Higher genus CMC surfaces in the 3-sphere via integrable systems

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Due to the vanishing of the Hopf differential, all CMC spheres in S^3 are round. Recently, Brendle proved that the only embedded minimal torus in the 3-sphere is the Clifford torus, and Andrews and Li confirmed the Pinkall-Sterling conjecture that all embedded CMC tori are the unduloidal rotational Delaunay tori.

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There is no theory yet which describes the whole moduli space of higher genus CMC surfaces in the 3-sphere nor are there methods which produce the *generic* surface.

The family of flat connections

For every conformal CMC immersion f from a Riemann surface M into the S^3 there exists an associated family of flat $\mathrm{SL}(2,\mathbb{C})$-connections

$$\lambda \in \mathbb{C}^* \mapsto \nabla^\lambda = \nabla + \lambda^{-1}\Phi - \lambda\Phi^*$$

on a hermitian rank 2 bundle $V \to M$ which is unitary along $S^1 \subset \mathbb{C}^*$ and trivial at two points $\lambda_1, \lambda_2 \in S^1$.

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The mean curvature H of the immersion is given by $H = i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}$.

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Conversely, for every family of flat $\mathrm{SL}(2, \mathbb{C})$-connections $\lambda \in \mathbb{C}^* \mapsto \nabla^{\lambda}$ on a Riemann surface satisfying the properties as above there exists a CMC immersion $f\colon M \to S^3$ which is the gauge between the two trivial connections ∇^{λ_1} and ∇^{λ_2}.

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It became apparent that in many situations it is easier to deal with families of flat connections (systems of ODE's) than with a CMC immersion itself.

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- All connections ∇^λ have abelian monodromy: the family of flat connections is given via its spectral data which parametrize the flat parallel eigenline bundles (Hitchin).The line bundle connections are determined by meromorphic differentials on the spectral curve with double poles and integer periods.
- The generic connection ∇^λ is irreducible: this is always the case for CMC immersions from compact Riemann surfaces of genus $g \geq 2$ [Ohnita, H., Gerding].

General construction of higher genus CMC surfaces

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The idea is to apply loop group factorization methods as in the theory of Dorfmeister, Pedit and Wu.

Connections and holomorphic structures

Connections ∇ on a complex vector bundle V over a Riemann surface split into a complex linear and a complex anti-linear part

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Two flat connections with the same holomorphic structure differ by a so-called Higgs field, a holomorphic endomorphism-valued 1-form.

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Equivalently, for generic flat SL$(2, \mathbb{C})$-connections ∇, the parallel endomorphisms are exactly the holomorphic endomorphisms with respect to $\bar{\partial}^{\nabla}$. Therefore, CMC tori can also be described by polynomial families of parallel endomorphisms - *the polynomial Killing field*.

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- do only consider holomorphic subbundles (resp. anti-holomorphic) and not parallel ones;
- do only consider families of Higgs fields instead of polynomial Killing fields;
- let the spectral curve parametrize those objects instead of the parallel ones.

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The underlying Riemann surface structure is given by the equation

$$y^3 = \frac{z^2 - 1}{z^2 - \alpha},$$

where $\alpha = -1$ corresponds to the Lawson surface itself. The Hopf differential is a multiple of $\frac{(dz)^2}{(z^2-\alpha)(z^2-1)}$.

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The fiber over a Lawson symmetric holomorphic structure $\bar{\partial}$ consist of the line of Lawson symmetric Higgs fields $\Psi \in H^0(M, K\,\mathrm{End}_0(V, \bar{\partial}))$.

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These eigenline bundles are given by the elements of the Jacobian of the torus $\tilde{M}/\mathbb{Z}_3$, and we obtain a two-to-one correspondence between $Jac(\tilde{M}/\mathbb{Z}_3)$ and the moduli space $\mathcal{M}$ of Lawson symmetric holomorphic structures.

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$$d + \begin{pmatrix} \pi a dz - \pi x d\bar{z} & c\frac{\theta(z-2x)}{\theta(z)} e^{-4\pi i x \operatorname{Im}(z)} dz \\ c\frac{\theta(z+2x)}{\theta(z)} e^{4\pi i x \operatorname{Im}(z)} dz & -\pi a dz + \pi x d\bar{z} \end{pmatrix}$$

where θ is the theta-function of $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ with simple zero at 0 and

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For every x there exists a unique $a = a^u(x)$ such that the corresponding flat $\mathrm{SL}(2,\mathbb{C})$-connection is unitary. We call a^u the Narasimhan-Seshadri section.

The spectral curve

In order to parametrize the family of flat connections $\lambda \mapsto \nabla^\lambda$ we need a spectral curve Σ together with maps $\mathcal{L}\colon \Sigma \to Jac(\tilde{M}/\mathbb{Z}_3)$ and $\mathcal{D}\colon \Sigma \to \mathcal{A}_1(\tilde{M}/\mathbb{Z}_3)$

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$$\begin{array}{ccc} \Sigma & \xrightarrow{\mathcal{L}} & Jac(\tilde{M}/\mathbb{Z}_3) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \Pi} \\ \mathbb{C} & \xrightarrow[{[\bar\partial^\lambda]}]{} & \mathcal{M} \end{array}$$

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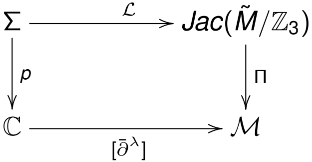
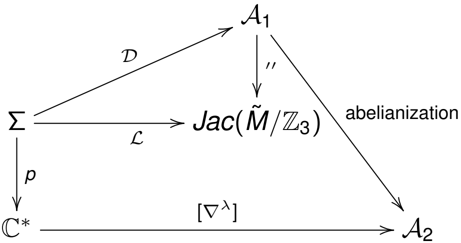


and



commute.

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for all $\mu \in \Sigma$ with $p(\mu) \in S^1$ and the closing condition

$$\mathcal{L}(\mu_k) = L(\frac{\gamma_1 + \gamma_2}{2} - 0)$$

for $p(\mu_k) = \lambda_k$, where γ_1 and γ_2 are the periods of the torus $\tilde{M}/\mathbb{Z}_3$.

Theorem (H.)

Spectral data $(\Sigma, \mathcal{L}, \mathcal{D})$ *as above satisfying the reality condition and the extrinsic closing condition give rise to a unique Lawson symmetric CMC surface.*

Experiments (joint work with Nick Schmitt)

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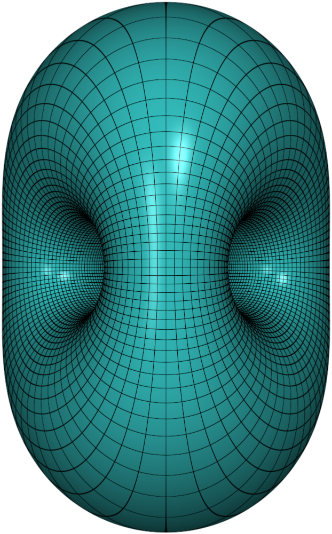
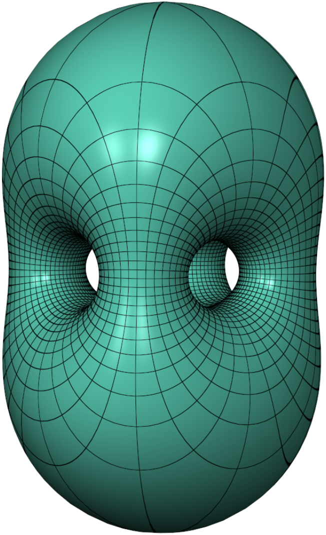
- a visualization of the conformal immersion in the xlab software of Schmitt.

The Lawson surface $\xi_{2,1}$ again

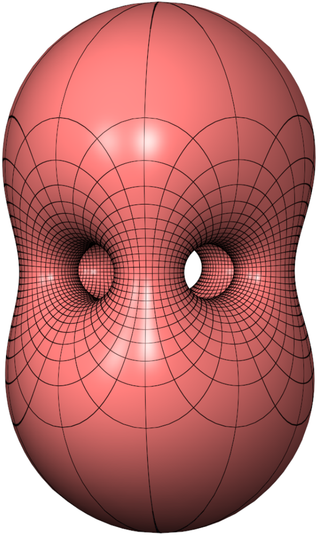


The theory and the experiments can be extended easily to the Lawson surfaces $\xi_{g,1}$ for arbitrary genus g.

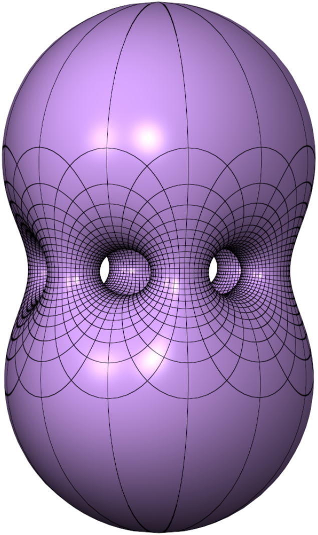
The Lawson surface $\xi_{3,1}$



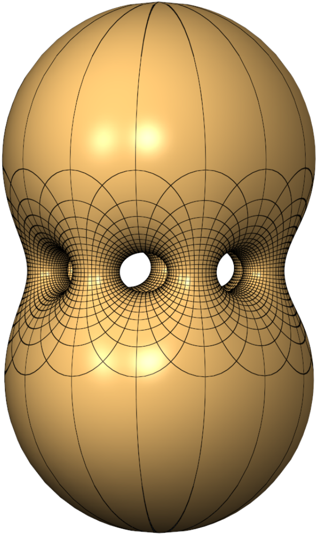
The Lawson surface $\xi_{4,1}$



The Lawson surface $\xi_{5,1}$



The Lawson surface $\xi_{6,1}$



Deformations of Lawson symmetric CMC surfaces

Physical idea: Break a space-orientation reversing symmetry of the Lawson surface in order to study deformations of $\xi_{2,1}$ through Lawson symmetric CMC surfaces by increasing the pressure outside.

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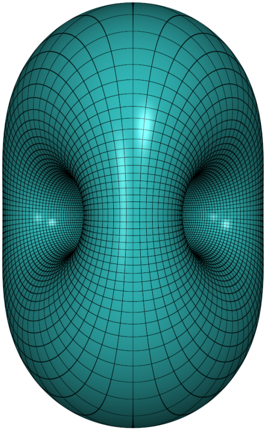
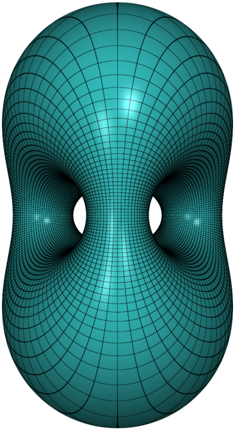
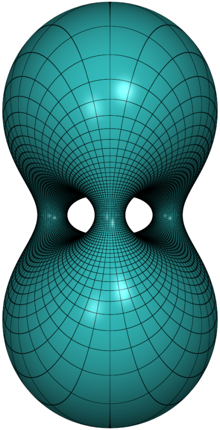
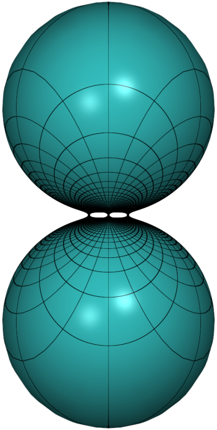
For tori, this yields the homogeneous CMC tori of rectangular conformal type. They may be obtained by the Whitham flow of the Clifford torus.

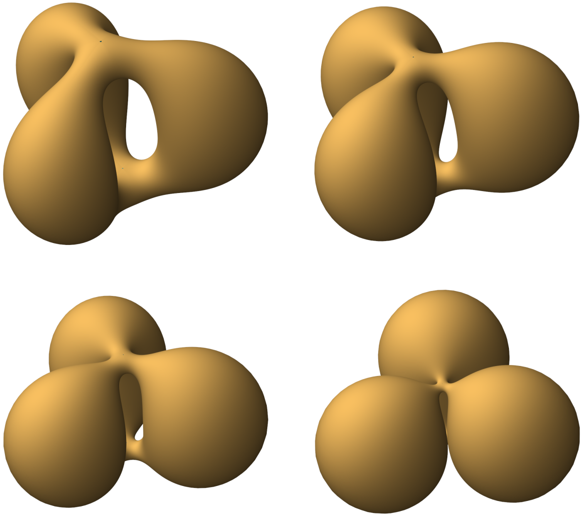
For higher genus, the idea is to use a modified isospectral flow of the spectral data of the Lawson surfaces $\xi_{g,1}$ such that the asymptotic, the reality condition and the extrinsic closing conditions remain valid.



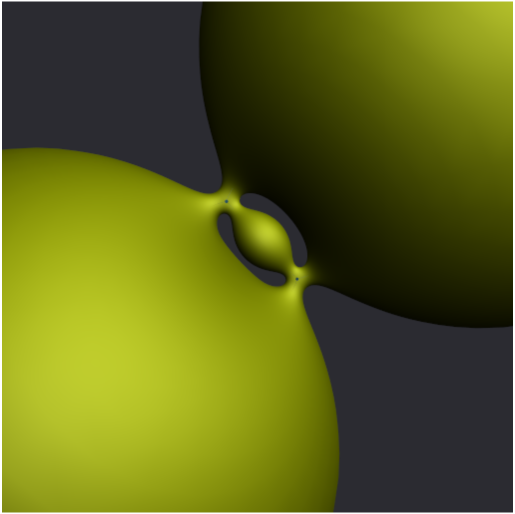


A distinct family of Lawson symmetric CMC surfaces of genus 2.



The button perspective



The moduli space of Lawson symmetric CMC surfaces of genus 2