$1-d/3$	0			



Asymptotic degree

Consider tilings with constant number of edges $n = |f|$. Let v_k be the number of vertices of degree k. Then

- $|V| = \sum_k v_k$, $2|E| = n|F|$
- $|E| = \frac{1}{2} \sum_k k\, v_k$ (double counting formula)
- $\chi = |V| - \frac{n-2}{n} |E|$

$$\bar{d} = \frac{\sum_k k\, v_k}{\sum_k v_k} = \frac{2|E|}{|V|} = \frac{2n}{n-2} \left(1 - \frac{\chi}{|V|} \right) \xrightarrow[|V| \to \infty]{} \frac{2n}{n-2}$$

to be compared with

$$\chi = \sum_{p \in V} K_c(p) = |V| \left(1 - d\, \frac{n-2}{2n} \right)$$

Exceptional tilings of $\mathbb{T}^2$

The torus possesses regular tilings by triangles, quads and hexagons.
An exceptional vertex is a vertex with different degree from the expected one (e.g. $d = 6$ for triangles).
Gauss–Bonnet implies : average degree remains constant (e.g. $d = 6$).

Theorem (Izmestiev, Kusner, Rote, Springborn & Sullivan)

There are no triangle tilings of the torus with two exceptional degrees $(5, 7)$.

Proof: uses geometric with equilateral (hence **flat** metric), $\alpha = \pi\, d/3$.

Parallel transport and holonomy

If γ is a *smooth* curve on S (no curved vertex), then one can define parallel transport of vector X along γ (using unfolding)
If γ is a loop, the transported vector $\tau_\gamma(X)$ is a rotated image of X, the same for any X. If γ is a loop around a vertex p, the rotation angle is exactly $\alpha(p)$.
The group generated by the rotations τ_γ for all loops γ is the holonomy group.

Lemma

For the equilateral metric with identical faces (with n faces, $n = 3, 4, 6$), the holonomy is a subgroup of $\mathbb{Z}_6$, $\mathbb{Z}_4$ or $\mathbb{Z}_3$.

Proof.

- X does not change within a face
- the angle (X, *edge of exit*) changes from (X, *edge of entry*) by an integral multiple of $\frac{\pi}{3}$ or $\frac{\pi}{2}$ or $\frac{2\pi}{3}$

Shortest geodesic

Let $\mathbb{T}^2$ be endowed with the triangle equilateral metric and two exceptional vertices $p_\pm$ of degrees 5 and 7 (angle defects $\pm\frac{\pi}{6}$. Consider γ a shortest geodesic (non nec. unique).

- γ does not meet p_+
- since γ does not meet p_+, it can be translated until it meets p_-, with side angles $\pi + \phi, \pi + \psi$, $\phi + \psi = \pi/6$ ($\phi, \psi \geq 0$)
- if ϕ or ψ are non zero, the holonomy is not in $\mathbb{Z}_6$, $\mathbb{Z}_4$ or $\mathbb{Z}_3$
- otherwise suppose $\phi = 0$, then one side of γ is flat: we can translate γ to γ' meeting again p_- (and p_- *only*)
- γ, γ' delimit a flat cylinder, bounded by curves with positive angles θ_1, θ_2, such that $\theta_1 + \theta_2 = \pi/6$, with p_- on both boundaries
- let δ link p_- to itself along the cylinder. Then a small perturbation of δ will have θ_1 holonomy.