From constant mean curvature surfaces to overdetermined elliptic problems

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The problem:

To classify domains $\Omega \in \mathbb{R}^n$ that support a positive solution of the over-determined elliptic system

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \\ \dfrac{\partial u}{\partial \nu} = \text{constant} & \text{on} \quad \partial\Omega \end{cases}$$

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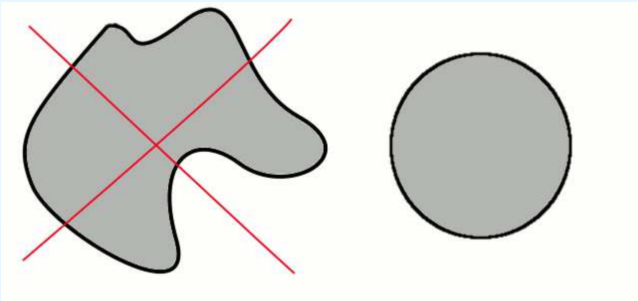
Theorem (Serrin, 1971). If Ω is bounded it is a ball



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Definition: If such problem is solvable, Ω is a f**-extremal domain**

Constant mean curvature surfaces

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Recall : the mean curvature $H(p)$ in a point p of a given hypersurface is the sum (or the mean) of the principal curvatures at p.

Conjecture of Berestycki, Caffarelli and Nirenberg

Communication on Pure and Applied Mathematics, (1997).

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \nu} = \text{constant} & \text{on } \partial\Omega\,, \end{cases}$$

EXTRA HYPOTHESIS

$\mathbb{R}^n \backslash \overline{\Omega}$ connected

u bounded

$$\Downarrow$$

Ω is a half space, or a ball, or a cylinder $\mathbb{R}^j \times B$ (where B is a ball) or the complement of one of these three exemples.

Many results from EDP's community

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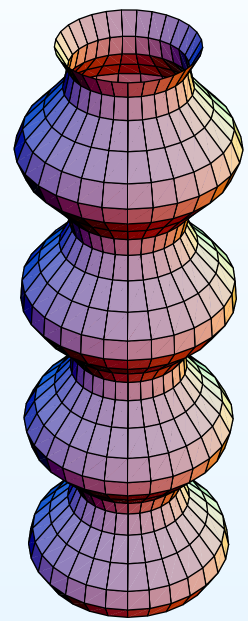
Farina-Valdinoci (*Arch. Rat. Mech.* $\&$ *An.*)

Some rigidity results for epigraphs in $\mathbb{R}^2$ for all functions f, and in $\mathbb{R}^3$ for some classes of functions f.

Coming back to constant mean curvature surfaces

In $\mathbb{R}^n$ there are non compact surfaces with constant mean curvature ! For exemple the Delaunay surfaces...



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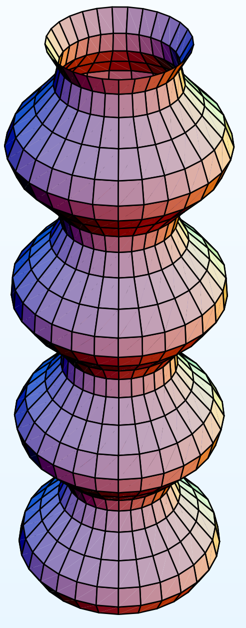

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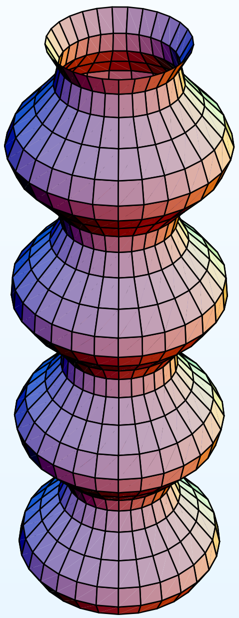


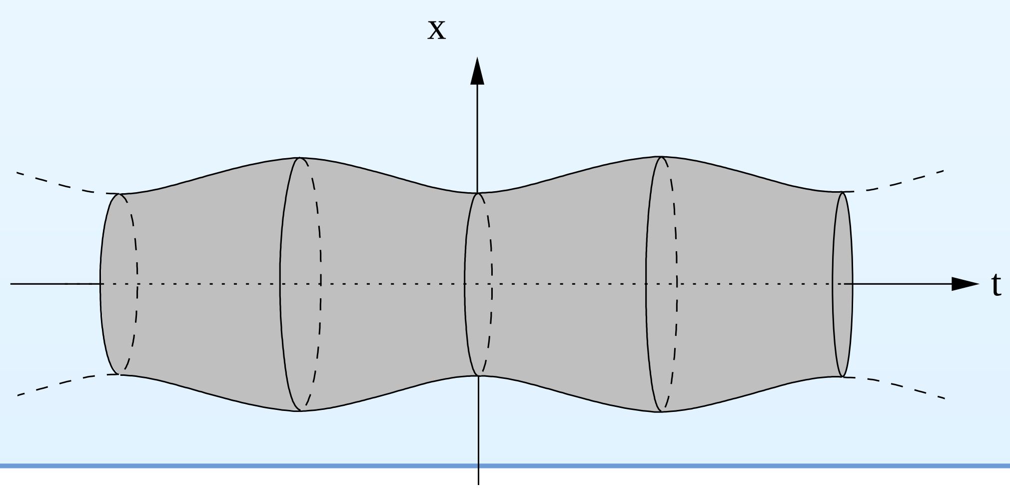
It is a one parameter smooth family of constant mean curvature surfaces in $\mathbb{R}^3$.

They are periodic perturbations of a cylinder and are surfaces of revolution

A parallel result on overdetermined elliptic problems

Theorem (S. 2010 $\&$ Schlenk-S. 2011): For $n \geq 2$ there exists a smooth family of periodic perturbations Ω of the cylinder $B^{n-1} \times \mathbb{R}$ (B^{n-1} = unit ball), with boundary of revolution, where there exists a periodic and positive solution to

$$\begin{cases} \Delta u + \lambda\, u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \nu} = \text{constant} & \text{on } \partial\Omega \end{cases}$$



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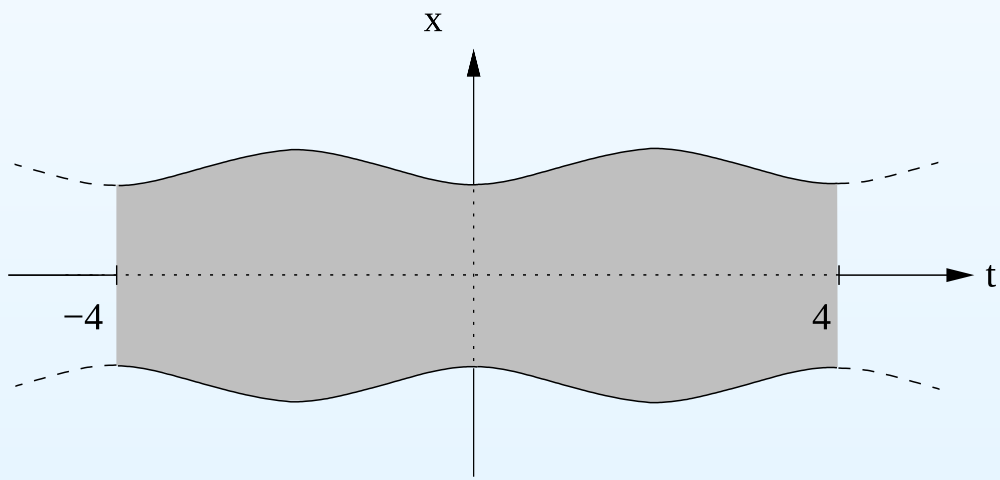
The result is true also in dimension 2. But this is not a counterexemple to the conjecture.



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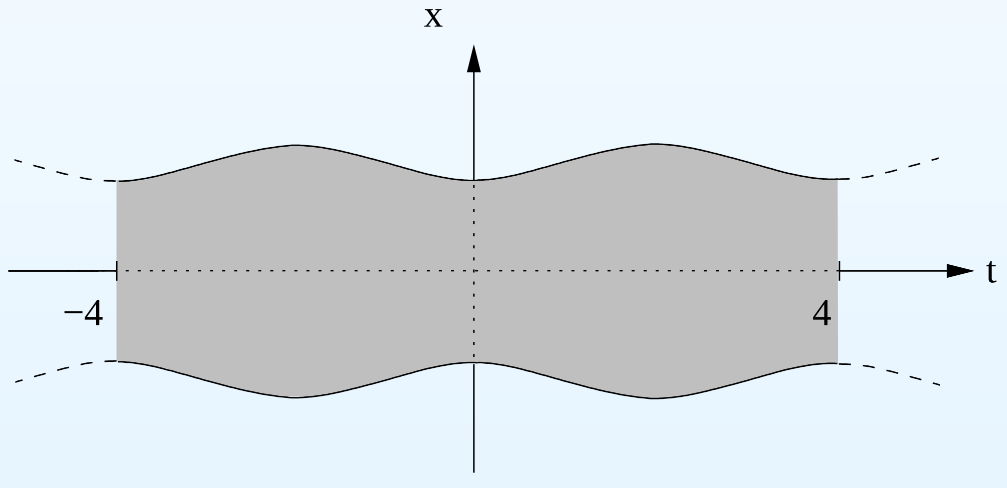

The complement of such domain is not connected.

The conjecture in dimension 2

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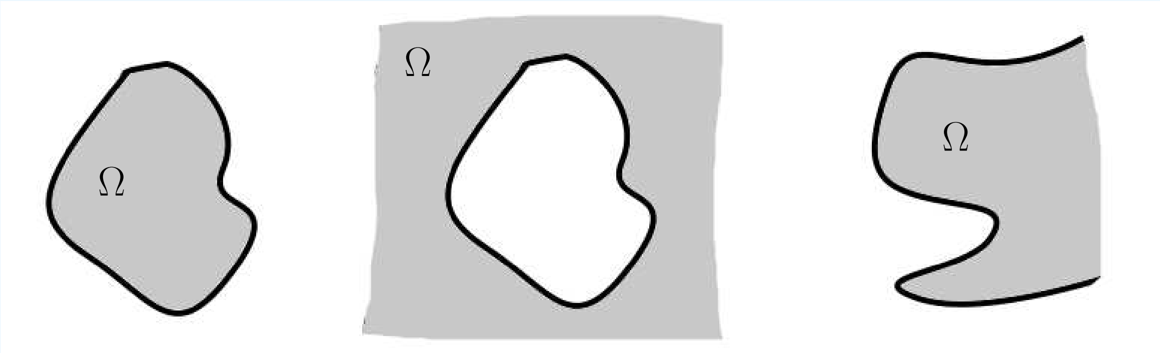
Theorem (Ros-S. 2013)

The conjecture of Berestycki-Caffarelli-Nirenberg in dimension 2 is true for all function f such that $f(t) \geq \lambda\, t$ for a $\lambda > 0$.

The proof of the theorem (1)

Step 1. If $\mathbb{R}^2 \backslash \overline{\Omega}$ is connected then there are only three possibilities for Ω:

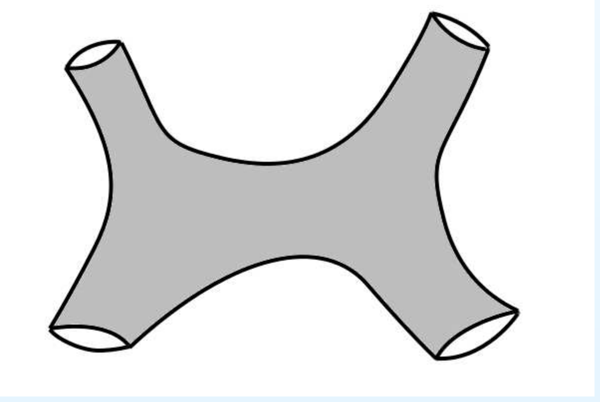
1. Ω is bounded (Done by Serrin !)
2. Ω is an exterior domain (Easy case, EDP techniques)
3. $\partial\Omega$ is an open curve that separated $\mathbb{R}^2$ in two connected components, and Ω is one of such components (Hard case)



Meeks, 1989, J. Diff. Geom.

Definition. A surface has finite topology if

1. it is a compact surface
2. outside of a big ball, the surface is done of a finite number of noncompact components diffeomorphic to $S^{n-1} \times \mathbb{R}_+$, called ends.



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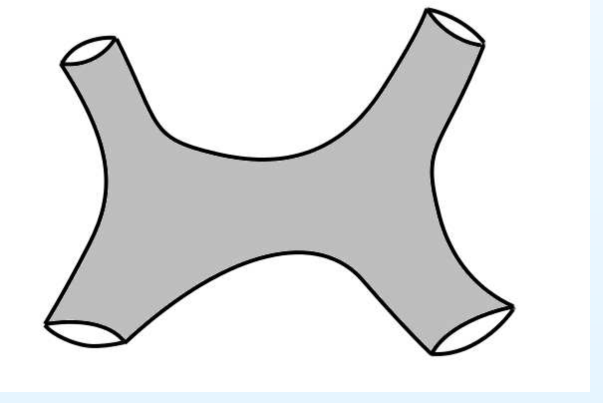

Theorem. Let S be a properly embedded finite topology nonzero CMC surface in $\mathbb{R}^3$. Then S cannot have only one end.

The proof of the theorem

Definition. We say that a domain has finite topology if

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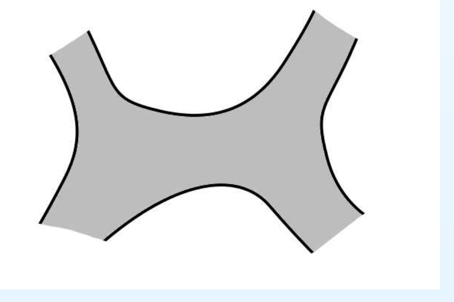

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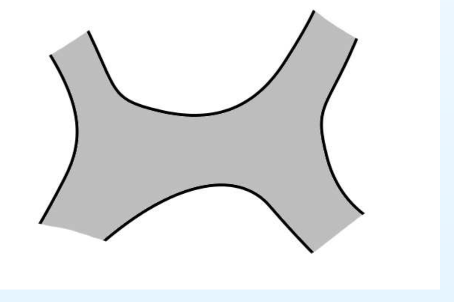
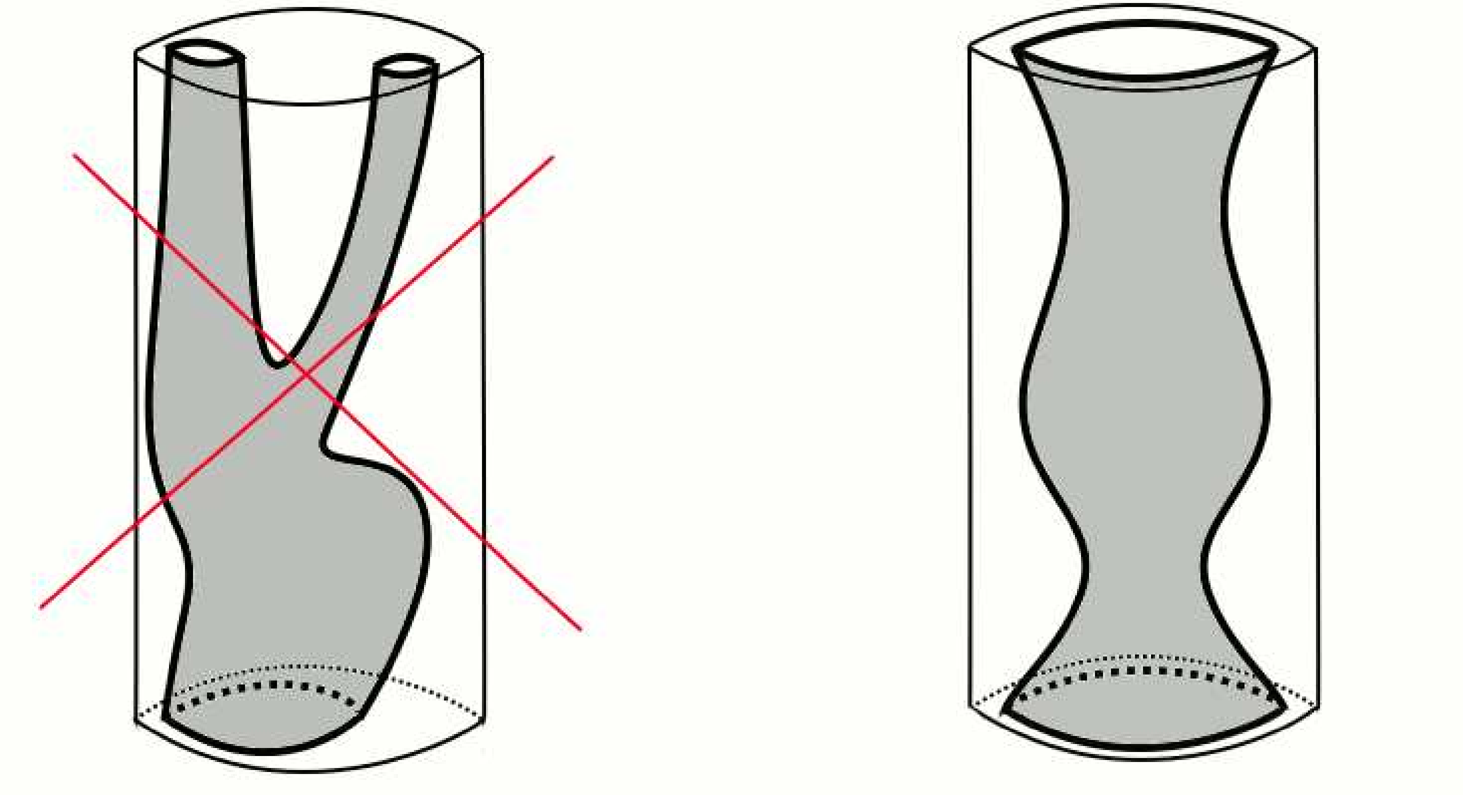


Proposition (Ros-S.). If $f(t) \geq \lambda t$ for some $\lambda > 0$ and Ω is an f-extremal domain, then Ω cannot have only one end.

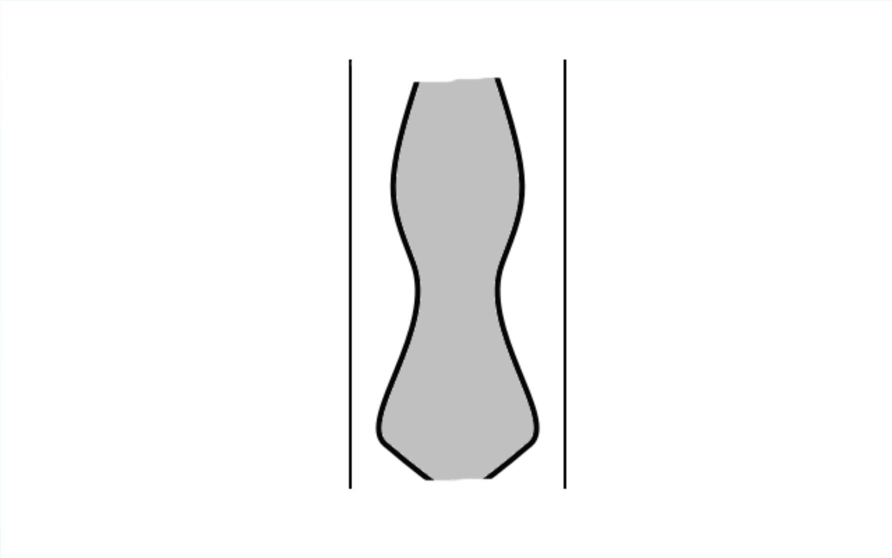
Theorem: Korevaar, Kusner $\&$ Solomon, 1989

Let S be a properly embedded finite topology nonzero CMC surface in $\mathbb{R}^3$ contained in a cylinder. Then S is surface of revolution.



An other result

Theorem (Ros-S. 2013). Let Ω be an f-extremal domain of $\mathbb{R}^2$ with bounded curvature. If Ω is contained in a half-plane, then Ω is either a ball or a half-plane or there exists a positive function $\varphi : \mathbb{R} \longrightarrow]0, \infty[$ such that Ω is $\{|y| < \varphi(x)\}$.

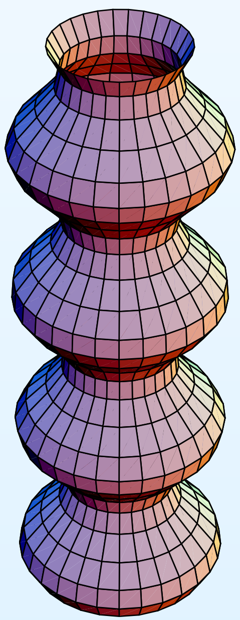


Corollary. Proof of the conjecture of Berestycki-Caffarelli-Nirenberg in the half-plane under the assumption of bounded curvature.

Generalization to $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ (Morabito-S.)

Existence of Delaunay type domains with a positive solution of

$$\begin{cases} \Delta u + \lambda\, u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \nu} = \text{constant} & \text{on } \partial\Omega \end{cases}$$



Open problems and conjectures (Ros-S.)

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Basic conjecture:

a class of CMC hypersurfaces in $\mathbb{R}^{n+1}$ (which?)

$$< - - >$$

domains in $\mathbb{R}^n$ that support a positive solution to the problem

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for $f(t) = \lambda\, t$ (and maybe others?)

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4. Correspondence between some kind of minimal surfaces and harmonic overdetermined problems.

Classification of harmonic overdetermined solutions

Theorem (Traizet 2013).

minimal bigraph in $\mathbb{R}^3$ $\longleftrightarrow$

domains in $\mathbb{R}^2$ that support a positive solution to the problem

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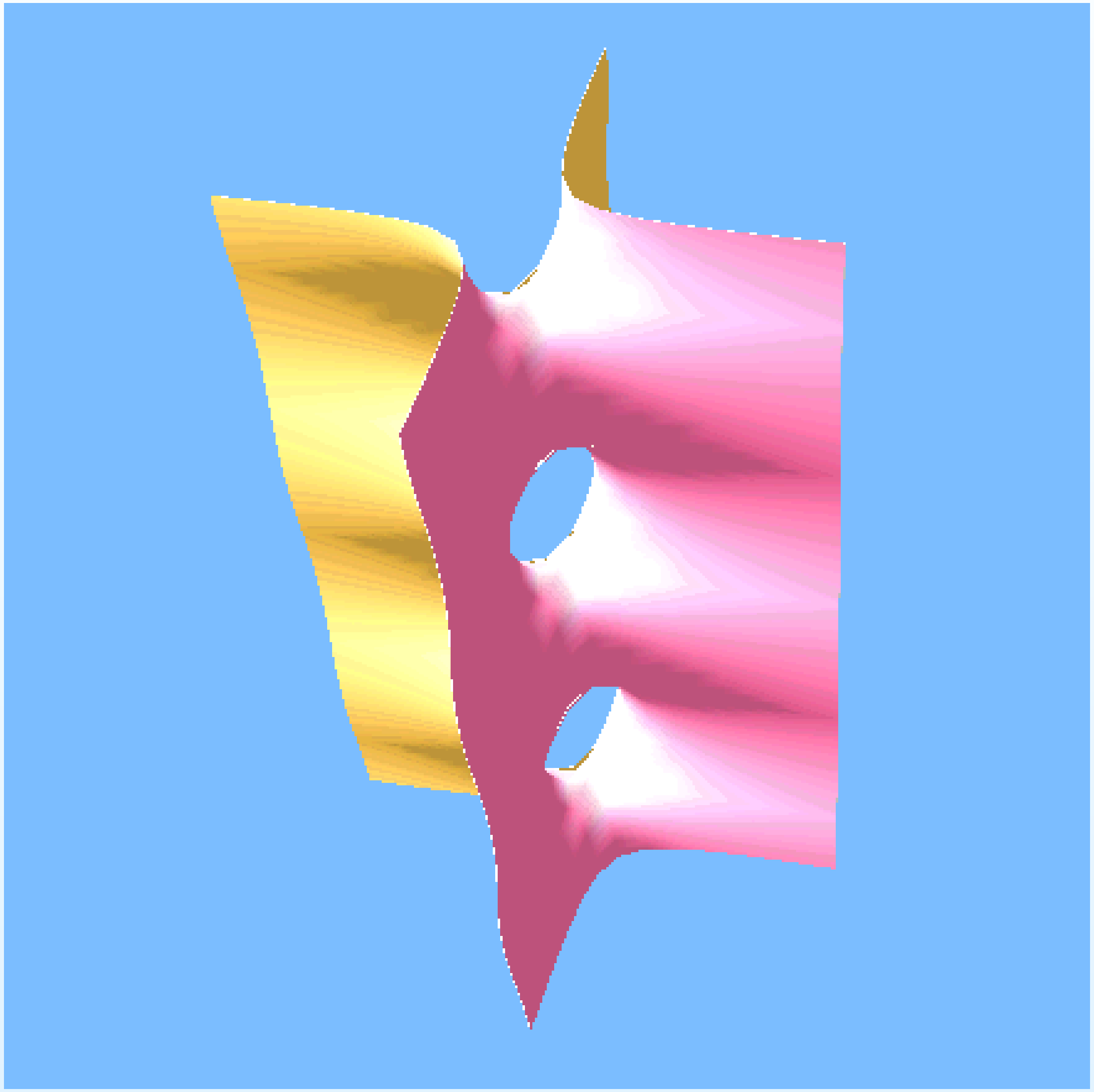
with the hypothesis that $\partial\Omega$ has a finite number of components, at least in the quotient if Ω is periodic

The Scherk simply periodic minimal surface



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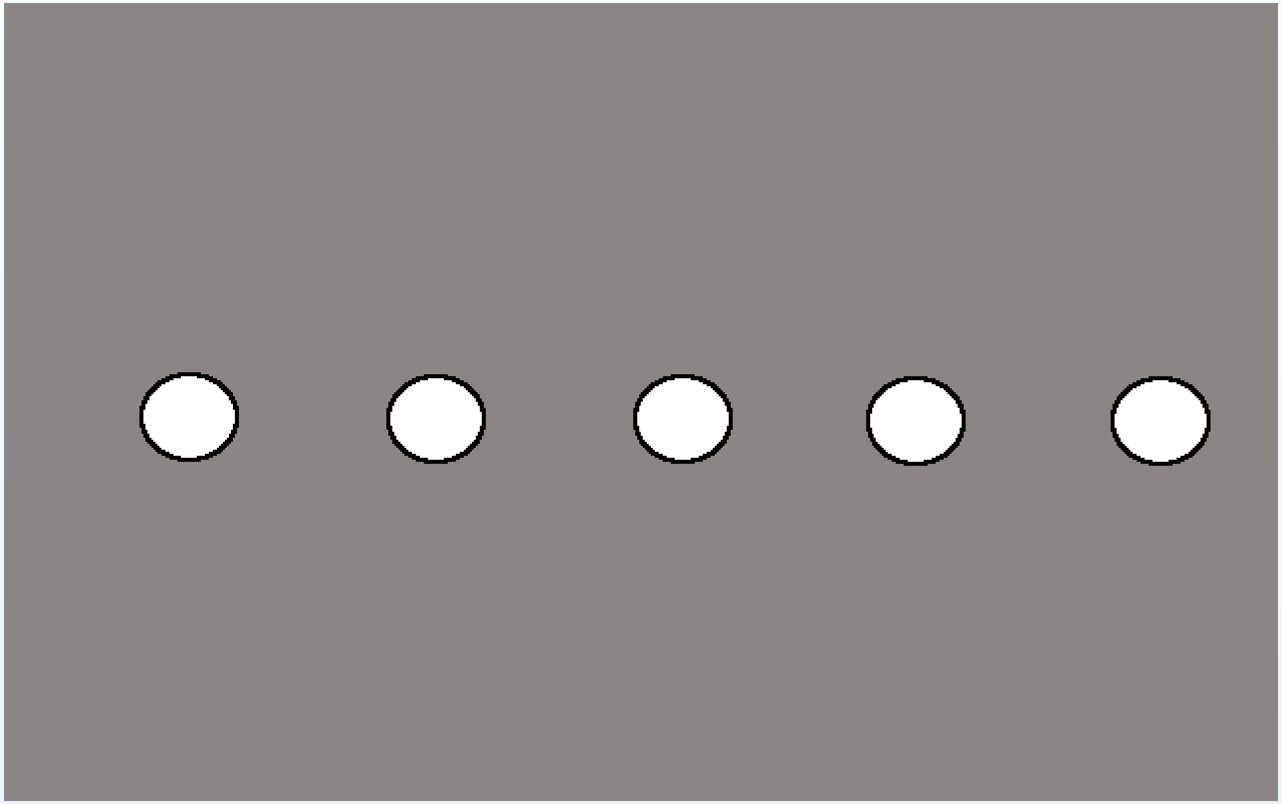

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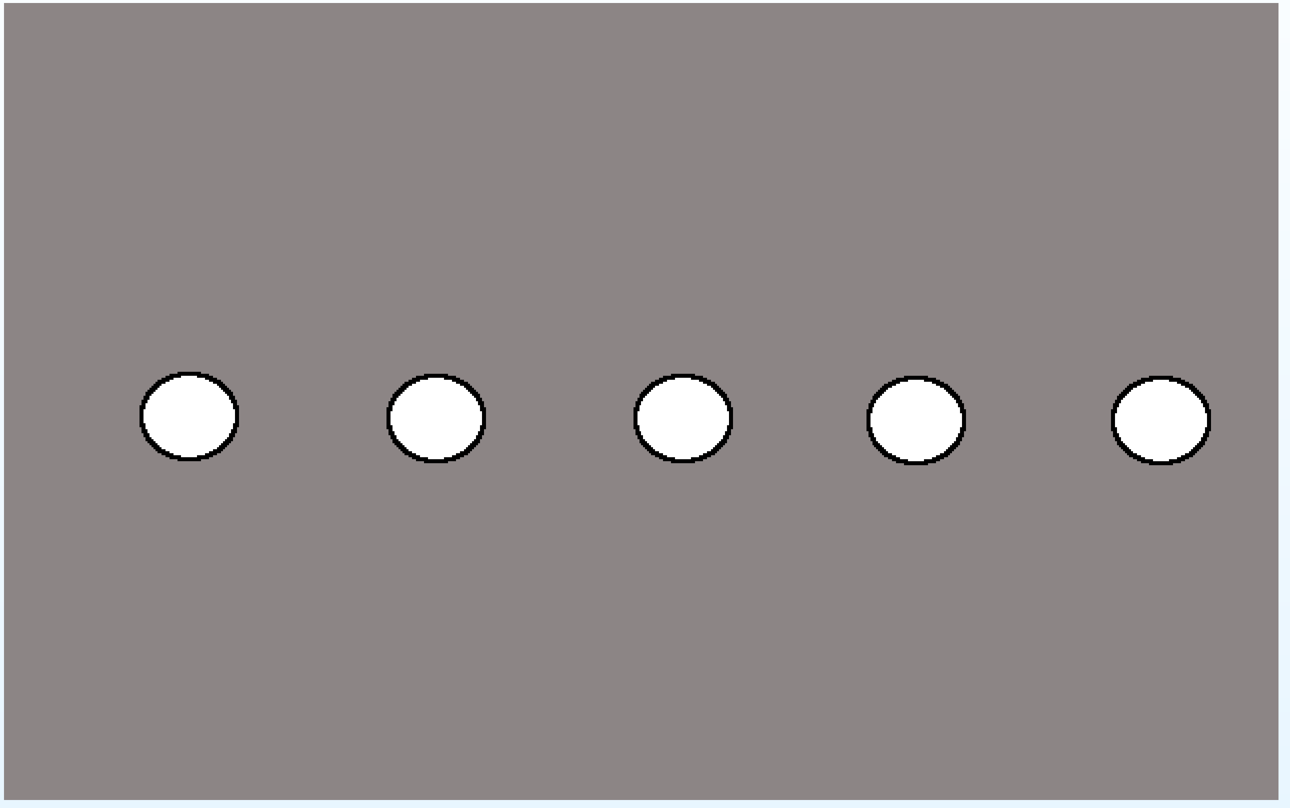

Such domain was found by the physicians Baker, Saffman and Sheffield in 1979 as a solution to an equilibrium problem in hydrodynamics of vortices !

We give the idea of the proof of:

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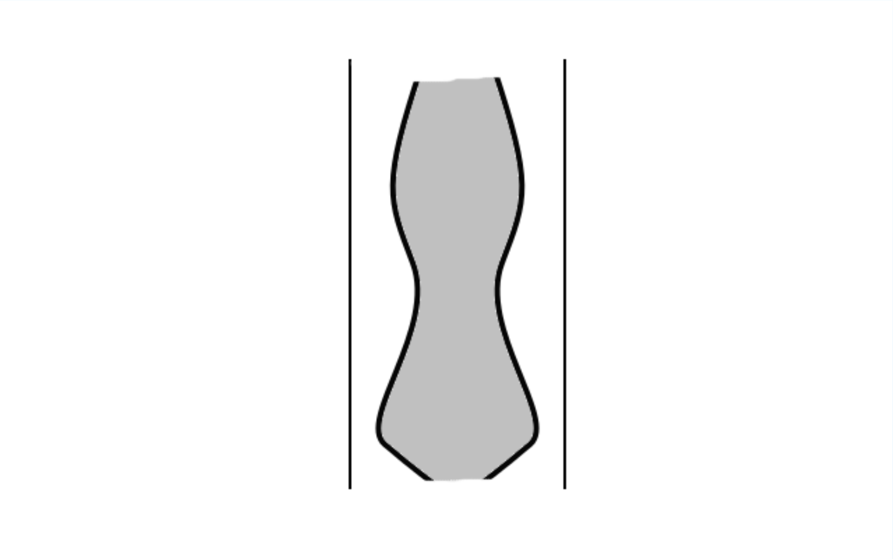
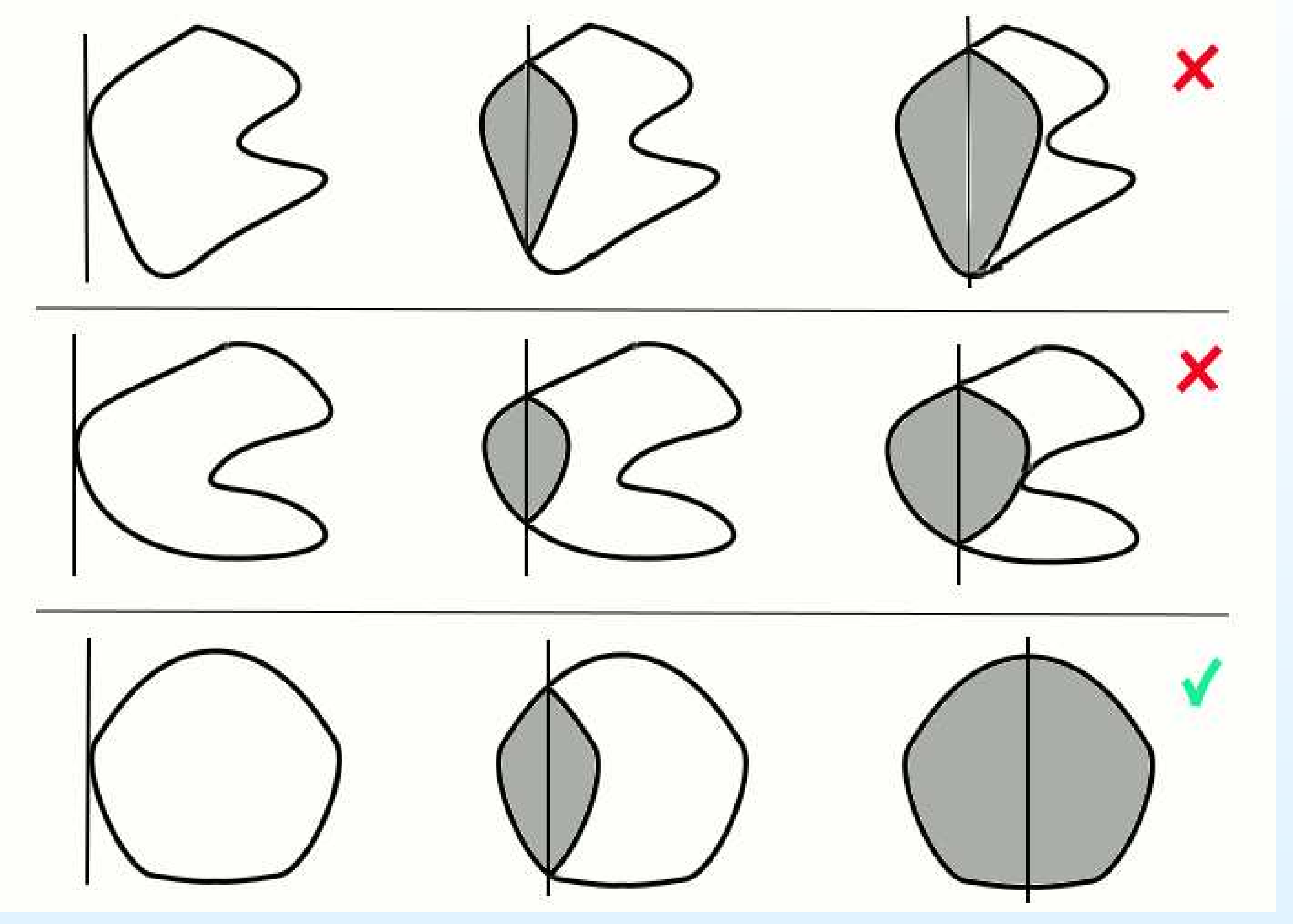
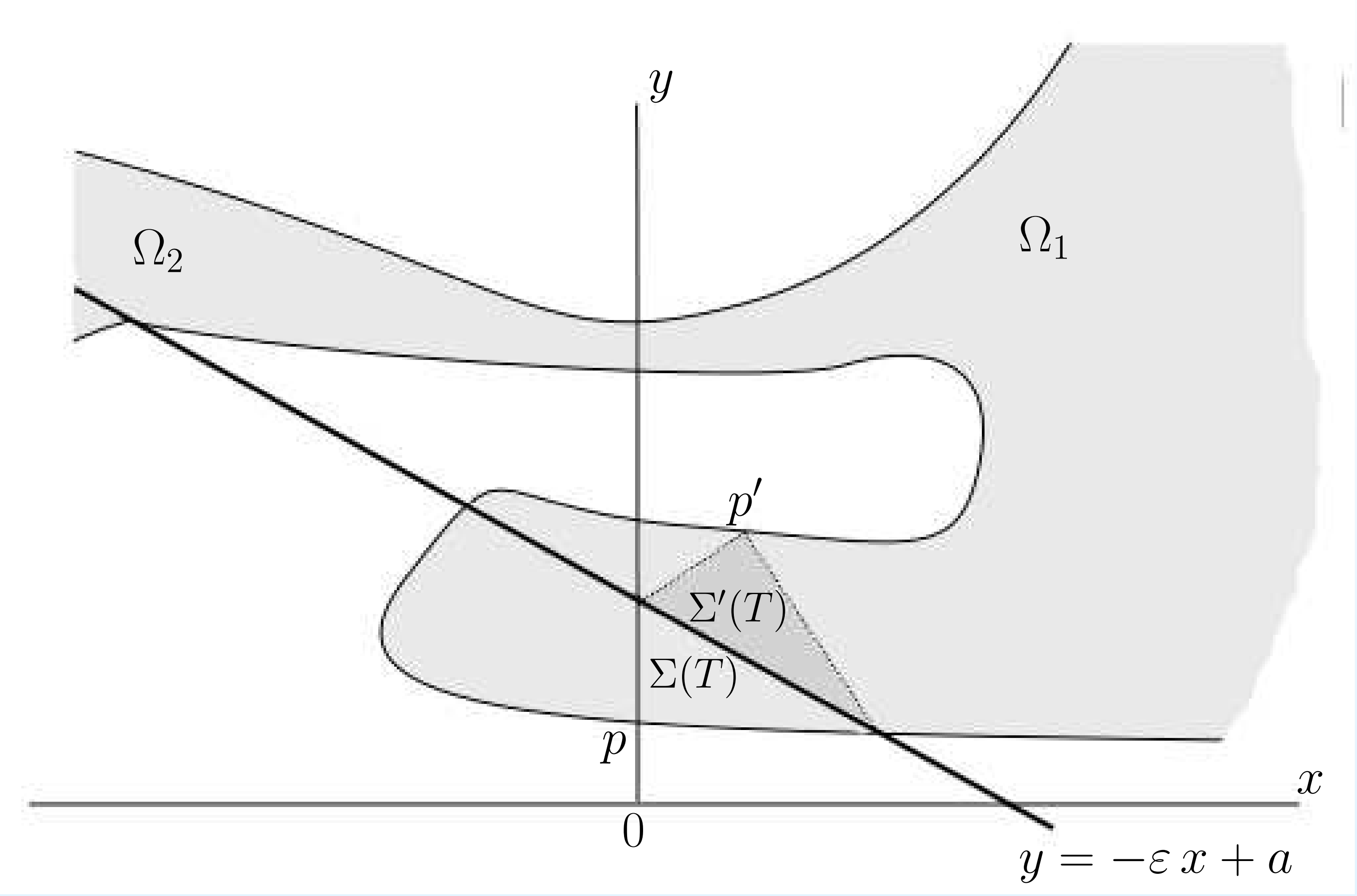


Step 1: The moving plane argument



Step 2: The tilted moving plane argument



Step 3: The implication of the moving plane argument

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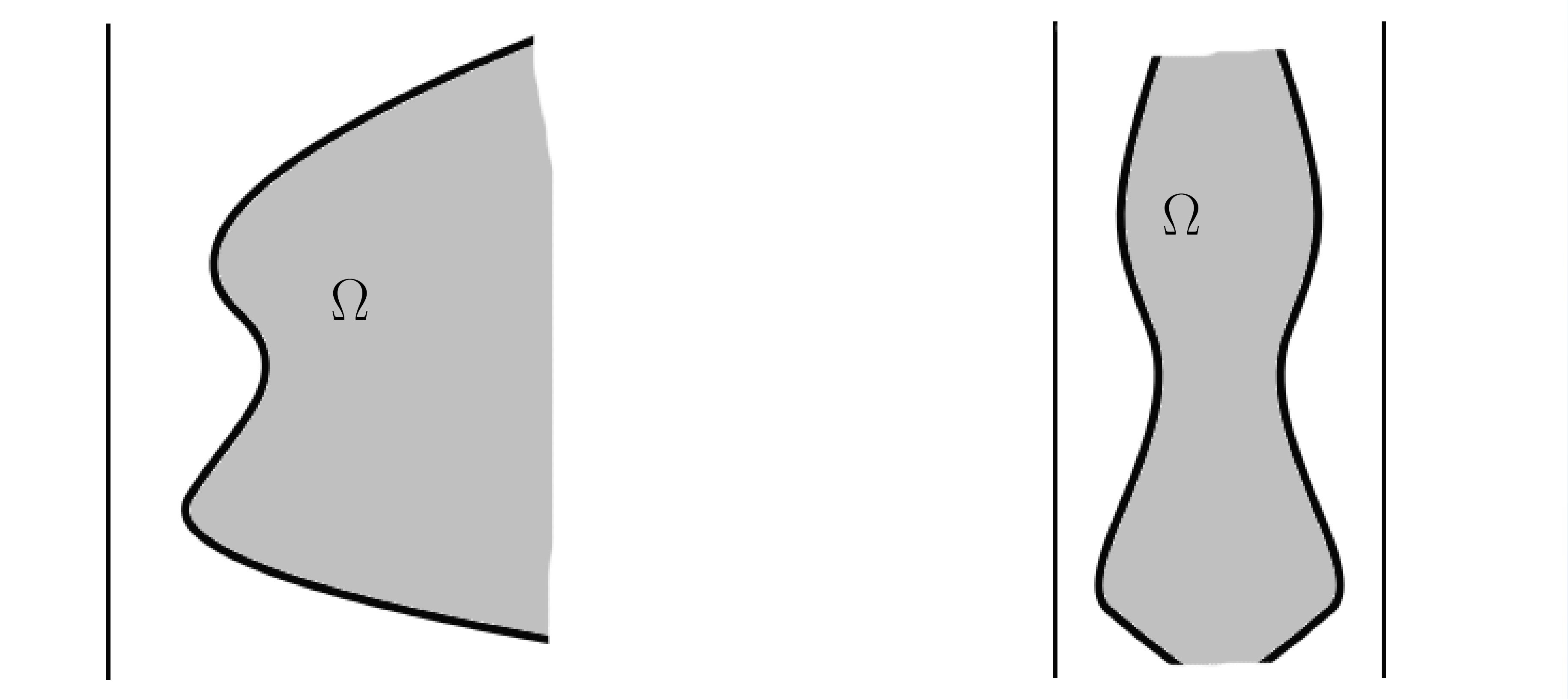

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