# Multiplicity results for $p$-Laplacian problems

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## Model problem

$1 < p < +\infty$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$

$$(P) \qquad \begin{cases} -\Delta_p u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*Background material:* Rabinowitz, Peral, Dinca-Jebelean-Mawhin (super $(p-1)$-polynomial growth)

- $\Omega \subset \mathbb{R}^N$ open bounded with smooth boundary $\partial\Omega$ $(N \geq 3)$
- $g : \Omega \times \mathbb{R} \to \mathbb{R}$ subcritical
  $(p^* = \frac{pN}{N-p}$ if $p \in ]1, N[$, $p^* = +\infty$ otherwise)

$$\lim_{|t|\to+\infty} \frac{g(x,t)}{|t|^{p-2}t} \in \mathbb{R} \quad \text{uniformly with respect to } x \in \Omega$$

The existence of (non-trivial) solutions is related to the interaction between $g$ and $\sigma(-\Delta_p)$

## Overview

* $p = 2$ semilinear case: $W_0^{1,2}(\Omega)$, $\sigma(-\Delta)$, $(\lambda_k)_k$
  - *existence results:*
    Amann-Zehnder, Landesman-Lazer, Ahmad-Lazer-Paul
  - *multiplicity results:*
    Rabinowitz, P.Bartolo-Benci-Fortunato, Chang

* $p \neq 2$ quasilinear case: $W_0^{1,p}(\Omega)$, $\sigma(-\Delta_p)$
  - *existence results:*
    Arcoya-Orsina, Drábek-Robinson, Li-Zhou, Liu-Li
  - *multiplicity results:*
    Li-Zhou, Perera-Szulkin

# About $\sigma(-\Delta_p)$

The spectral properties of the $p$-Laplacian in $W_0^{1,p}(\Omega)$ are still mostly unknown

- eigenvalues $(\mu_k)_k$ in García-Peral 1987 via the Krasnoselskii genus
  eigenvalues $(\mu'_k)_k$ in Perera-Szulkin 2005 via the cohomological index of Fadell and Rabinowitz
  $(\mu_k)_k$ and $(\mu'_k)_k$ are unbounded, increasing and $\mu'_k \geq \mu_k$
- the first eigenvalue is characterized as

$$\mu_1 = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, \mathrm{d}x}{\int_\Omega |u|^p \, \mathrm{d}x}$$

  (positive, simple, isolated and has a positive eigenfunction $\varphi_1$)

eigenvalues do not provide for $W_0^{1,p}(\Omega)$ a decomposition similar to that of $W_0^{1,2}(\Omega)$

## Quasi–eigenvalues for $-\Delta_p$

In Candela-Palmieri:
$(\eta_h)_h$ increasing and diverging sequence
corresponding functions $(\psi_h)_h$ generate the whole $W_0^{1,p}(\Omega)$
$\psi_1 \equiv \varphi_1$, $\eta_1 = \mu_1$

$$W_0^{1,p}(\Omega) = V_h \oplus W_h \quad \text{for all } h \in \mathbb{N}$$

where $V_h = \mathrm{span}\{\psi_1, \ldots, \psi_h\}$

$$\eta_{h+1} \int_\Omega |w|^p \, \mathrm{d}x \leq \int_\Omega |\nabla w|^p \, \mathrm{d}x \quad \forall h \in \mathbb{N} \text{ and } w \in W_h$$

if $p = 2$, $(\eta_h)_h$ agrees with $(\lambda_h)_h$

for all $h \in \mathbb{N}$: $\eta_h \leq \mu_h$

## Quasi–eigenvalues for $-\Delta_p$

In Li-Zhou:
$(\nu_k)_k$ increasing and diverging sequence
for all $k \in \mathbb{N}$

$$\mathcal{W}_k = \{V : V \text{ is a subspace of } W_0^{1,p}(\Omega), \varphi_1 \in V, \dim V \geq k\}$$

$$\nu_k = \inf_{V \in \mathcal{W}_k} \sup_{u \in V \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, \mathrm{d}x}{\int_\Omega |u|^p \, \mathrm{d}x}$$

$\nu_1 = \mu_1$
if $p = 2$, $(\nu_k)_k$ agrees with $(\lambda_k)_k$
for all $k \in \mathbb{N}$: $\nu_k \geq \mu_k$

# Setting of the problem

Let $l_\infty \in \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ s.t.

$$g(x,t) \;=\; l_\infty |t|^{p-2}t + f(x,t) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}$$

Problem $(P)$ becomes

$$(P_\infty) \qquad \begin{cases} -\Delta_p u - l_\infty |u|^{p-2}u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Moreover

$f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$

$$\lim_{|t| \to +\infty} \frac{f(x,t)}{|t|^{p-2}t} \;=\; 0$$

## Setting of the problem

The weak solutions of $(P_\infty)$ are the critical points of the $C^1$ functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{l_\infty}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x$$

on $W_0^{1,p}(\Omega)$, with $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$

Moreover

- $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = l_0 \in \mathbb{R}$
- $l_\infty \notin \sigma(-\Delta_p)$
- $f(x,\cdot)$ is odd for $x \in \Omega$

## Semilinear case: a multiplicity result

Under the previous assumptions, if

- there exist $h, k \in \mathbb{N}$ s.t.

$$\min\{l_0 + l_\infty, l_\infty\} \;<\; \lambda_h \;<\; \lambda_k \;< \max\{l_0 + l_\infty, l_\infty\}$$

with $(\lambda_k)_k$ (distinct) eigenvalues of $-\Delta$ in $W_0^{1,2}(\Omega)$, then

$(P_\infty)$ has at least $\dim(M_h \oplus \ldots \oplus M_k)$ distinct pairs of non-trivial solutions

where $M_j$ is the eigenspace corresponding to the eigenvalue $\lambda_j$ of $-\Delta$ in $W_0^{1,2}(\Omega)$

## Quasilinear case: a multiplicity result

Under the previous assumptions, if

- there exist $h, k \in \mathbb{N}$, with $h \geq k$, s.t.

$$\min\{l_0 + l_\infty, l_\infty\} < \ \eta_h \leq \ \nu_k \ < \max\{l_0 + l_\infty, l_\infty\}$$

with $(\eta_k)_k$ and $(\nu_k)_k$ sequences of quasi–eigenvalues of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, then

$(P_\infty)$ has at least $k - h + 1$ distinct pairs of non-trivial solutions

Previous results: $l_0 + l_\infty = 0$ (Li-Zhou) or also $l_0 + l_\infty \notin \sigma(-\Delta_p)$ (Perera-Szulkin)

## Main tools of the proof

- Genus (Coffman) and pseudo-index (Benci) related to the genus
- $V, W$ closed subspaces of $X$; if

$$\dim V < +\infty \quad \text{and} \quad \operatorname{codim} W < +\infty$$

then, for all odd bounded homeomorphism $h$ on $X$ and for all open bounded symmetric neighbourhood $B \subset X$ of $0$:

$$\gamma(V \cap h(\partial B \cap W)) \geq \dim V - \operatorname{codim} W$$

- the functional $J$ satisfies a variant of the Palais-Smale condition at level $c \in \mathbb{R}$: any sequence $(u_n)_n \subseteq X$ s.t.

$$\lim_{n \to +\infty} J(u_n) = c$$

$$\lim_{n \to +\infty} \|\mathrm{d}J(u_n)\|_{X'}(1 + \|u_n\|_X) = 0$$

converges in $X$, up to subsequences

## The proof

- Using the $\nu_k$: there exists $V^\sigma \in \mathcal{W}_k$ with $\dim V^\sigma = k$ s.t.

$$J(u) \leq c_\infty \qquad \forall u \in V^\sigma$$

- using the assumption on $l_0$ and $\eta_k$ setting $S_\rho = \{u \in W_0^{1,p}(\Omega) : \|u\| = \rho\}$ if $\rho$ is small enough

$$J(u) \geq c_0 \qquad \forall u \in S_\rho \cap W_{h-1}$$

- $(S_\rho \cap W_{h-1}, \mathcal{H}^*, \gamma^*)$ pseudo-index theory

$$\gamma^*(V^\sigma) = \min_{h \in \mathcal{H}^*} \gamma(V^\sigma \cap h^{-1}(S_\rho \cap W_{h-1})) \geq \dim V^\sigma - \text{ codim } W_{h-1}$$

- Abstract theorem in P.Bartolo-Benci-Fortunato adapted to Banach spaces

## Remarks

For all $k \in \mathbb{N}$

$$\eta_k \leq \mu_k \leq \nu_k$$

Under additional assumptions:

- existence results
- resonant case
- $l_0 \in \{\pm\infty\}$

# Problems with broken symmetry

Let $h : \Omega \times \mathbb{R} \to \mathbb{R}$, $h$ continuous, $\varepsilon \in \mathbb{R}$, $g$ odd

$$(P_\varepsilon) \qquad \begin{cases} -\Delta_p u = g(x,u) + \varepsilon h(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- $g$ is super $(p-1)$-linear and subcritical at infinity
  - $h(x,u) = h(x)$, $\varepsilon = 1$
    $p = 2$: Struwe, Bahri-Berestycki, Rabinowitz, Bahri-Lions, Tanaka, Bolle-Ghoussoub-Teherani
    $p \neq 2$: García-Peral ($\Omega = ]0,1[^N$)
  - $h : \Omega \times \mathbb{R} \to \mathbb{R}$, $\varepsilon$ "small"
    Li-Liu (if $p \neq 2$ $h(x,\cdot)$ also odd); Hirano-Zou (only $p = 2$)
- $g$ $(p-1)$-linear at infinity
  - $h(x,\cdot)$ odd, non resonant case
    $p = 2$: Li-Liu

## Bolle-Ghoussoub-Teherani type problem

$$(P_\varphi) \qquad \begin{cases} -\Delta_p u = |u|^{q-2}u + h & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

$1 < p < q < p^*$, $\Omega$ any smooth domain, $\varphi \in C^2(\overline{\Omega})$

If $2 < p < q < \frac{pN(p+1)}{pN+N-p}$, then $(P_\varphi)$ has infinitely many solutions for any $h \in C(\overline{\Omega})$

Problem set on a Banach space, nonlinear operator, regularity of solutions

## The idea of the proof

Bolle's method: it is considered a continuous path of functionals $(J_\theta)_{\theta\in[0,1]}$ starting at a symmetric functional (corresponding to $h = 0 = \varphi$) and ending at the non even functional associated to the problem

Setting $u = v + \varphi$ on $\Omega$, problem $(P_\varphi)$ becomes

$$(P) \qquad \begin{cases} -\Delta_p(v+\varphi) = |v+\varphi|^{q-2}(v+\varphi) + h & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

The weak solutions of $(P)$ are the critical points of the $C^1$ functional

$$J_1(v) = \frac{1}{p}\int_\Omega |\nabla(v+\varphi)|^p \,\mathrm{d}x - \frac{1}{q}\int_\Omega |v+\varphi|^q \,\mathrm{d}x - \int_\Omega hv \,\mathrm{d}x$$

on $W_0^{1,p}(\Omega)$

## The idea of the proof

- $(J_\theta)_{\theta \in [0,1]}$ verifies some conditions
- $(c_k)_k$ sequence of mini-max values for $J_0$

then

- either $J_1$ has infinitely many critical points
- or a certain bound on $c_{k+1} - c_k$ holds

$\exists\ L > 0$ s.t.

$$c_k \geq L k^{\frac{pq}{N(q-p)}-1} \quad \forall k \geq k_0$$

thus we get an absurd by

$$c_k \leq L k^p$$

if $p = 2$ better bound by Tanaka (using Morse Theory)

## Remarks

If $p = 2$, problem $(P_\varphi)$ has infinitely many solutions for all $q \in ]2, \frac{2N}{N-1}[$ (Bolle-Ghoussoub-Teherani)

Without additional assumptions, it is still an open problem whether there exist infinitely many solutions for $q$ up to $2^* = \frac{2N}{N-2}$, also for $\varphi = 0$

$1 < p < q < p^*$ and $\frac{q}{q-1} < \frac{pq}{N(q-p)} - 1$
Then problem $(P_0)$ has infinitely many solutions for any $f \in L^{p'}(\Omega)$ ($p'$ is the conjugate exponent of $p$)

in this case

$$c_k \leq L k^{\frac{q}{q-1}}$$

# The radial case

$R > 0$, $\xi \in \mathbb{R}$

$$(P_\xi) \qquad \begin{cases} -\Delta_p u = |u|^{q-2}u + h & \text{in } B_R \\ u = \xi & \text{on } \partial B_R \end{cases}$$

$B_R$ is the open ball centered in $0$ with radius $R$ in $\mathbb{R}^N$
$2 < p < q < p^*$ and $h \in C_{\mathrm{rad}}(\overline{B}_R)$
if $p = 2$ optimal result in Candela-Palmieri-Salvatore
Setting $u = v + \xi$ the weak radial solutions of $(P_\xi)$ are the critical points of the $C^1$ functional

$$J_1(v) = \frac{1}{p}\int_{B_R} |\nabla v|^p \, \mathrm{d}x - \frac{1}{q}\int_{B_R} |v+\xi|^q \, \mathrm{d}x - \int_{B_R} hv \, \mathrm{d}x \quad \text{on} \quad W^{1,p}_{\mathrm{rad}}$$

# Better estimates

If $2 < p < q < \bar{p}$ $\exists\, C$ s.t.

$$J_0(v) \geq C\left(I_{2,r}(v)\right)^{\frac{p}{2}}$$

with

$$I_{2,r}(v) = \frac{1}{2}\int_{B_R} |\nabla v|^2 \, \mathrm{d}x - \frac{D}{r}\int_{B_R} |v|^r \, \mathrm{d}x$$

if $v \in W^{1,p}_{\mathrm{rad}}$ verifies

$$\int_{B_R} |\nabla v|^p \, \mathrm{d}x >> \left(\int_{B_R} |v|^r \, \mathrm{d}x\right)^{\frac{p}{2}}$$

and $r = \frac{q(pN-2N+2p)-pN(p-2)}{p^2} \in ]2, 2^*[$

$\exists \tilde{L} > 0$ s.t.

$$c_k \geq \tilde{L} k^{\frac{pr}{2(r-2)}} \quad \forall k \geq \tilde{k}$$

## The radial result

If $2 < p < q < \bar{q}$ where

- $\bar{q} = \max\left\{\frac{pN(p+1)}{pN+N-p}, \bar{p}\right\}$ with $\bar{p} := \frac{pN(pN-2N+4)}{(N-2)(pN-2N+2p)}$, if $N \geq 4$
- $\bar{q} = \max\left\{\frac{3p(p+1)}{2p+3}, 4\right\}$, if $N = 3$

Then, for any $h \in C_{\mathrm{rad}}(\overline{B}_R)$ problem $(P_\xi)$ has infinitely many radial solutions

$\bar{q} < p^*$
Improvement in the radial case if $\bar{q} = \bar{p}$:
for $N = 4$
for $N \geq 5$ if $p < \frac{2N}{N-4}$
(for $N = 3$: $\bar{q} = 4$ if $p < 3$)

## A perturbed problem

Under the assumptions of the symmetric case, for any continuous $h$

$$(P_\varepsilon) \qquad \begin{cases} -\Delta u - l_\infty u = f(x,u) + \varepsilon h(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- $(P_\varepsilon)$ may not have a variational structure: truncation argument
- lack of symmetry ($h$ could be not odd): topological relevant critical values

Work in progress for $p \neq 2$

## Our results

Under the same assumptions of the unperturbed case, the number $\bar{m}$ of distinct critical values of $J$ is stable under small perturbations $(1 \leq \bar{m} \leq \dim(M_h \oplus \ldots \oplus M_k))$

- for $\varepsilon$ small $(P_\varepsilon)$ has at least a solution
- moltiplicity results (only $f$ odd):
  for $\varepsilon$ small, $(P_\varepsilon)$ has at least $\bar{m}$ distinct pairs of solutions
  but if $h$ is not odd, a further assumption is needed
- the problem can also be resonant
- cases $l_0 = 0$, $l_0 \in \{\pm\infty\}$

## Critical and essential levels

If $c \in \mathbb{R}$ is a critical level of a functional $I$, does $G$ "closed to $I$" have a critical value near $c$? This is not true for every critical level Reeken, Degiovanni-Lancelotti, Mawhin-Willem

Let $I \in C^1(X, \mathbb{R})$, $a, b \in \mathbb{R}, a \leq b$ and $I^c = \{u \in X : I(u) \leq c\}$. The pair $(I^b, I^a)$ is *trivial* if for every neighborhood $[\alpha', \alpha'']$ of $a$ and $[\beta', \beta'']$ of $b$ there exist two closed subsets $A$ and $B$ of $X$ s.t. $I^{\alpha'} \subseteq A \subseteq I^{\alpha''}$, $I^{\beta'} \subseteq B \subseteq I^{\beta''}$ and $A$ is a strong deformation retract of $B$

A real number $c$ is an *essential value* of $I$ if for every $\varepsilon > 0$ there exist $a, b \in ]c - \varepsilon, c + \varepsilon[$ $(a < b)$ s.t. the pair $(I^b, I^a)$ is not *trivial*

"odd" definitions

## Critical and essential values

- an essential value $c$ is a critical value if $(PS)_c$ holds
- $0$ is not an essential value for

$$\phi(x) = \begin{cases} (x+1)^3 & \text{if } x < -1 \\ 0 & \text{if } -1 \leq x \leq 1 \\ (x-1)^3 & \text{if } x > 1 \end{cases}$$

- values arising from mini-max procedures are essential ones
- if $I \in C^1$ and $(PS)$ holds, from the Deformation Lemma, if $c$ is not a critical value, "near" there exists a trivial pair
  in some sense we require that the reversed implication holds

## The idea of the proof

Truncation argument:

- continuous cut functions for $j \in \mathbb{N}$
- for $\varepsilon$ small, functionals $J_{j,\epsilon}$
- the critical levels of $J$ are critical ones for $J_{j,\epsilon}$, indeed

Let $c \in \mathbb{R}$ be a *topologically relevant critical value* (i.e. an essential one) of $I \in C^1(X, \mathbb{R})$. Then, for every $\eta > 0$ there exists $\delta > 0$ s.t. every $G \in C^1(X, \mathbb{R})$ satisfying $(PS)$ in $]c-\eta, c+\eta[$ with

$$\sup\{|I(u) - G(u)| : u \in X\} < \delta$$

admits a critical value in $]c-\eta, c+\eta[$

the critical points of $J_{j,\epsilon}$ are uniformly bounded with respect to $j$ and $\varepsilon$