The Minkowski problem and constant curvature surfaces in $\mathbb{R}^3$

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Joint work with Antonio Alarcón

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Aim

The aim of this talk is to classify the family of surfaces with

- *positive constant Gauss curvature* in $\mathbb{R}^3$,
- Gauss map a *diffeomorphism* onto a finitely punctured $\mathbb{S}^2$, and
- *extrinsic conformal structure* a circular domain in $\overline{\mathbb{C}}$.

We will derive some applications:

- Harmonic diffeomorphisms between certain domains of $\mathbb{S}^2$.
- Capillary surfaces in $\mathbb{R}^3$.
- A Hessian equation of Monge-Ampère type on $\mathbb{S}^2$.

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- A. Alarcón and R. Souam, *The Minkowski problem, new constant curvature surfaces in $\mathbb{R}^3$, and some applications.* Preprint 2012 (arXiv:1206.6066).

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- $z = u + \imath v$ conformal parameter on $\mathcal{R}$.
- Gálvez-Martínez 2000 The Gauss map $N : \mathcal{R} \to \mathbb{S}^2$ satisfies

$$X_u = N \times N_v \quad \text{and} \quad X_v = -N \times N_u,$$

hence it is a harmonic local diffeomorphism.
Conversely...

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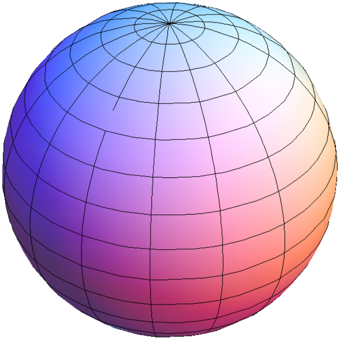
- The *outer* parallel surface at distance 1 to a K-surface is an *H-surface* with $H = 1/2$ and *intrinsic* conformal structure $\mathcal{R}$.

K-surfaces of revolution

- K-surfaces of revolution are classified.



- The round sphere is the only complete K-surface in $\mathbb{R}^3$. There are no complete ends in the theory.

Peaked spheres

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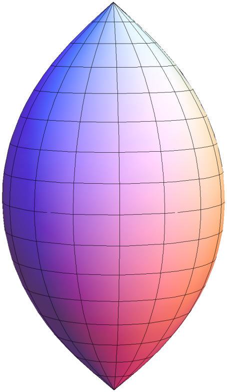

Peaked spheres

- Gálvez-Hauswirth-Mira 2010 studied the family of K-surfaces with isolated singularities (peaked spheres).
  - The extrinsic conformal structure is a circular domain $\mathcal{R} \subset \mathbb{S}^2 \equiv \overline{\mathbb{C}}$.
  - There is no peaked sphere with exactly one singularity.
  - The only peaked spheres with exactly two singularities are the rotational ones.
  - For $n > 2$ there is a $3n - 6$ parameter family of peaked spheres with exactly n singularities, which can be parameterized by the *intrinsic conformal structures*.
  - The Gauss map is solution to the Neumann problem for harmonic diffeomorphisms

$$N : \overline{\mathcal{R}} \to \mathbb{S}^2, \quad \left.\frac{\partial N}{\partial \mathbf{n}}\right|_{\partial \overline{\mathcal{R}}} = 0.$$

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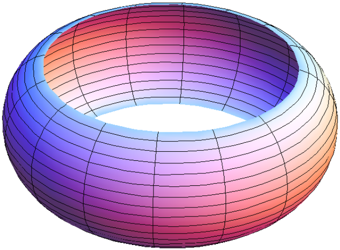

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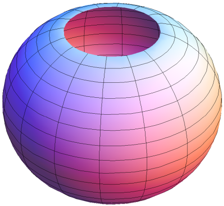

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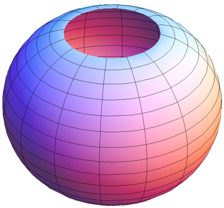

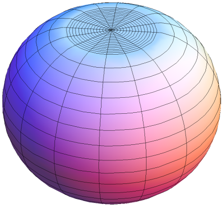

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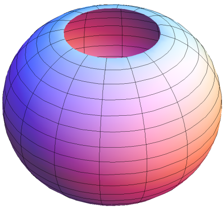

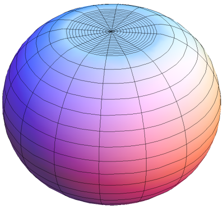

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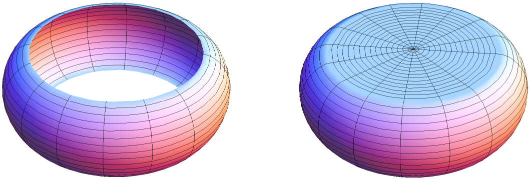

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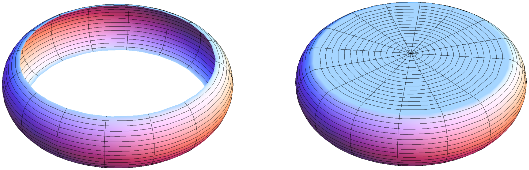

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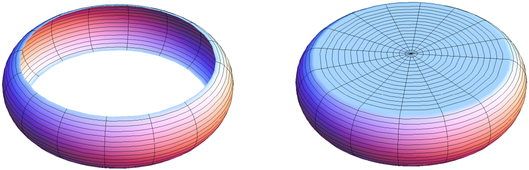
- The extrinsic conformal structure is a circular domain $\mathcal{R} \subset \overline{\mathbb{C}}$.
- The Gauss map is a (harmonic) diffeomorphism

$$N : \mathcal{R} \to \mathbb{S}^2 - \{(0,0,1),(0,0,-1)\},$$

hence extends continuously to $\overline{\mathcal{R}}$ being constant over each connected component of $\partial\overline{\mathcal{R}}$.



Aim

- Are there K-surfaces satisfying that properties but with $m \in \mathbb{N}$ ends?

Question

Let $\{p_1, \ldots, p_m\} \subset \mathbb{S}^2$.
Do there exist K*-surfaces whose extrinsic conformal structures are circular domains* $\mathcal{R} \subset \overline{\mathbb{C}}$ *and their Gauss maps harmonic diffeomorphisms* $\mathcal{R} \to \mathbb{S}^2 - \{p_1, \ldots, p_m\}$*?*

Classification Result

- Let $\{p_1, \ldots, p_m\}$ be a subset of $\mathbb{S}^2$.

Theorem (Alarcón-S., 2012)

The following statements are equivalent:

(i) *There exists a K-surface $S \subset \mathbb{R}^3$ whose extrinsic conformal structure is a circular domain $\mathcal{R} \subset \overline{\mathbb{C}}$ and its Gauss map is a (harmonic) diffeomorphism*

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(ii) *There exist* positive *real constants $a_1, \ldots, a_m$ such that*

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- There is no such S for $m = 1$.
- If $m = 2$ then $p_2 = -p_1$.

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Furthermore, if S is as above and γ_j denotes the component of $\overline{S} - S$ corresponding to p_j via the Gauss map, then

(I) *γ_j is a Jordan curve contained in an affine plane $\Pi_j \subset \mathbb{R}^3$ orthogonal to p_j, and*

(II) *$\mathscr{S} := S \cup (\cup_{j=1}^m \overline{D}_j)$ is the boundary surface of a smooth convex body in $\mathbb{R}^3$, where D_j denotes the bounded component of $\Pi_j - \gamma_j$.*

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In addition, given $(a_1, \ldots, a_m)$ satisfying (ii)*, there exists a* unique, *up to translations, K-surface S satisfying* (i) *such that $\mathrm{Area}(D_j) = a_j$ for all j.*

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- If $m = 2$ then S is rotational.

Convexity and the equilibrium condition

- Let $\{p_1, \ldots, p_m\} \subset \mathbb{S}^2$.
- Assume there exists a K-surface $S \subset \mathbb{R}^3$ satisfying (i) and let us show that $\{p_1, \ldots, p_m\}$ and S satisfy (ii), (I), and (II).
- Denote by $N_S : S \to \mathbb{S}^2 - \{p_1, \ldots, p_m\}$ the outer Gauss map of S.

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- Fix $j \in \{1, \dots, m\}$ and assume that $p_j = (0, 0, 1) \in \mathbb{R}^3$.
- Consider

$$H = \{p \in S \colon x_3(N_S(p)) \in [1 - \epsilon, 1)\}$$

  for $\epsilon > 0$.
- Assume that H is a topological annulus with boundary and a local graph in the x_3-direction at any point.
- Write $(X_1, X_2, X_3) : H \to \mathbb{R}^3$ the inclusion map, and $(N_S)|_H = (N_1, N_2, N_3) : H \to \mathbb{S}^2 \cap \{x_3 \in [1 - \epsilon, 1)\} \subset \mathbb{R}^3$.

Convexity and the equilibrium condition.

- The *Legendre transform* of H,

$$\mathcal{L} = \left(\frac{N_1}{N_3}, \frac{N_2}{N_3}, \frac{N_1}{N_3} X_1 + \frac{N_2}{N_3} X_2 + X_3 \right) : H \to \mathbb{R}^3,$$

defines a strongly positively curved surface $\mathcal{L}(H)$ with boundary in $\mathbb{R}^3$, that is a local graph in the x_3-direction at any point.

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- $\mathscr{S}$ has a unique supporting plane at every point $\Rightarrow \mathscr{S}$ bounds a smooth convex body $\Rightarrow \mathscr{S}$ is $\mathcal{C}^1$ and embedded.

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- $0 = \int_{\mathscr{S} - S} N_S(p) = \int_{\cup_{j=1}^m D_j} N_S(p) = \sum_{j=1}^m \mathrm{Area}(D_j) p_j.$
- $\mathcal{R}$ is a circular domain $\Rightarrow \mathrm{Area}(D_j) > 0\ \forall j.$

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- Let $X : \mathbb{S}^2 \to \mathbb{R}^3$ be an immersion such that $X(\mathbb{S}^2)$ is a closed strictly convex surface in $\mathbb{R}^3$.
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- Minkowski observed that κ must satisfy

$$\int_{\mathbb{S}^2} \frac{p}{\kappa(p)}\, dp = 0.$$

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- Define $\kappa : \mathbb{S}^2 \to \mathbb{R}$, $\kappa = K \circ N_X^{-1}$, where $K : \mathbb{S}^2 \to \mathbb{R}$ denotes the Gauss curvature function of X.
- Minkowski observed that κ must satisfy

$$\int_{\mathbb{S}^2} \frac{p}{\kappa(p)} \, dp = 0.$$

- Minkowski asked for the converse.

Existence. The generalized Minkowski problem

- Let S be a compact convex surface in $\mathbb{R}^3$, not necessarily smooth; i.e., S is the boundary of a general convex body in $\mathbb{R}^3$.

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  for any Borel subset $E \subset \mathbb{S}^2$.
- If S is $\mathcal{C}^2$ and strictly convex, then:

$$\mu(S) = \frac{1}{\kappa}\mu_{\mathbb{S}^2},$$

  where $\mu_{\mathbb{S}^2}$ denotes the canonical Lebesgue measure on $\mathbb{S}^2$.

Existence. The generalized Minkowski problem

- If S is a polyhedron, then:

$$\mu(S) = \sum_{j=1}^{n} c_j \, \delta_{\nu_j},$$

where δ_{ν_j} is the Dirac measure at ν_j and c_j is the Euclidean area of the face of S with outer normal ν_j.

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- The convex surface $\mathscr{S}$ in (II) agrees with the solution to the *generalized Minkowski problem* for the Borel measure

$$\mu(\mathscr{S}) = \mu_{\mathbb{S}^2} + \sum_{j=1}^{m} a_j \delta_{p_j} \quad \big(a_j = \mathrm{Area}(D_j)\big).$$

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$$\int_{\mathbb{S}^2} \mathrm{i}_{\mathbb{S}^2}\, \mu = 0 \in \mathbb{R}^3$$

and $\mu(H) > 0$ for any hemisphere $H \subset \mathbb{S}^2$.
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- This gives the uniqueness part of the theorem.
- The theorem gives no information about the regularity of $\partial\mathscr{K}$.

Existence. The Minkowski problem

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$$\int_{\mathbb{S}^2} \frac{p}{\kappa(p)}\, dp = 0.$$
Then there exists a unique, up to translations, *smooth* embedding $X : \mathbb{S}^2 \to \mathbb{R}^3$ such that $X(\mathbb{S}^2)$ is a closed *strictly convex* surface and the curvature function $K : \mathbb{S}^2 \to \mathbb{R}$ of X is given by
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- Idea: Construct approximate solutions and take limits.

Existence

- Let $\{p_1, \ldots, p_m\} \subset \mathbb{S}^2$.
- Assume there exists positive constants $(a_1, \ldots, a_m)$ such that

$$\sum_{j=1}^{m} a_j p_j = 0 \in \mathbb{R}^3,$$

and let us show a K-surface S satisfying (i) (hence (I) and (II)) with $\mathrm{Area}(D_j) = a_j$ for all j.

Existence

- Denote by $B(p, r)$ the metric ball in $\mathbb{S}^2$ centered at $p \in \mathbb{S}^2$ with radius $r > 0$, and by $A(p, r) = B(p, 2r) - \overline{B(p, r)}$.
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- Then $\displaystyle\int_{\mathbb{S}^2} \frac{p}{\kappa_n(p)}\, dp = 0$ and the Minkowski problem can be solved for $\kappa_n : \mathbb{S}^2 \to \mathbb{R}$.
- There exists a smooth embedding $X_n : \mathbb{S}^2 \to \mathbb{R}^3$ such that
  1. $\mathscr{S}_n := X_n(\mathbb{S}^2)$ is a closed smooth strictly convex surface,
  2. the Gauss map of X_n is the identity map of $\mathbb{S}^2$,
  3. the curvature function of X_n agrees κ_n; in particular $S_n := X_n(\Sigma_n)$ is a K-surface,
  4. $\mathrm{Area}(X_n(B(p_j, 1/n))) = a_j$ for all j.

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Claim

There exists $\xi > 0$ (not depending on n) such that $\mathbb{B}(\xi) \subset \mathscr{K}_n \subset \mathbb{B}(1/\xi)$ $\forall n$. (adapting arguments in Cheng-Yau 1976*.)*

5. Blaschke selection theorem $\Rightarrow$ $\{\mathscr{K}_n\}_{n\in\mathbb{N}}$ converges in the Hausforff distance to a convex body $\mathscr{K}$.

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- the extrinsic conformal structure of S is a circular domain in $\overline{\mathbb{C}}$.

Application: Harmonic diffeomorphims

- Liouville There is no non-constant harmonic map $\mathbb{C} \to \mathbb{D}$, with the Euclidean metric.
- Heinz 1952 There is no harmonic diffeomorphism $\mathbb{D} \to \mathbb{C}$ with the Euclidean metric.

Question (Schoen-Yau 1985)

Are Riemannian surfaces which are related by a harmonic diffeomorphism quasiconformally related?
In particular, are there harmonic diffeomorphisms from $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}^2$?

- Collin-Rosenberg 2010 There exists an entire minimal graph Σ over $\mathbb{H}^2$ in the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$ with the conformal type of $\mathbb{C}$.
  In particular, the vertical projection $\Sigma \to \mathbb{H}^2$ is a harmonic diffeomorphism from $\mathbb{C}$ into $\mathbb{H}^2$.

Application: Harmonic diffeomorphims

Theorem (Alarcón-S.; CMH, in press)

- *For any* $m \in \mathbb{N}$, $m \geq 2$, *and any subet* $\{p_1, \ldots, p_m\} \subset \mathbb{S}^2$ *there exist a circular domain* $\mathcal{R} \subset \overline{\mathbb{C}}$ *and a harmonic diffeomorphism* $\mathcal{R} \to \mathbb{S}^2 - \{p_1, \ldots, p_m\}$.

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- The harmonic diffeomorphism $\mathcal{R} \to \mathbb{S}^2 - \{p_1, \ldots, p_m\}$ appears as the vertical projection $\Sigma \to \mathbb{S}^2 - \{p_1, \ldots, p_m\}$, where Σ is a maximal graph over $\mathbb{S}^2 - \{p_1, \ldots, p_m\}$ in the Lorentzian manifold $\mathbb{S}^2 \times \mathbb{R}_1$, with $\Sigma \cong \mathcal{R}$.
- Such a maximal graph Σ is constructed by solving Dirichlet problems.

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- Such a maximal graph Σ is constructed by solving Dirichlet problems.

- The non-existence of harmonic diffeomorphisms $\mathbb{D} \to \mathbb{S}^2 - \{p\}$ follows from K-surface theory.

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Corollary

Let $\{p_1, \ldots, p_m\} \subset \mathbb{S}^2$ *with* $\sum_{j=1}^m a_j p_j = 0 \in \mathbb{R}^3$ *for some positive numbers* $a_1, \ldots, a_m$.
Then there exists a circular domain $\mathcal{R}$ *in* $\overline{\mathbb{C}}$ *and a harmonic diffeomorphism* $\mathcal{R} \to \mathbb{S}^2 - \{p_1, \ldots, p_m\}$.

- The Gauss map of the K-surface S in (i) is such a harmonic diffeomorphism.

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- The Gauss map of the K-surface S in (i) is such a harmonic diffeomorphism.
- We do not know if those harmonic diffeomorphisms given as Gauss maps of K-surfaces in $\mathbb{R}^3$ and those given as vertical projections of maximal graphs in $\mathbb{S}^2 \times \mathbb{R}_1$ are the same or not.

The support function

- The *support function* of S,

$$h : \mathbb{S}^2 - \{p_1, \ldots, p_m\} \to \mathbb{R}, \quad h(p) = \langle p, N_S^{-1}(p) \rangle,$$

satisfies $\quad \big(\det(\nabla^2 h + h\mathrm{I})\big) \circ N_S = 1 \ \big(= 1/K \big) \quad$ on S.

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- A parameterization $X : \mathbb{S}^2 - \{p_1, \ldots, p_m\} \to \mathbb{R}^3$ of S is given by

$$X(p) = \nabla h(p) + h(p)p.$$

Application: A Hessian equation

- Fully nonlinear, elliptic second order partial differential equations of the form

$$\mathscr{F}[h] := F\big(\nabla^2 h + A(\cdot, h, \nabla h)\big) = B(\cdot, h, \nabla h) \quad \text{on } \Omega \subset \mathcal{M},$$

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  have been objective of considerable interest in recent years.

- The space of solutions to the Hessian one equation

$$\det \nabla^2 h = 1 \quad \text{on } \mathbb{R}^2 - \{q_1, \dots, q_k\}$$

  was described by Gálvez-Martínez-Mira 2005 (Jorgens 1955 for $k = 1$).

- Jorgens 1954 The only solutions for $k = 0$ are quadratic polynomials.

Application: A Hessian equation

- Two solution u and v of the Hessian equation

$$\det\left(\nabla^2 h + h\mathrm{I}\right) = 1 \quad \text{on } \mathbb{S}^2 - \{p_1, \ldots, p_m\}$$

  are *equivalent*, $u \sim v$, if $u - v$ is the restriction to $\mathbb{S}^2 - \{p_1, \ldots, p_m\}$ of a linear function of $\mathbb{R}^3$.

Application: A Hessian equation

- Two solution u and v of the Hessian equation

$$\det\left(\nabla^2 h + h\mathrm{I}\right) = 1 \quad \text{on } \mathbb{S}^2 - \{p_1, \ldots, p_m\}$$

are *equivalent*, $u \sim v$, if $u - v$ is the restriction to $\mathbb{S}^2 - \{p_1, \ldots, p_m\}$ of a linear function of $\mathbb{R}^3$.

Corollary

The space of equivalence classes of solutions of the above equation, under $\sim$*, with non-removable singularities at the points* $\{p_1, \ldots, p_m\}$*, is in bijection with the set*

$$\Big\{(a_1, \ldots, a_m) \in \mathbb{R}^m \,\big|\, a_j > 0 \;\forall j,\; \sum_{j=1}^{m} a_j p_j = 0\Big\}.$$

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- $h : \mathbb{S}^2 - \{p_1, \ldots, p_m\} \to \mathbb{R}$ is solution to the equation iff it is the support function of a surface S as those in the theorem.
- The only solution for $m = 0$ is the round sphere and there is no solution for $m = 1$.

Application: Capillary surfaces

- A capillary surface in a region $\mathcal{B}$ in $\mathbb{R}^3$ is a compact CMC surface meeting $\partial\mathcal{B}$ at a constant angle $\gamma \in [0, \pi]$ along its boundary. They model liquid drops inside a container in the absence of gravity.
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- The outer parallel surface at distance 1 to a K-surface is a CMC surface with $H = 1/2$ (i.e., an H-surface).

Application: Capillary surfaces

- Let $\{p_1, \dots, p_m\}$ be a subset of $\mathbb{S}^2$.

Corollary

The following statements are equivalent:

(i) *There exists a* (positively curved) *H-surface $S \subset \mathbb{R}^3$ whose* intrinsic *conformal structure is a circular domain $\mathcal{R} \subset \overline{\mathbb{C}}$ and its Gauss map is a harmonic diffeomorphism*

$$\mathcal{R} \to \mathbb{S}^2 - \{p_1, \dots, p_m\}.$$

(ii) *There exist* positive *real constants $a_1, \dots, a_m$ such that*

$$\sum_{j=1}^{m} a_j p_j = 0 \in \mathbb{R}^3.$$

Application: Capillary surfaces

Corollary

Furthermore, if S is as above and γ_j denotes the component of $\overline{S} - S$ corresponding to p_j via the Gauss map, then

(I) *γ_j is a Jordan curve contained in an affine plane $\Pi_j \subset \mathbb{R}^3$ orthogonal to p_j, and*

(II) *$\mathscr{S} := S \cup (\cup_{j=1}^{m} \overline{D}_j)$ is the boundary surface of a smooth convex body in $\mathbb{R}^3$, where D_j denotes the bounded component of $\Pi_j - \gamma_j$.*

In addition, given $(a_1, \ldots, a_m)$ satisfying (ii), *there exists a* unique, *up to translations, H-surface S satisfying* (i) *such that $\mathrm{Area}(D_j) = a_j$ for all j.*

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- In particular, S is an embedded H-surface of genus zero which meets tangentially all the faces of the polyhedral region determined by the affine planes Π_j, $j = 1, \ldots, m$.