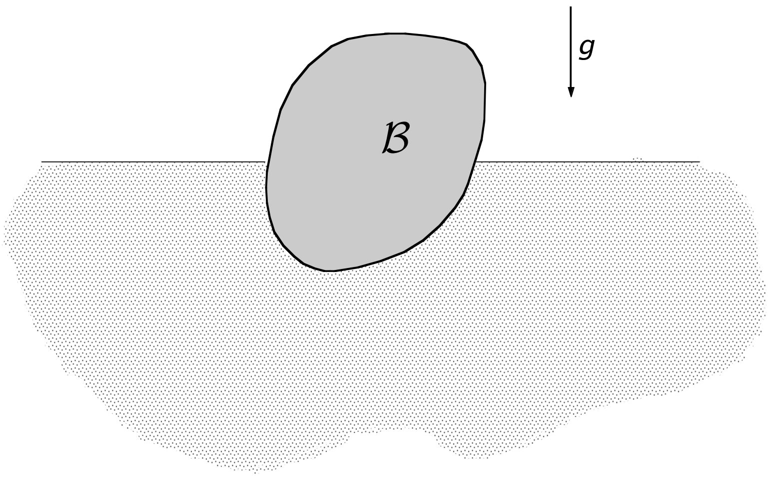
Floating (or sinking) Bodies

Robert Finn

Place rigid body $\mathcal{B}$ of density ρ into bath of fluid of density ρ_0 in vertical gravity field g, assuming no interaction of adjacent materials. Then $\mathcal{B}$ floats if $\rho < \rho_0$ and sinks if $\rho > \rho_0$ (Archimedes Law).



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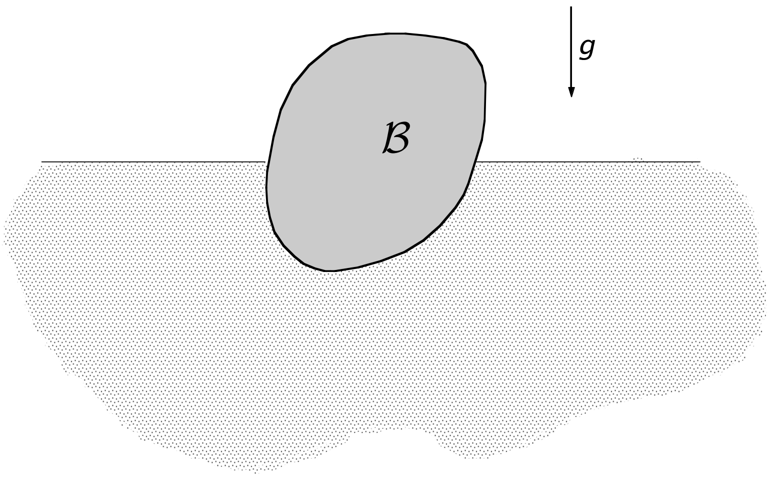
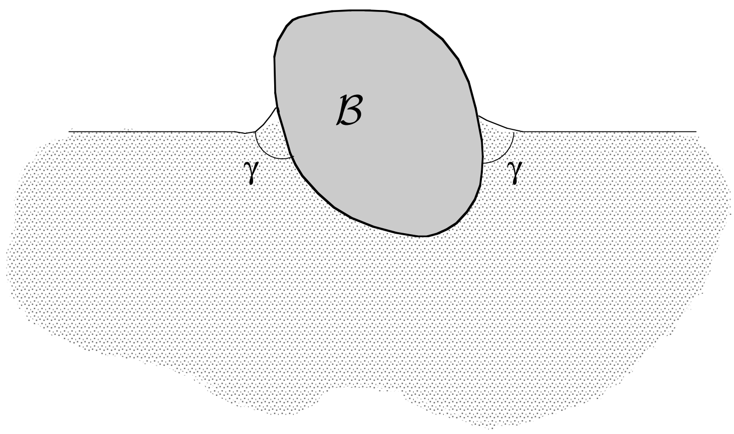


In fact adjacent materials do interact, and the change can be dramatic, with very different behavior.
Aristoteles already knew 350 BC that heavy bodies can float on water. Now known that if $\mathcal{B}$ floats then orientation and height of $\mathcal{B}$ adjusts, and outer free surface changes, to meet $\mathcal{B}$ in “contact angle“ γ, depending on materials.



Nothing further for almost 2000 years, when Galileo questioned Aristoteles’ reasoning in his „Discorsi“, about 1600. Then 200 years later Laplace took up the topic, using the then new concept of surface tension, and the relatively new magic weapon of the Calculus, obtaining some remarkable although not yet complete mathematical results.

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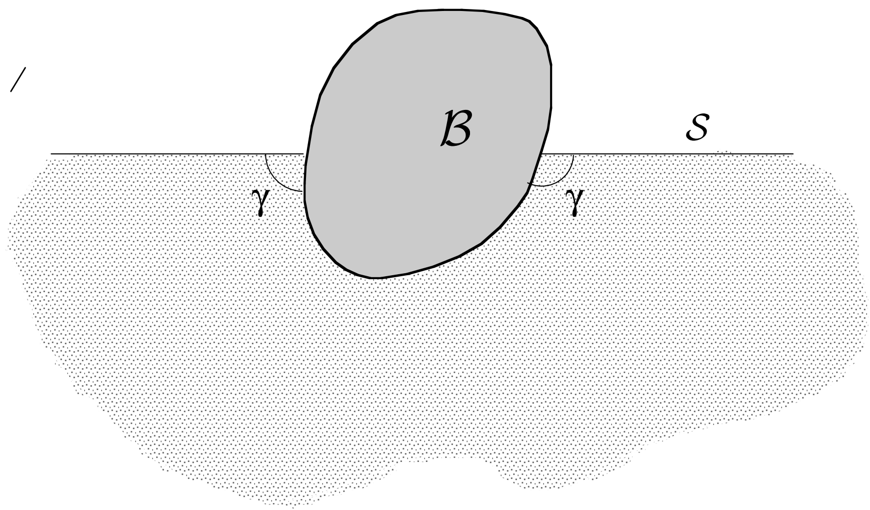

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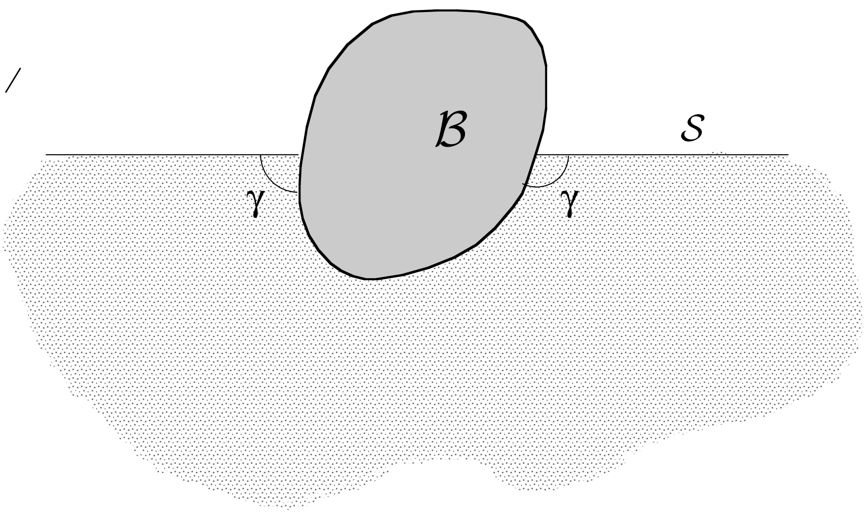

Theorem: *There exist at least four positions of $\mathcal{B}$ yielding prescribed γ.*

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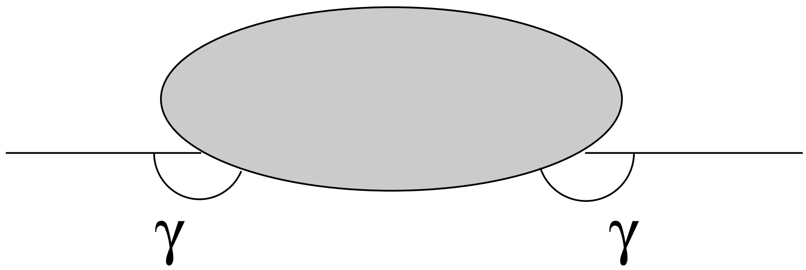
Proof: From „Four vertex theorem“ of differential geometry

Example: Ellipse yields exactly four



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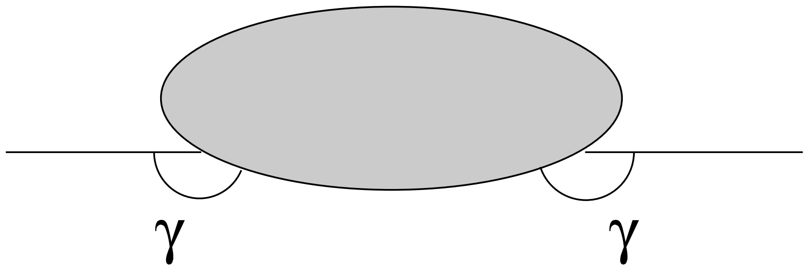

Question: What are the shapes that will float in every orientation?

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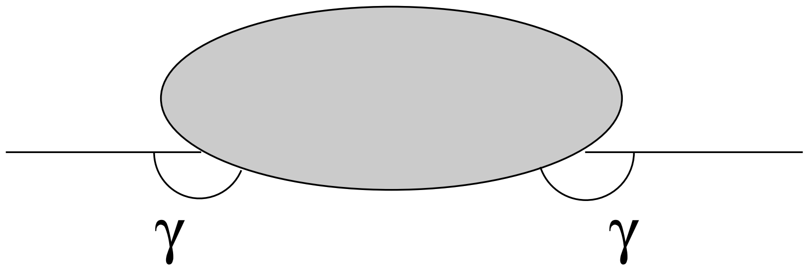

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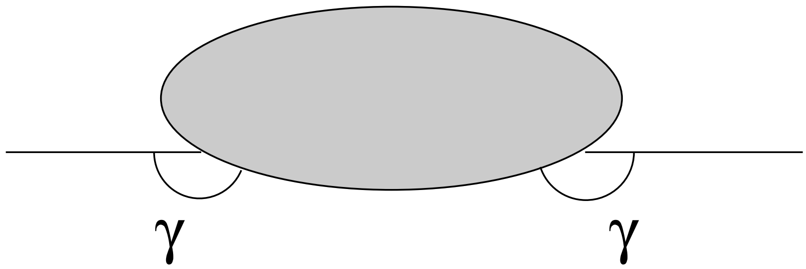

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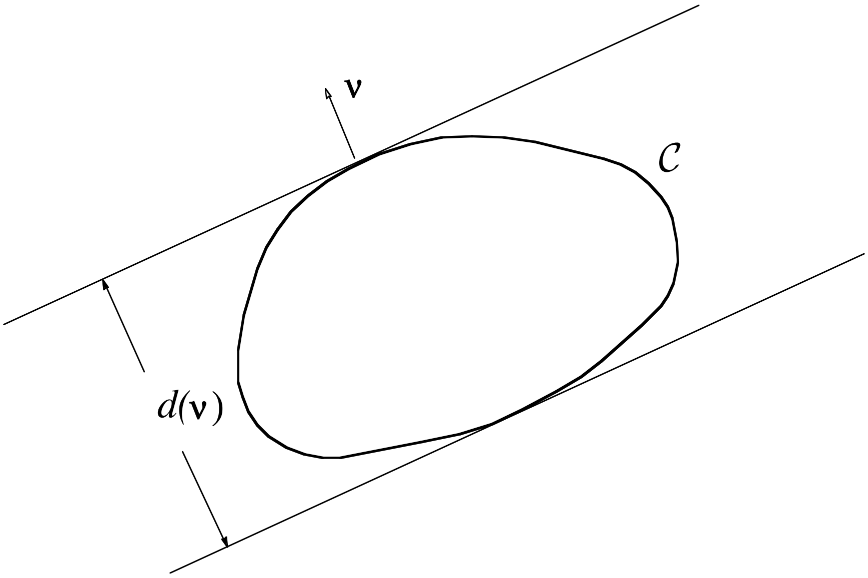
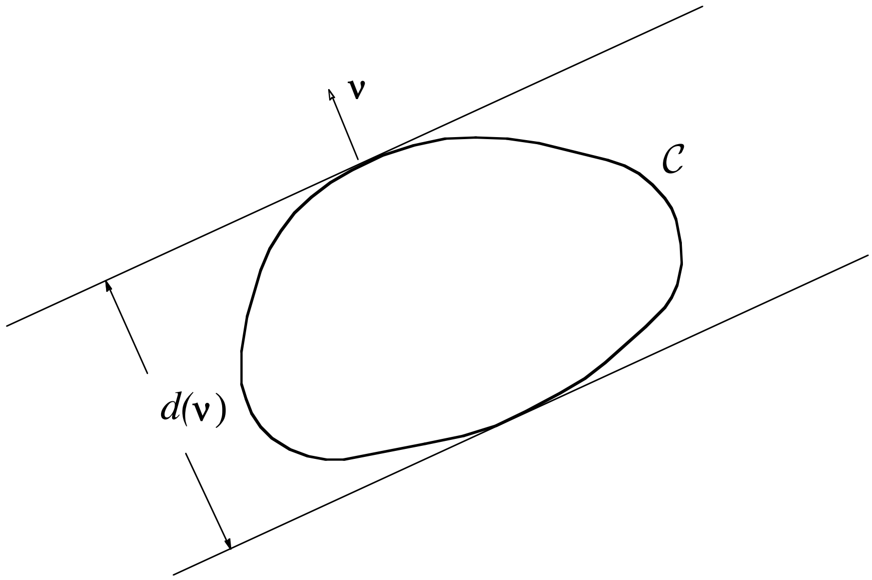
Counterexample: If $\gamma=\pi/2$ then any curve of constant width will work!



$d(\boldsymbol{\nu})$ = width in direction $\boldsymbol{\nu}$

Defn. „*constant width*“ if $d(\boldsymbol{\nu})$ independent of $\boldsymbol{\nu}$.



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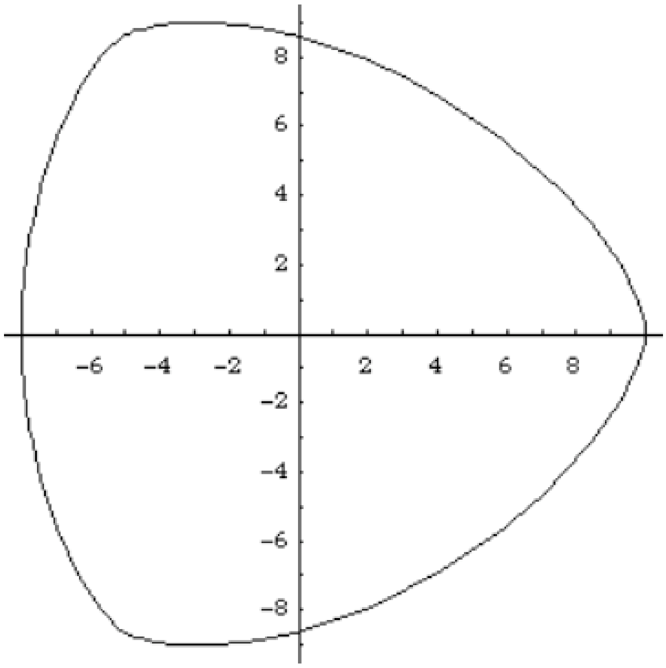
N.B. There is some indication that the disaster on the spacecraft Challenger happened at least in part because the engineers assumed it self-evident that all curves of constant width are circles. The engineers determined the „circularity“ of the port to a fuel module by measuring its width in three different directions.

Example due to Rabinowitz (1997):

$$x = 9\cos\vartheta + 2\cos 2\vartheta - \cos 4\vartheta$$
$$y = 9\sin\vartheta - 2\sin 2\vartheta - \sin 4\vartheta$$



Procedure yields non-countable family of counterexamples.

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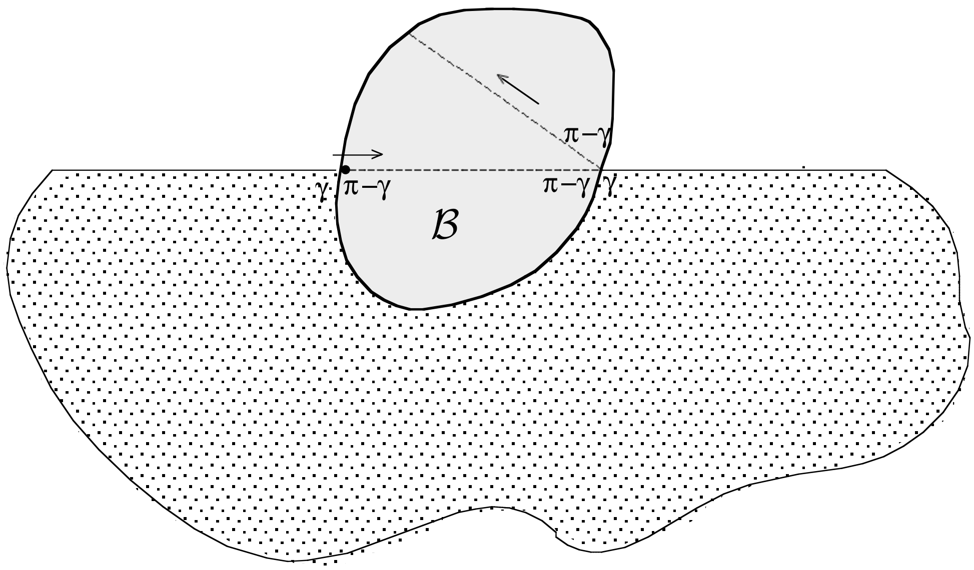

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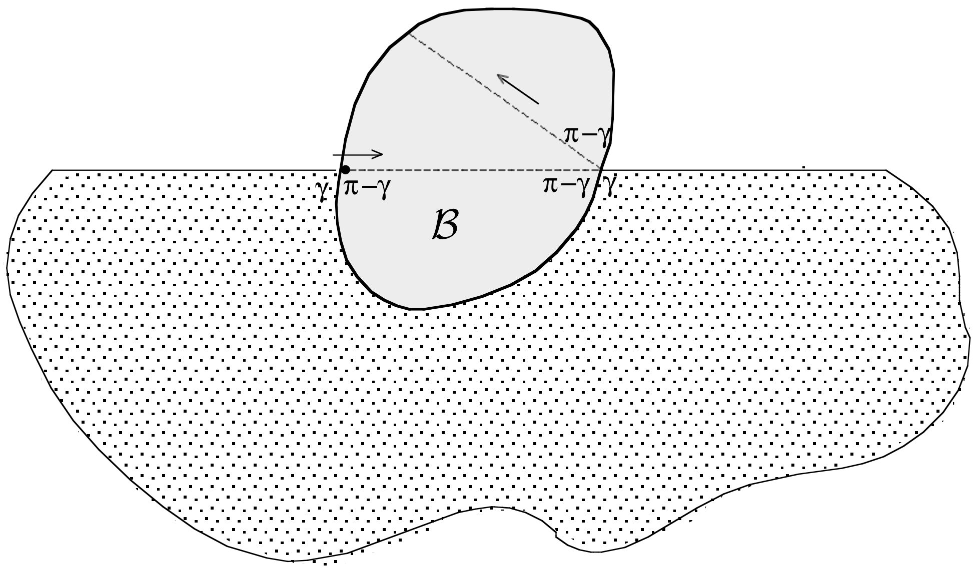

There is a theorem by E. Gutkin, that *a non-circular billiard table with invariant curve at angle* α *exists if and only if* α *satisfies*

$$\tan n\alpha = n\tan\alpha$$

for some $n \geq 3$.

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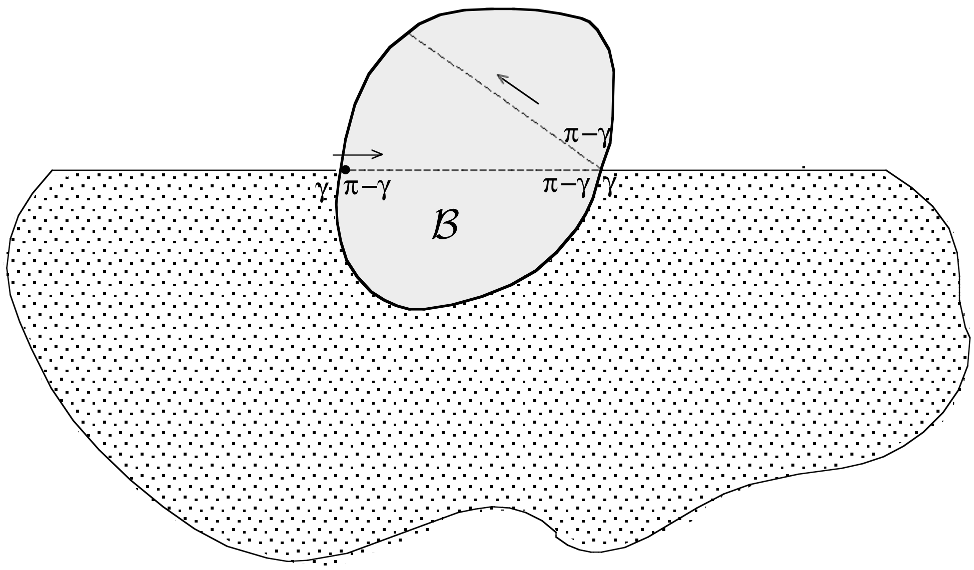

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We conclude that *the set of contact angles γ, for which a non-circular body can be found that will float in every orientation with angle γ, is countable and everywhere dense.*

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Proof: Based on **Joachimstal's Theorem** in differential geometry: *If two surfaces meet at a constant angle, and if the intersection curve is a curvature line on one of the surfaces, then it is a curvature line on both.*

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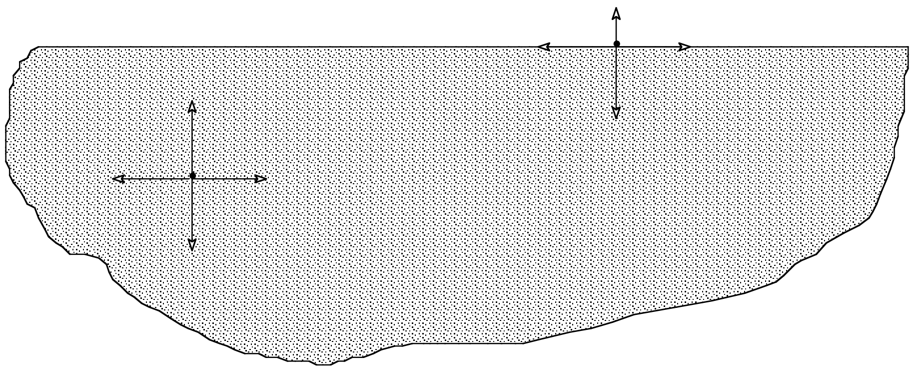
There is also a surface energy

$\mathcal{E}_{\mathcal{S}}$ = work needed to create surface interface, due to unequal attractions.

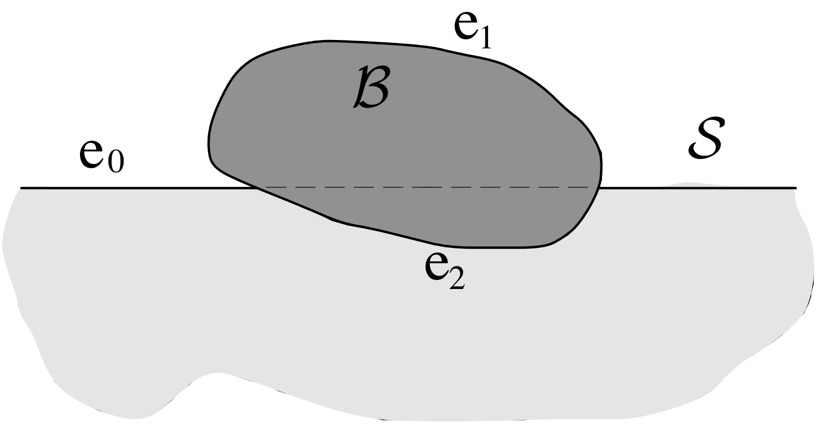


$\mathcal{E}_{\mathcal{S}} = e\mathcal{S}$, where e = energy per unit area, depending only on materials. For fluid/fluid interface, $e \equiv \sigma$ = surface tension.

We seek to minimize the total energy,

$$\mathcal{E} = \mathcal{E}_{\mathcal{S}} + \mathcal{E}_g.$$

Let e_0, e_1, e_2 be surface energy densities on outer, upper, lower interfaces. Start again with 2-D case, with $g = 0$.



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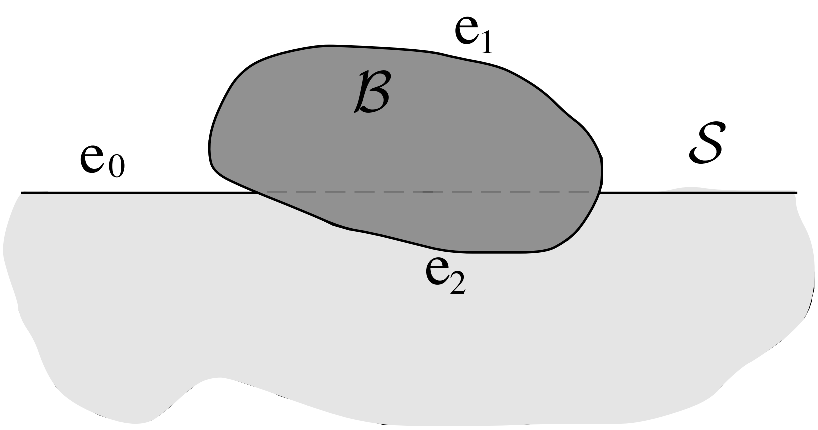

Theorem: *If $\mathcal{B}$ is strictly convex, and if $g = 0$, then a strictly minimizing partly wetting configuration exists if and only if* $\left|\frac{e_1 - e_2}{e_0}\right| < 1$.

Theorem: *Whether or not $g = 0$, if $|e_1 - e_2| < |e_0|$ then at a local minimum the contact angle γ is determined by*

$$e_1 = e_2 + e_0 \cos\gamma.$$

Example: Two of the ellipse positions minimize, the others are unstable.

If $g \neq 0 \Rightarrow$ basic changes to previous considerations. If the tank infinitely deep, a fully submerged $\mathcal{B}$ must sink if its density $\rho > \rho_0$. *Thus there can be no global minimum for the energy* $\mathcal{E}$. But conceivably a partially wetted position can occur, with a local minimum for $\mathcal{E}$.

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Behavior controlled by limitations on how much the outer surface $\mathcal{S}$ can bend. In 2-D can determine the most general $\mathcal{S}$ explicitly, as solution of

$$(\sin\psi)_x = \kappa u$$

Here ψ = inclination angle, $\kappa = \rho g/\sigma$. We get

$$x = x_0 + \frac{1}{\sqrt{2\kappa}} \int_0^{\varphi} \frac{\sin\varphi}{\sqrt{1-\sin\varphi}} d\varphi$$

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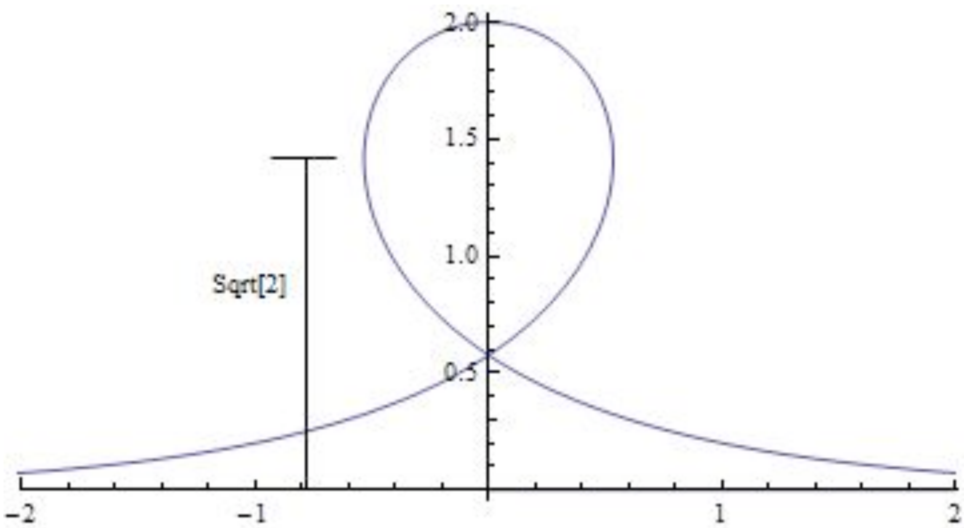

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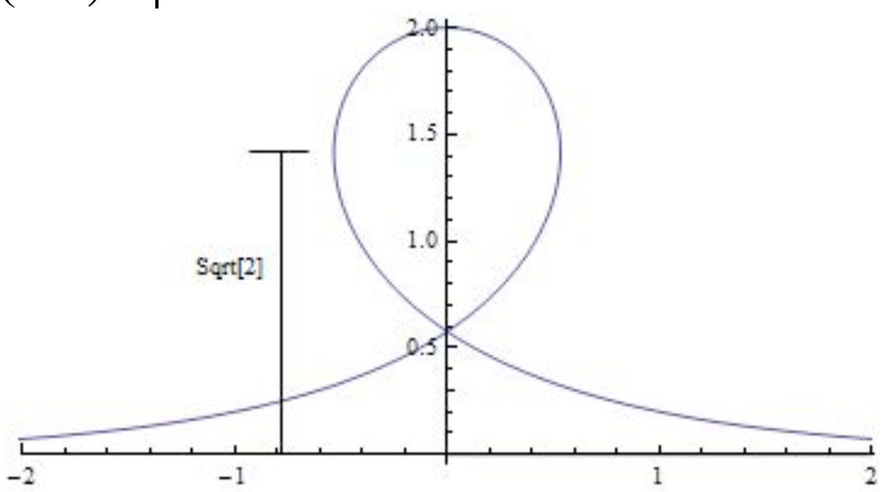

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We conclude: *Let h_c be height of centroid of $\mathcal{B}$, and $d = \mathrm{dia}\{\mathcal{B}\}$. If $|h_c| > 2\sqrt{e_0/(\rho_2 - \rho_1)g} + d$ then $\mathcal{B}$ disjoint from $\mathcal{S}$, hence $|\mathcal{E}_g| < A + B\sqrt{e_0}$* when floating, where A, B depend only on shape of body and gravitational quantities, not on surface energies. So from bottom to top of possible positions

$$|\delta\mathcal{E}_g| < 2\,(A + B\sqrt{e_0}).$$

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Choose e_0 big enough that $2\,(A + B\sqrt{e_0}) < s_{\mathcal{B}}\, e_0$. Then

choose e_1, e_2 so that $\delta\mathcal{E}_{\mathcal{S}} = -(e_2 - e_1)\, s_{\mathcal{B}} < -2\,(A + B\sqrt{e_0})$.

Then $\delta\mathcal{E}_{\mathcal{S}} < -|\delta\mathcal{E}_g| \Rightarrow \delta\mathcal{E} < 0 \Rightarrow$ local minimum.

Further

$$\frac{e_2 - e_1}{e_0} > \frac{2\,(A + B\sqrt{e_0})}{s_{\mathcal{B}} e_0}$$

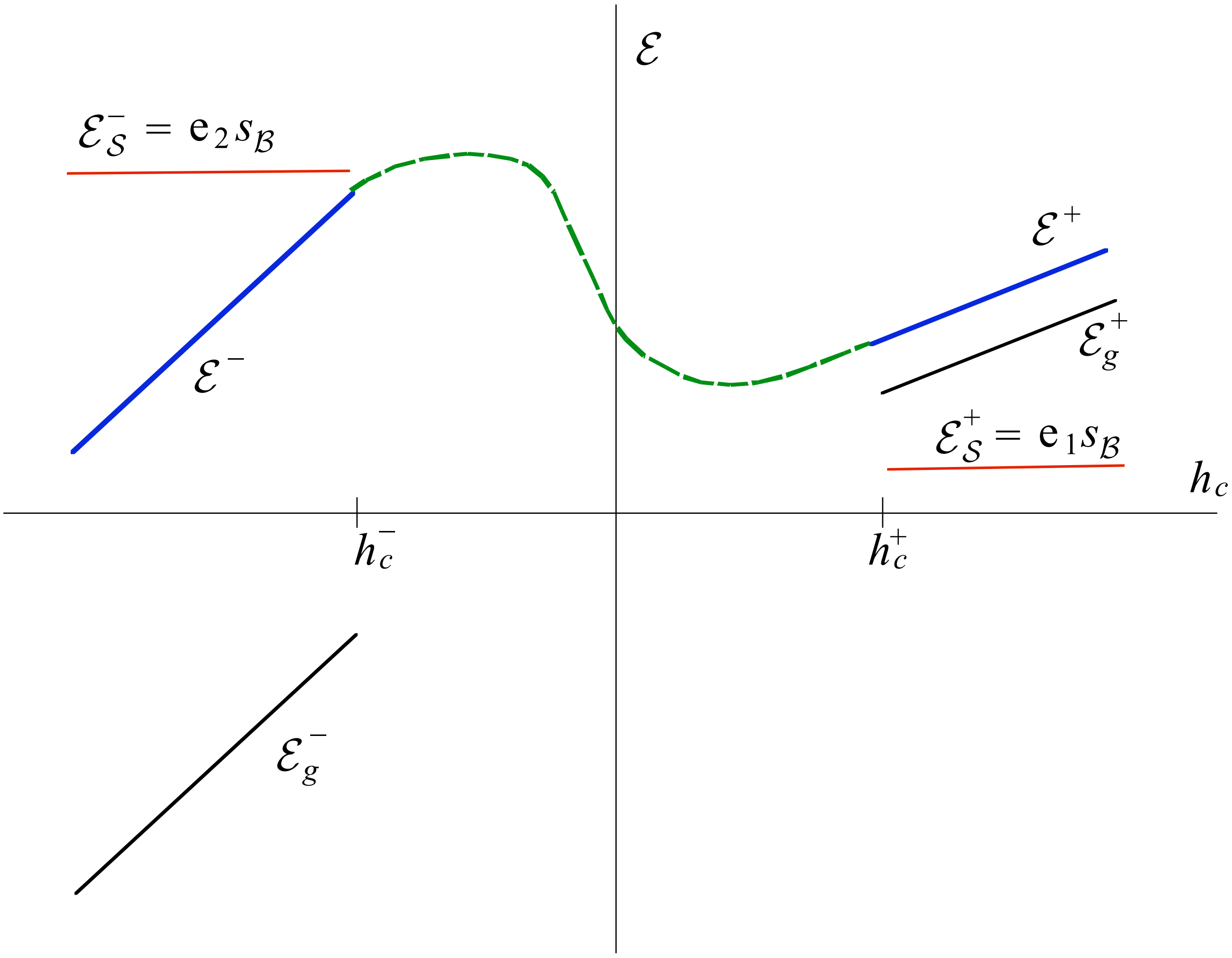
The right side $< 1 \Rightarrow$ we can scale e_1, e_2 to get

$$1 > \frac{e_2 - e_1}{e_0} > \frac{2\,(A + B\sqrt{e_0})}{s_{\mathcal{B}} e_0}$$

and the minimizing configuration achieved with

$$e_1 = e_2 + e_0 \cos\gamma.$$

N.B. This result requires $\gamma > \pi/2$.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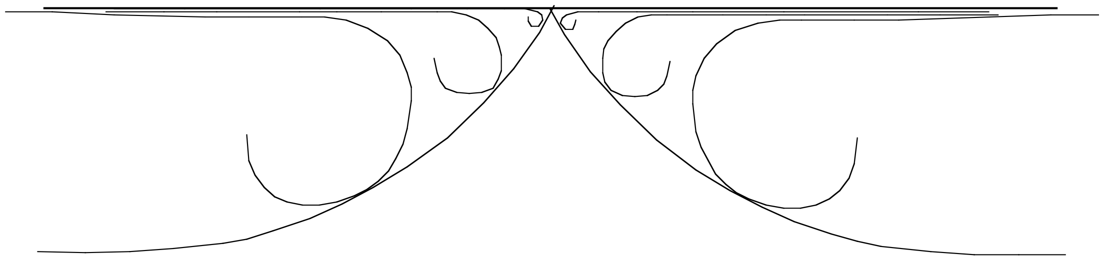
What happens in 3-D? More complicated since many more possible configurations for $\mathcal{S}$. Restrict attention to the family of symmetric solns:



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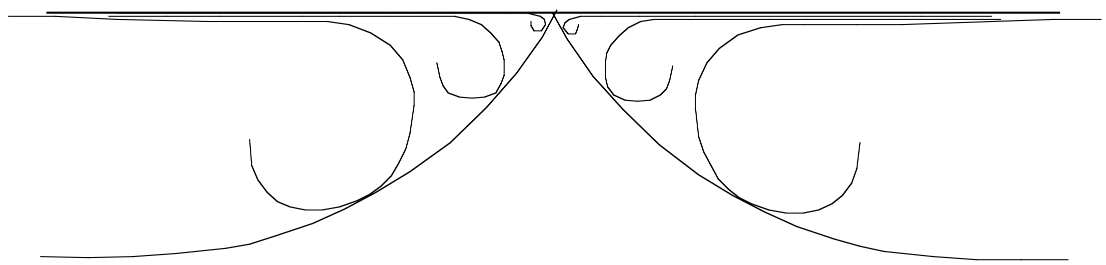

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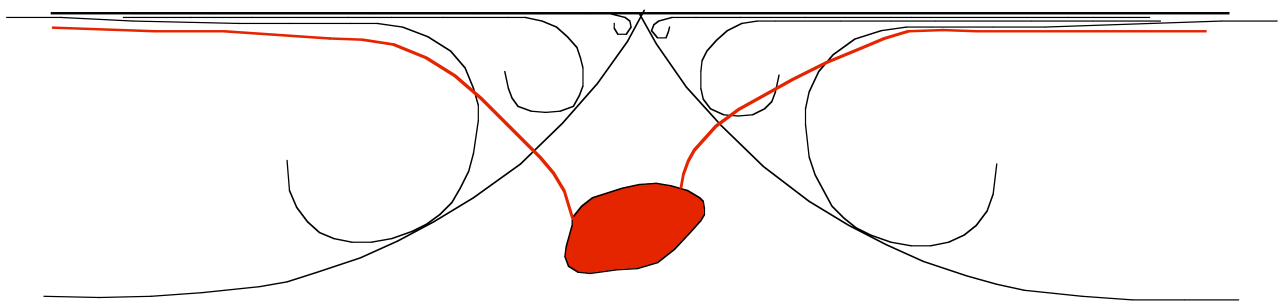
Thus, the red configuration is not possible, regardless of the density or surface properties of the body.



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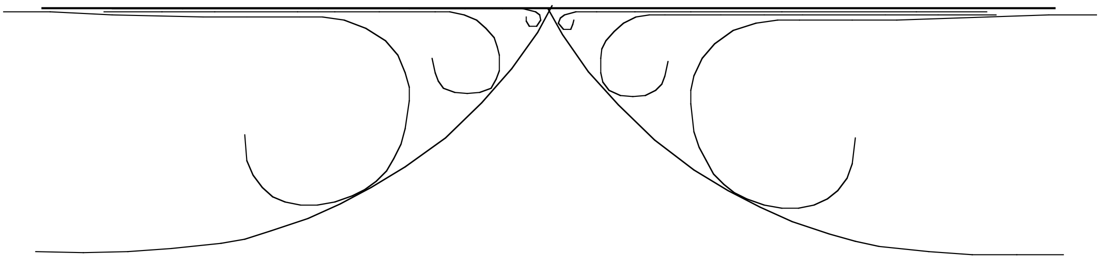

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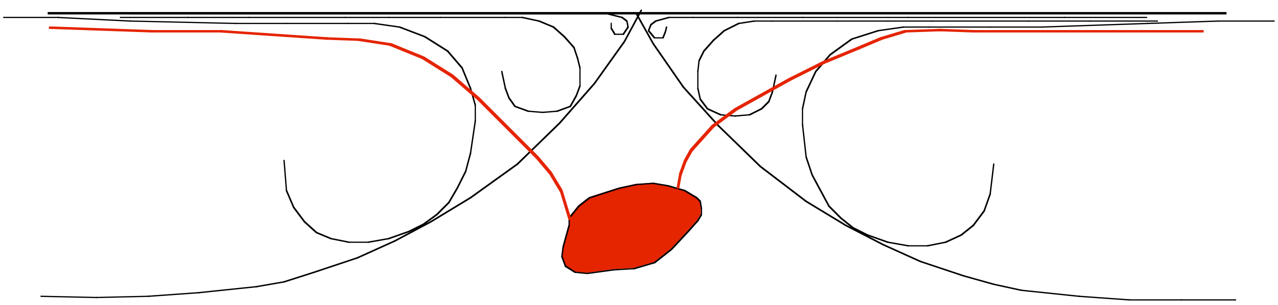

If a small body floats it must be closer to the surface than must a large one of the same shape.

Theorem: *Any body of given density and with* $\gamma > \pi/2$ *can be made to float by scaling its size down sufficiently.*

Theorem (2 or 3 D): *If one holds $\mathcal{B}$ rigidly far enough above or below the rest level at infinity, then $\mathcal{B}$ will be disjoint from $\mathcal{S}$. If one then moves $\mathcal{B}$ rigidly vertically from top to bottom (or reverse), the motion of $\mathcal{S}$ will be discontinuous.*

Proof: The height of $\mathcal{S}$ is given by a solution of

$$\mathrm{div}\, Tu = \kappa u, \quad Tu \equiv \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \quad \kappa > 0.$$

The only global solution for $\mathcal{S}$ that vanishes at infinity is the flat one $u \equiv 0$.

Historical comments

Apparently the first recorded discussion of capillarity phenomena is due to Aristoteles c. 350 B.C, and addressed exactly the question that has since been most ignored. He wrote:

A large flat body, even of heavy material, will float on water, but a long thin one such as a needle will always sink.

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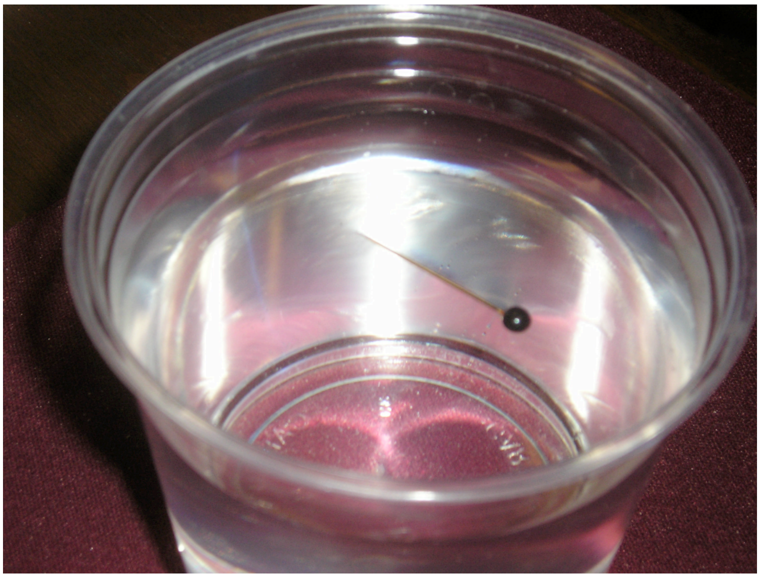
2. Experimental response:



Formal calculations suggest that a thin metal needle with γ as small as 16 deg can be made to float in equilibrium on water. Thus my own criterion, although perhaps the first sufficient one known, may not be necessary, as it requires $\gamma > \pi/2$. Conceivably it is necessary for stable equilibria, at this point I have no idea.

In a somewhat other direction, the theory predicts an absolute maximum diameter for which a needle of given material can float in the earth's gravity field, regardless of contact angle. The result indicates that a typical small paper clip can be made to float, but that a large paper clip will always sink. I could confirm the prediction with a kitchen sink experiment.

This may be where Aristoteles went wrong; a needle in his time may have had a much larger diameter than the typical modern household product that I used for my picture.

The end. Thank you for listening!