Sobre experimentos realizados en el Max Planck Institute (MIPKG): bifurcando cilindros

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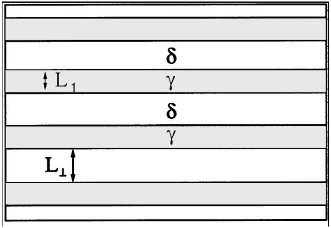
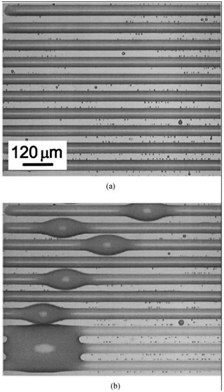
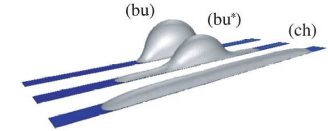
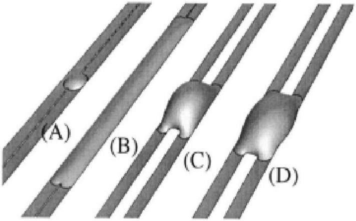
Motivation: experiments about the adhesion of liquids along strips.

[Reference: Gau, H., Herminghaus, S., Lenz, P. & Lipowsky, R.: Liquid microchannels on structured surfaces. *Science* (1999)]

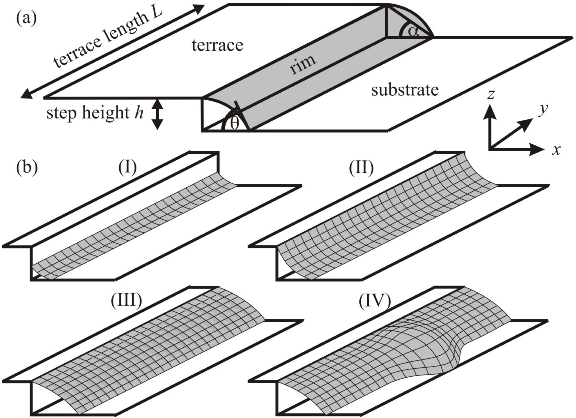




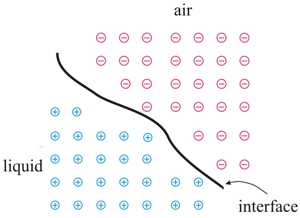


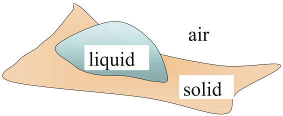


Non uniqueness of solutions ⇝ bifurcation



[Reference: Kusumaatmaja, H., Mutihac, C., Lipowsky, R. & Riegler, H.: How capillary instabilities affect nucleation processes, (2010) preprint]





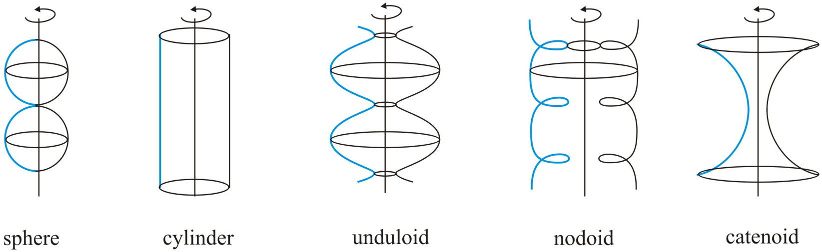
An interface is the boundary between two different media

In equilibrium the shape of the liquid drop

$$2H\tau = P_A - P_L$$

The interface is a surface of constant mean curvature

Surfaces of revolution with constant mean curvature



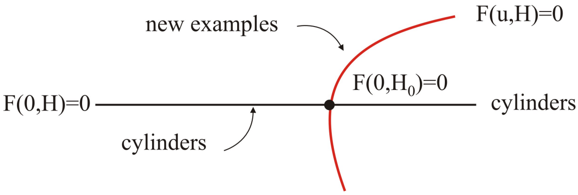
Objective

A mathematical proof of such evidences

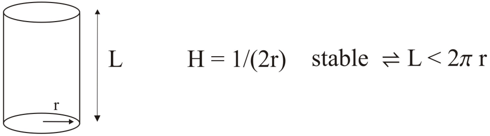
Non uniqueness of solutions $\rightsquigarrow$ bifurcation.

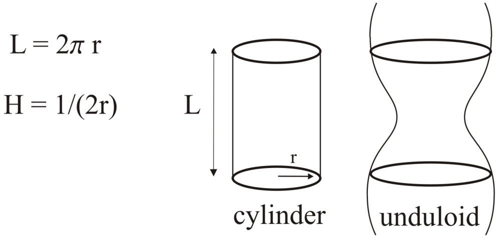
The existence of new surfaces must occur when the stability of cylinders fails $\leftrightsquigarrow$ study the stability of cylinders.

Study a eigenvalue problem $\rightsquigarrow$ dimension of solutions of the Jacobi equation.

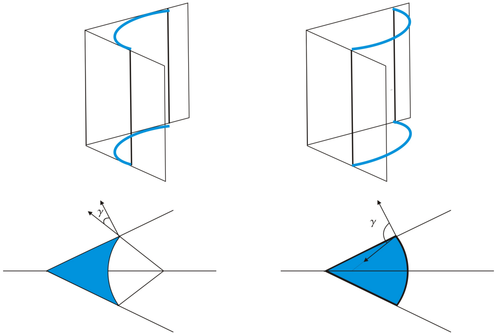


For cylinders: the Plateau-Rayleigh instability criterio $\Rightarrow$ existence of new periodic constant mean curvature (unduloids) [Schlenk & Sicbaldi]



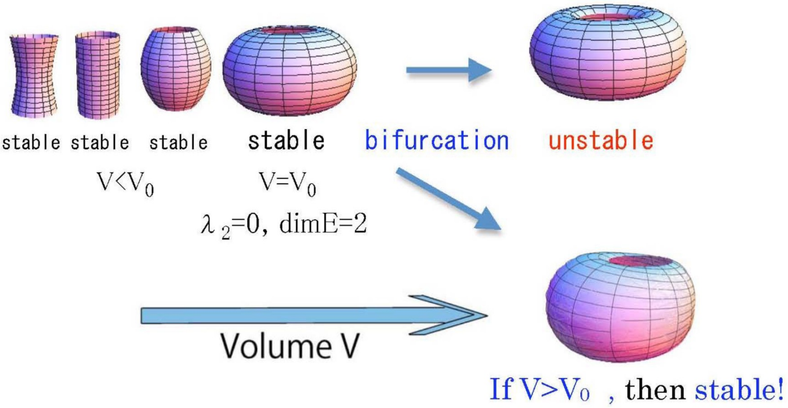


- Vogel: bifurcating cylinders in wedges with constant contact angle in both sides of the wedge.



Given a convex cylinder in a wedge with a given contact angle, there exists a volume such that there are a family of capillary surfaces in the wedge that bifurcate from the given cylinder.

- Patnaik (a student of Wente): prescribing two coaxial circles in parallel planes.
- Mazzeo, Pacard, Rossman, Grosse-Brauckmann: bifurcation of nodoids.
- Koiso-Palmer-Piccione: bifurcation nodoids.



The stability of a CMC surface

$$E''(0) = -\int_M u(\Delta u + |\sigma|^2 u)\ dM + \int_{\partial M} u\Big(\frac{\partial u}{\partial \nu} - qu\Big) ds,$$

$$q = \frac{1}{\sin\gamma}\tilde{\sigma}(\tilde{\nu},\tilde{\nu}) + \cot\gamma\sigma(\nu,\nu).$$

The eigenvalue problem

$$\begin{cases} Lu + \lambda u = 0 & \text{on } M \qquad L = \Delta + |\sigma|^2 \\ u = 0 & \text{on } \partial M \end{cases} \tag{1}$$

Lemma

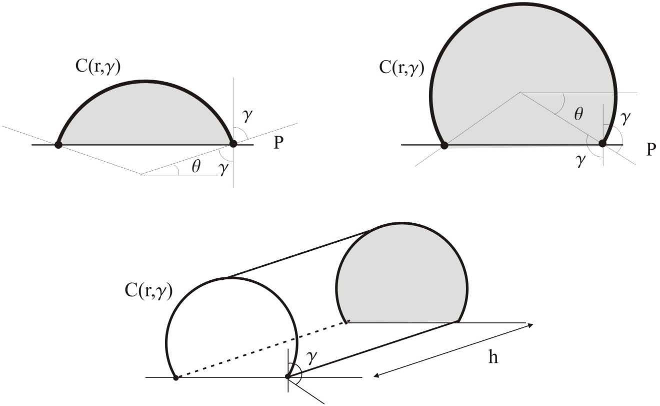
1. *A countable set of eigenvalues $\lambda_1 < \lambda_2 \leq \ldots$, with $\lambda_n \to +\infty$*
2. *If $\lambda_1 \geq 0$, the immersion ϕ is stable.*
3. *If $\lambda_2 < 0$, the immersion ϕ is unstable.*
4. $L^2(M) = \bigoplus_{n=1}^{\infty} E_{\lambda_n}$.

$C(r,\gamma)$: the piece of cylinder over P whose boundary lies in P.

γ is the contact angle; $\theta = \pi/2 - \gamma$.

- Concave cylinders: $\gamma \in (0, \pi/2)$.
- Convex cylinders: $\gamma \in (\pi/2, \pi)$.

$\partial C(r,\gamma) = L_1 \cup L_2$.



Stability problem in truncated pieces $0 \leq x \leq h$ and to vary h.

Use separation of variables: $C(r,\theta) \leftrightarrows [0,h] \times [\frac{\pi}{2}-\gamma, \frac{\pi}{2}+\gamma]$ with variables (t,s).

$$u(t,s) = \sum_{n=1}^{\infty} g_n(s) \sin(\frac{n\pi}{h} t)$$

$$g_n(\frac{\pi}{2}-\gamma) = g(\frac{\pi}{2}+\gamma) = 0$$

$$\Delta = \partial_{tt} + \frac{1}{r^2}\partial_{ss}, \quad |\sigma|^2 = 4H^2 - 2K = \frac{1}{r^2}.$$

$$L(u) + \lambda u = \sum_{n=1}^{\infty} \Big(\frac{1}{r^2} g_n'' + (\frac{1}{r^2} - \frac{n^2\pi^2}{h^2} + \lambda) g_n\Big) \sin(\frac{n\pi}{h} t).$$

Solve:

$$g_n''(s) + r^2\Big(\frac{1}{r^2} - \frac{n^2\pi^2}{h^2} + \lambda\Big)g_n(s) = 0$$

$$g_n(\frac{\pi}{2} - \gamma) = g_n(\frac{\pi}{2} + \gamma) = 0$$

According the sign of $C = r^2(\frac{1}{r^2} - \frac{n^2\pi^2}{h^2} + \lambda)$

Proposition

1. *If $\gamma \in (0, \pi/2]$, the cylinder $C(r,\gamma)$ is stable.*
2. *Assume $\gamma \in (\pi/2, \pi)$. Consider a cylinder $C(r,\gamma)$ of length h. Then $\lambda_1 \geq 0$ if and only if $h \leq h_0$, where*

   $$h_0 = \frac{2\pi r\gamma}{\sqrt{4\gamma^2 - \pi^2}}.$$

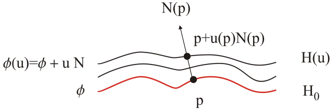
   In such case, the surface is stable.
3. *A cylinder $C(r,\gamma)$ with $\gamma \in (\pi/2, \pi)$ is unstable.*

The result

Let $\gamma \in (\pi/2, \pi)$. There exists bifurcation when the length of the cylinder is

$$T = 2h_0 = \frac{4\pi r\gamma}{\sqrt{4\gamma^2 - \pi^2}}.$$

The cylinder bifurcates in a family of non-rotational surfaces with constant mean curvature with the same boundary.



Let $\phi : M \to \mathbb{R}^3$ be with constant mean curvature H_0.

Let V be an open of $0 \in C_0^{2,\alpha}(M)$ such that for any $u \in V$, the normal graph

$$\phi_u = \phi + uN$$

Define $F : V \times \mathbb{R} \to C^\alpha(M)$ by

$$F(u,H) = 2(H - H(u)), \quad H(u) = \text{ mean curvature of } \phi_u.$$

- $F(0, H_0) = 0$.
- ϕ_u has constant mean curvature iff there exists $H \in \mathbb{R}$ such that

$$F(u,H) = 0.$$

Lemma

The functional F is Fréchet differentiable with respect u and H and

$$D_uF(0,H)v = -L(v), v \in C_0^2(M).$$

Lemma

Given $\lambda \in \mathbb{R}$ and $f \in L^2(M)$, we consider the equation

$$\lambda u - L(u) = f,\ u \in H_0^1(M).$$

1. *If λ is not an eigenvalue, there is a unique solution.*
2. *If λ is an eigenvalue, there is a solution if and only f is L^2-orthogonal to E_λ.*

Uniqueness of $F(u, H) = 0 \leftrightsquigarrow$ the Implicit Function Theorem $\leftrightsquigarrow$ solutions of the Jacobi equation $Lu = 0$.

Implicit Function Theorem: $D_u F(0, H_0)$?

$$D_u F(0, H_0)(v) = L[v] = \Delta v + (4H_0^2 - 2K)v$$

- If $D_u F(0, H_0)$ is bijective, there exists $\delta > 0$ and a unique map $\varphi : (H_0 - \delta, H_0 + \delta) \to C_0^{2,\alpha}(M)$ such that $\varphi(H_0) = 0$ and $F(\varphi(H), H) = 0$ for any $|H - H_0| < \delta$. In such case, the immersion defined by $\phi + \varphi(H)N$ has constant mean curvature H.

- Assume $\lambda = 0$ is not an eigenvalue ($\Leftrightarrow$ the only solutions of the Jacobi equation are trivial). Then

  $D_u F(0, H_0)$ is one-to-one

  and apply the Implicit Function Theorem.

  1. injective: if $v \in C_0^2(M)$ satisfies $D_u F(0, H_0)(v) = 0$, by Lemma, the solution is unique $\Rightarrow v = 0$.
  2. surjective: given $f \in L^2(M)$, $\exists v, D_u F(0, H)(v) = f$? (use Lemma)

Concave cylinders deform in concave cylinders

Assume $\lambda = 0$ is an eigenvalue. A particular case of the Implicit Function Theorem:

Lemma (Koiso)

Let $\phi : M \to \mathbb{R}^3$ be an immersion with constant mean curvature H_0. Assume

1. $\lambda = 0$ *is an eigenvalue of* L.
2. $E_0 = < u_0 >$.
3. $\int_M u_0 \, dM \neq 0$.

Then there exists uniqueness of deformation by CMC surfaces with the same boundary.

If the Implicit Function Theorem fails $\Rightarrow$ 0 is an eigenvalue of L with $\lambda_2 \leq 0$.

If $\lambda_2 = 0$, one seeks bifurcation when

1. $\dim(E_0) \geq 2$ or
2. if $\dim(E_0) = 1$ with $E_0 =< u_0 >$ and $\int_M u_0 \, dM = 0$.

Theorem (Crandall and Rabinowith)

Let $F : X \times I \to Y$ be a twice continuously Fréchet differentiable functional, where X and Y are Banach spaces, $I \subset \mathbb{R}$ and $H_0 \in I$. Suppose $F(0, H) = 0$ for all $H \in I$ and

1. *dim* $Ker(D_u F(0, H_0)) = 1$*:* $Ker(D_u F(0, H_0)) =< u_0 >$.
2. *cod*$($*rank* $D_u F(0, H_0)) = 1$*:* $F(0, H_0)$ *is a Fredholm operator of index zero.*
3. $D_H D_u F(0, H_0)(u_0) \notin$ *rank* $D_u F(0, H_0)$.

Then there exists bifurcation at $(0, H_0)$ *by* $s \longmapsto (u(s), H(s))$

1. $u(0) = 0,\ H(0) = H_0$.
2. $F(u(s), H(s)) = 0,$ *for any* $|s| < \epsilon$.
3. *in a neighbourhood of* $(0, H_0)$*, there are no more solutions.*

Here we take $X = V \subset C_0^{2,\alpha}(M)$ and $Y = C^{\alpha}(M)$. Fix a radius $r > 0$ (or $H_0 = 1/(2r)$).

We seek non trivial solutions of (1) that are T-periodic in the x-direction for some period $T > 0$. Use separation of variables: functions u defined in $\mathbb{R}/2\pi T\mathbb{Z} \times [\frac{\pi}{2} - \gamma, \frac{\pi}{2} + \gamma]$.

Write u as a Fourier expansion on $\sin(2\pi nt/T)$ and $\cos(2\pi nt/T)$:

$$u(t,s) = \sum_{n=1}^{\infty} g_n(s) \sin(\frac{2\pi n}{T} t).$$

Convex cylinders

1. If $h \leq h_0$, the eigenvalues are all non-negative.
2. If $h \in (h_0, 2h_0)$, the first eigenvalue is negative but the other $\lambda_{k,n}$ are all positive.
3. If $h = 2h_0$, $\lambda_2 = 0$. Then $T = 2h_0 \frac{4\pi r\gamma}{\sqrt{4\gamma^2 - \pi^2}}$.

Proof of $\dim(E_0) = 1$.

The functions g_n satisfy

$$g_n''(s) + c^2 g_n(s) = 0, \quad c^2 = r^2\Big(\frac{1}{r^2} - \frac{4n^2\pi^2}{T^2} + \lambda\Big).$$

Then

$$g_n(s) = \sin(\frac{k\pi(s-\theta)}{\pi - 2\theta}), \quad k \in \mathbb{N}, \theta = \frac{\pi}{2} - \gamma.$$

Denote for k, n the eigenfunctions

$$u_{k,n}(t,s) = \sin\big(\frac{k\pi(s-\theta)}{\pi-2\theta}\big)\sin\big(\frac{2\pi n}{T}t\big),$$

$$\lambda_{k,n} = \frac{1}{r^2}\Big(\frac{(k^2-n^2)\pi^2 + n^2(\pi-2\theta)^2}{(\pi-2\theta)^2} - 1\Big).$$

Then 0 is an eigenvalue for $k = n = 1$, that is, $\lambda_{1,1}$ and

$$E_0 = < u_{1,1} > = < \sin\big(\frac{\pi(s-\theta)}{\pi-2\theta}\big)\sin\big(\frac{2\pi}{T}t\big) > .$$

In particular, $\dim(E_0) = 1$.

Proof of $\mathrm{cod}(\mathrm{rank} D_u F(0, H_0)) = 1$.

$$D_u F(0, H)(v) = L(v) = v_{uu} + 4H^2 v_{ss} + 4H^2 v.$$

Calculate $\mathrm{Im}(L)$:

$$\begin{aligned} f \in \mathrm{Im}(L) &\Leftrightarrow \exists v : L(v) = f \quad (0 \text{ is an eigenvalue of } L) \\ &\Leftrightarrow \int_M u_{1,1} v \; dM = 0 \text{ for any } v \in \mathrm{Ker}(L) \end{aligned}$$

$$\mathrm{Im}(L) = < u_{1,1} >^{\perp} \Rightarrow \mathrm{cod}(\mathrm{rank} D_u F(0, H_0)) = 1.$$

Proof of $D_H D_u F(0,H)(u_{1,1}) \notin \mathrm{Im}(D_u F(0,H))$.

Computation of $D_H D_u F(0, H_0)$:

$$D_H D_u F(0,H)(v) = 8H(v_{ss} + v).$$

$$\begin{aligned} D_H D_u F(0,H)(u_{1,1}) &= 8H\left(1 - \frac{\pi^2}{4\gamma^2}\right)\Big(\sin\left(\frac{\pi(s-\theta)}{\pi - 2\theta}\right)\sin\left(\frac{2\pi t}{T}\right)\Big) \\ &= 8H\left(1 - \frac{\pi^2}{4\gamma^2}\right)u_{1,1}. \end{aligned}$$

If $D_H D_u F(0,H)(u_{1,1}) \in \mathrm{Im}(D_u F(0,H))$, by using Lemma

$$\int_M u_{1,1} D_H D_u F(0,H)(u_{1,1}) \, dM = 0.$$

Thus

$$\int_M 8H\left(1 - \frac{\pi^2}{4\gamma^2}\right)u_{1,1}^2 \, dM = 0,$$

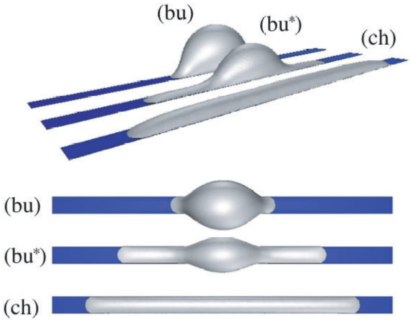
contradiction! because $\gamma \neq \pm\pi/2$.

The surfaces obtained close to the value H_0 are: embedded, periodic with period T and lie in one side of P. By Alexandrov reflection method:

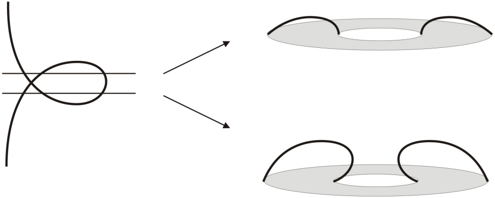
Corollary

The new cmc surfaces have an axial symmetry.

(...) giving a mathematical support about the experiments.



In progress: patterns made by ring-shaped domains (MIPKG) $\leftrightsquigarrow$ bifurcation of nodoids



[Reference: Lenz, P. Fenzl, W. & Lipowsky, R.: Wetting of ring-shaped surface domains. *Europhys. Lett.* (2001)]

Reference: R. López, Bifurcation of cylinders for wetting and dewetting models with striped geometry

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