Holomorphicity of real Kaehler submanifolds

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Joint work with:

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Throughout this talk we denote by

$$f\colon M^{2n} \to \mathbb{R}^{2n+p}$$

a real Kaehler submanifold. This means that (M^{2n}, J) is a connected Kaehler manifold of complex dimension $n \geq 2$ isometrically immersed into Euclidean space with substantial codimension p even locally.

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It is well-known that real Kaehler submanifolds in low codimension are generically holomorphic. By that we mean that $p = 2q$ is even and that

$$f\colon M^{2n} \to (\mathbb{C}^{n+q}, \tilde{J}) \equiv \mathbb{R}^{2n+p}$$

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is holomorphic.

If f is holomorphic, then its second fundamental form

$$\alpha(X, Y)\colon TM \times TM \to N_f M$$

satisfies

$$\alpha(X, JY) = \tilde{J}\alpha(X, Y) = \alpha(JX, Y)$$

for all $X, Y \in TM$.

A real Kaehler submanifold is called pluriharmonic if

$$\alpha(X, JY) = \alpha(JX, Y)$$

for all $X, Y \in TM$. This is equivalent to

$$A_\xi \circ J = -J \circ A_\xi \ \text{ for all } \ \xi \in \Gamma(N_f M)$$

where

$$\langle A_\xi X, Y \rangle = \langle \alpha(X, Y), \xi \rangle.$$

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Theorem

A real Kaehler submanifold $f : M^{2n} \to \mathbb{R}^{2n+p}$ is minimal if and only if it is pluriharmonic.

Dajczer and Rodríguez - 1986.

Associated family

Minimal real Kaehler submanifolds have been intensively studied since it was shown that they posses several of the basic properties of minimal surfaces.

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Theorem

Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a minimal simply connected real Kaehler submanifold. Then there is a one-parameter associated family of minimal isometric immersions $f_\theta\colon M^{2n} \to \mathbb{R}^{2n+p}$, $\theta \in [0,\pi)$, such that $f_0 = f$.

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The associated family is given by the line integral

$$f_\theta(x) = \int_{x_0}^{x} f_* \circ (\cos\theta I + \sin\theta J)$$

where x_0 is any fixed point of M^{2n}. Since

$$f_{\theta*}(x) = f_* \circ (\cos\theta I + \sin\theta J)$$

then all f_θ have the same Gauss map.

Holomorphicity

Theorem

Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a simply connected minimal real Kaehler submanifold. If f is holomorphic, then its associated family $\{f_\theta\}_{\theta\in[0,\pi)}$ satisfies $f_\theta = f$ for any $\theta \in [0,\pi)$.

Conversely, if there exists $\theta_1 \neq \theta_2 \in [0,\pi)$ such that f_{θ_1} and f_{θ_2} are congruent, then $p = 2\ell$ and f is holomorphic.

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Theorem

Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a simply connected minimal real Kaehler submanifold. The map $F\colon M^{2n} \to \mathbb{R}^{4n+2p} = \mathbb{R}^{2n+p} \oplus \mathbb{R}^{2n+p}$ given by

$$F = (1/\sqrt{2})f \oplus (1/\sqrt{2})f_{\pi/2}$$

is a holomorphic isometric immersion with respect to the standard complex structure on $\mathbb{R}^{4n+2p}$ given by $\tilde{J}(X,Y) = (Y,-X)$.

Dajczer and Gromoll – 1985.

Weierstrass type representation

Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a minimal real Kaehler submanifold. Given a simply connected coordinate chart U of M^{2n} with $z_j = x_j + iy_j$, define the maps $\varphi_j\colon U \to \mathbb{C}^m$, $1 \leq j \leq n$, by

$$\varphi_j = \sqrt{2}\, f_{z_j} = (1/\sqrt{2})\left(f_{x_j} - if_{y_j}\right).$$

Then the φ_j satisfy the following conditions:

(i) The vectors $\varphi_1, \ldots, \varphi_n$ are linearly independent at any point in U,

(ii) The functions φ_j are holomorphic,

(iii) The subspace $\operatorname{span}\{\varphi_1, \ldots, \varphi_n\} \subset \mathbb{C}^m$ is isotropic,

(iv) The integrability conditions $\partial\varphi_j/\partial z_k = \partial\varphi_k/\partial z_j$, $1 \leq j, k \leq n$.

And if $F\colon U \to \mathbb{C}^m$ is the holomorphic representative of f, then

$$\varphi_j = F_{z_j}, \ \ 1 \leq j \leq n. \tag{1}$$

That a subspace $V \subset \mathbb{C}^m$ is *isotropic* means that $u \cdot v = 0$ for all $u, v \in V$ where "$\cdot$" denotes the standard symmetric inner product in $\mathbb{C}^{2n+p}$.

Weierstrass type representation

Proposition

Consider maps $\varphi_1, \ldots, \varphi_n \colon U \to \mathbb{C}^m$ *on a simply connected open subset* U *of* $\mathbb{C}^m$ *that satisfy conditions* (i) *to* (iv)*. Then there is a holomorphic map* $F \colon U \to \mathbb{C}^m$ *such that* (1) *holds. If* $f \colon U \to \mathbb{R}^m$ *is defined by*

$$f = \sqrt{2}Re[F]$$

then $M^{2n} = (U, f^*\langle\, ,\, \rangle)$ *is a Kaehler manifold and* f *is a minimal real Kaehler submanifold whose holomorphic representative is* F.

Arezzo, Pirola and Solci – 2004.

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Real Kaehler hypersurfaces have been locally classified by Dajczer and Gromoll by means of the Gauss parametrization in terms of a pseudoholomorphic surface in a sphere and a smooth function on the surface. The minimal ones can also be parametrized by using the above result.

The type number

Let V be an n-dimensional vector space and $T_1, \ldots, T_r \in End(V)$. The type number of $\{T_1, \ldots, T_r\}$ is the largest integer τ for which there exist τ vectors $X_1, \ldots, X_\tau$ in V such that the τr vectors

$$\{T_i X_j,\ 1 \leq i \leq r,\ 1 \leq j \leq \tau\}$$

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Let V and W be vector spaces of finite dimension with positive definite inner products and let $\beta \colon V \times V \to W$ be a symmetric bilinear form. For any given $\xi \in W$, let $B_\xi \colon V \to V$ be defined by

$$\langle B_\xi X, Y \rangle = \langle \beta(X, Y), \xi \rangle.$$

The type number of β is defined as the type number of $B_{\xi_1}, \ldots, B_{\xi_r}$, where $\xi_1, \ldots, \xi_r$ is any basis of W.

Theorem

Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kaehler submanifold such that type number of the second fundamental form satisfies $\tau(x) \geq 3$ for all $x \in M^{2n}$. Then $p = 2\ell$ and f is holomorphic.

Notice that $\tau \geq 3$ forces the codimension of f to satisfy $3p \leq 2n$.

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It is a natural task to understand what can (locally) happen in low codimension under weaker, and algebraically simpler, assumptions on the second fundamental form than the one on the type number.

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Moreover, it is desirable to replace the type number assumption for something simpler and more meaningful. By the latter, we mean providing a workable condition as a starting point in order to study the real Kaehler submanifold that are not holomorphic.

The index of relative nullity

Introduced by Chern and Kuiper in 1958 the index of relative nullity $\nu(x)$ at $x \in M^n$ of an isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$ turned out to be a fundamental concept in the theory of isometric immersions.

It is the dimension of the kernel of the second fundamental form

$$\mathcal{N}(\alpha)(x) = \{Y \in T_xM : \alpha(Y, X) = 0 \text{ for all } X \in T_xM\}.$$

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If ν is constant along an open subset $U \subset M^n$, then the relative nullity distribution $x \in U \mapsto \mathcal{N}(\alpha)(x)$ is integrable, the leaves of the foliation are totally geodesic submanifolds of M^n and their images under f are open subsets of affine subspaces in $\mathbb{R}^{n+p}$.

The leaves or the relative nullity foliation on the open subset where ν is minimal are complete manifolds if M^m is complete.

Theorem

Assume that the index of relative nullity of a real Kaehler submanifold $f: M^{2n} \to \mathbb{R}^{2n+2}$ satisfies $\nu(x) < 2(n-2)$ at any point $x \in M^{2n}$. Then the submanifold is holomorphic along each connected component of an open dense subset of M^{2n}.

Dajczer – 1989

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Theorem

Assume that the index of relative nullity of a real Kaehler submanifold $f\colon M^{2n} \to \mathbb{R}^{2n+3}$ satisfies $\nu(x) < 2(n-3)$ at any $x \in M^{2n}$. Then there exists an open dense subset U of M^{2n} such that, along each connected component U' of U, the submanifold $f|_{U'}$ has a Kaehler extension, namely, there exist a real Kaehler hypersurface $j\colon N^{2n+2} \to \mathbb{R}^{2n+3}$ and a holomorphic isometric immersion $h\colon U' \to N^{2n+2}$ such that $f|_{U'} = j \circ h$.

Dajczer and Gromoll – 1997

- Yan and Zheng observed that both results hold under the slightly weaker assumption that the complex index of relative nullity

$$\nu^c(x) = \dim \mathcal{N}(\alpha)(x) \cap J\mathcal{N}(\alpha)(x),$$

satisfies the same pointwise inequalities required for $\nu(x)$.

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Theorem

Let $f\colon M^{2n} \to \mathbb{R}^{2n+4}$ a real Kaehler submanifold with $\nu^c(x) < 2(n-4)$ at any $x \in M^{2n}$. Then there is an open dense subset U of M^{2n} such that along each connected component U' of U, the submanifold $f|_{U'}$ has a Kaehler extension, namely, there exist a real Kaehler submanifold $j\colon N^{2n+2} \to \mathbb{R}^{2n+4}$ and an isometric immersion $h\colon U' \to N^{2n+2}$ that is holomorphic and such that $f|_{U'} = j \circ h$.
Moreover, the submanifold j may not be unique but it can be chosen to be minimal if f is minimal.

Yan and Zheng 2013

The complex s-nullity

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For real Kaehler submanifolds the complex version goes as follows.

The complex s-nullity $\nu_s^c(x)$, $1 \leq s \leq p$, at $x \in M^{2n}$ of a real Kaehler submanifold $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ is

$$\nu_s^c(x) = \max_{U^s \subset N_f M(x)} \{\dim(\mathcal{N}(\alpha_{U^s}) \cap J\mathcal{N}(\alpha_{U^s}))\},$$

where U^s denotes an s-dimensional subspace of $N_f M(x)$ and $\alpha_{U^s} = \pi_{U^s} \circ \alpha$ being $\pi_{U^s}\colon N_f M \to U^s$ the projection.

Notice that $\nu_p^c(x) = \nu^c(x)$.

The minimal case

Theorem

Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ *be a real Kaehler submanifold. Assume that the complex s-nullities satisfy* $\nu_s^c(x) < 2(n-s)$ *at any* $x \in M^{2n}$ *for all* $1 \leq s \leq p$*. Then* f *is a minimal immersion.*

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Notice that already for codimension $p = 4$ the composition of isometric immersions obtained in the aforementioned result by Yan and Zheng shows that the theorem does not hold if we drop just one of the assumptions on the complex s-nullities.

The holomorphic case

Theorem

Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kaehler submanifold. Assume $p \leq 11$ and that the complex s-nullities satisfy $\nu_s^c(x) < 2(n-s)$ at any point $x \in M^{2n}$ and for all $1 \leq s \leq p$. Then p is even and f is holomorphic.

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Examples of minimal but not holomorphic real Kaehler submanifold with codimension $p = 2$ constructed by Dajczer and Gromoll satisfy one of the assumptions on the complex s-nullities in the above result since $\nu_1^c(x) = 2n - 4$ but not the other one since $\nu_2^c(x) = 2n - 4$.

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It remains an open problem if the theorem still holds if the assumption $p \leq 11$ is dropped. In that respect, we observe that the core of the proof is a result in the theory of flat bilinear forms that only holds until dimension 11. In fact, we constructed examples showing that this result is false if the dimension is 12.

Flat bilinear forms

Let $W^{p,p}$ be a $2p$-dimensional real vector space endowed with an inner product of signature (p,p). Hence p is the dimension of the vector subspaces of $W^{p,p}$ of maximal dimension where the induced inner product is either positive or negative definite.

A vector subspace $L \subset W^{p,p}$ is called degenerate if $L \cap L^{\perp} \neq \{0\}$ and nondegenerate otherwise.

A bilinear form $\beta \colon V^n \times V^n \to W^{p,p}$ is called flat if

$$\langle \beta(X,Y), \beta(Z,T) \rangle - \langle \beta(X,T), \beta(Z,Y) \rangle = 0$$

for all $X, Y, Z, T \in V^n$. It is said that β is null when

$$\langle \beta(X,Y), \beta(Z,T) \rangle = 0$$

for all $X, Y, Z, T \in V^n$. Thus null bilinear forms are flat.

Lemma 1

Lemma

Let V^n and U^p denote real vector spaces where V^n carries $J \in End(V)$ satisfying $J^2 = -I$ and U^p is endowed with a positive definite inner product. Given a symmetric bilinear form $\alpha\colon V^n \times V^n \to U^p$ let $\gamma\colon V^n \times V^n \to U^p$ be given by

$$\gamma(X, Y) = \alpha(X, Y) + \alpha(JX, JY).$$

Assume that $\beta\colon V^n \times V^n \to W^{p,p} = U^p \oplus U^p$ defined by

$$\beta(X, Y) = (\gamma(X, Y), \gamma(X, JY))$$

be flat with respect to the inner product on $W^{p,p}$ given by

$$\langle\langle(\xi_1, \xi_2), (\eta_1, \eta_2)\rangle\rangle = \langle \xi_1, \eta_1\rangle_{U^p} - \langle \xi_2, \eta_2\rangle_{U^p}.$$

Then $\dim \mathcal{N}(\beta) = n - r$ where

$$r = \max\{\dim B_X(V)\colon X \in V^n\}.$$

Lemma 2

Lemma

Let V^n and U^p be real vector spaces where V^n carries $J \in End(V^n)$ satisfying $J^2 = -I$ and U^p is endowed with a positive definite inner product. Then let $\alpha\colon V^n \times V^n \to U^p$, $n \geq 2p$, be a symmetric bilinear form satisfying

$$\alpha(JX, Y) = \alpha(X, JY)$$

for $X, Y \in V^n$. Assume that $\beta\colon V^n \times V^n \to W^{p,p} = U^p \oplus U^p$ given by

$$\beta(X, Y) = (\alpha(X, Y), \alpha(X, JY)) \tag{2}$$

is flat with respect to the inner product on $W^{p,p}$ defined by

$$\langle\langle(\xi_1, \xi_2), (\eta_1, \eta_2)\rangle\rangle = \langle \xi_1, \eta_1\rangle_{U^p} - \langle \xi_2, \eta_2\rangle_{U^p}.$$

If $p \leq 11$ and $\mathcal{S}(\beta)$ is nondegenerate then $\dim \mathcal{N}(\beta) \geq n - 2p$.

A counterexample

Let $\gamma\colon V_0^m \times V_0^m \to W_0^{q,q}$, $m > 2q$, $q \geq 6$, be a flat symmetric bilinear form such that $\mathcal{S}(\gamma) = W_0^{q,q}$ and $\dim \mathcal{N}(\gamma) < m - 2q$. Explicit examples of such γ's have been constructed by Dajczer and Florit in 2004.

Set $V^n = V_0^m \oplus V_0^m$ with $J \in End(V)$ given by $J(X, Y) = (-Y, X)$ and let $W^{p,p} = W_0^{q,q} \oplus W_0^{q,q}$ be endowed with the inner product

$$\langle\langle (\zeta^1, \zeta^2), (\eta^1, \eta^2) \rangle\rangle = \langle \zeta^1, \eta^1 \rangle_{W_0^{q,q}} - \langle \zeta^2, \eta^2 \rangle_{W_0^{q,q}}.$$

Proposition

Let $\beta\colon V^n \times V^n \to W^{p,p}$ be the symmetric bilinear form given by

$$\beta((X, Y), (Z, T)) = (\gamma(X, Z) - \gamma(Y, T), \gamma(X, T) + \gamma(Y, Z)).$$

Then β is flat with $\mathcal{S}(\beta) = W^{p,p}$ and $\dim \mathcal{N}(\beta) < n - 2p$.

And β satisfies all of the required conditions in order to be a counterexample to Lemma 2.

The conformal case

Theorem

Let $f : M^{2n} \to \mathbb{R}^{2n+1}$, $n \geq 4$, be a conformal Kaehler submanifold where M^{2n} is free of flat points. Then f is conformally congruent to a real Kaehler hypersurface.

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Theorem

Let $f\colon M^{2n} \to \mathbb{R}^{2n+2}$, $n \geq 5$, be a conformal Kaehler submanifold where M^{2n} is free of flat points. Then there is an open dense subset M_0 of M^{2n} such that along any connected component N^{2n} of M_0 one of the following holds:

(i) *$f|_N$ is conformally congruent to a real Kaehler submanifold $g\colon N^{2n} \to \mathbb{R}^{2n+2}$.*

(ii) *$f|_N = h \circ g$ is a composition of a real Kaehler hypersurface $g\colon N^{2n} \to \mathbb{R}^{2n+1}$ and a conformal immersion $h\colon V \to \mathbb{R}^{2n+2}$ where $V \subset \mathbb{R}^{2n+1}$ is open and $g(N) \subset V$.*

GRACIAS POR SU PRESENCIA!