Geodesic completeness of compact Lorentzian manifolds with special holonomy

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Geodesic completeness

Let (M, g) be a semi-Riemannian Mannigfaltigkeit, its ∇ Levi-Civita connection.

- Geodesics $\gamma : (a, b) \to M$: $\nabla_{\dot\gamma}\dot\gamma|_t = 0$, $\forall \;\; t \in (a, b)$.
- Picard-Lindelöf $\implies$ For $v \in T_pM$ $\exists!$ maximal geodesic $\gamma : I_{max} \to M$ with

$$\gamma(0) = p, \quad \dot\gamma(0) = v.$$

- (M, g) is geodesically complete if $I_{max} = \mathbb{R}$ for all maximal geodesics.
- If (M, g) is Riemannian, then
  - geodesics are critical points of the energy functional

$$E(\gamma) = \int_a^b g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\,\mathrm{d}t$$

  - (M, d_g) metric space, $d_g(p, q) := \inf\left\{\ell_g(\gamma) : p \overset{\gamma}{\rightsquigarrow} q\right\}$
  - Hopf-Rinow Theorem: (M, d_g) complete $\iff$ (M, g) geodesically complete.

Theorem

Compact Riemannian manifolds are geodesically complete.

Completeness of compact Riemannian manifolds

- Define the vector field X on TM by the geodesic flow on TM

$$TM \ni (p, v) \quad \mapsto \quad X|_{(p,v)} := \tfrac{d}{dt}\big(\gamma(t), \dot\gamma(t)\big)|_{t=0},$$
$$\gamma \text{ geodesic with } \gamma(0) = p,\ \dot\gamma(0) = v \in T_pM$$

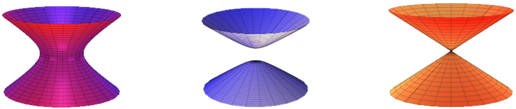
- γ geodesic $\implies g(\dot\gamma, \dot\gamma) \equiv R^2 > 0$, d.h.

$$\dot\gamma(t) \in \mathbf{S}^{n-1}_R \longleftarrow \text{ sphere in } T_{\gamma(t)}M \simeq \mathbb{R}^n \text{ of radius } R.$$

$\implies$ intgeral curves of X remain in *compact* set in TM.

- integral curves of X defined on $\mathbb{R}$ and hence $I_{max} = \mathbb{R}$.

For indefinite metrics, $\mathbf{Q}_C = \left\{X \in \mathbb{R}^{1,n-1} \mid g(X, X) = C\right\} =$



$\widetilde{M} := \mathbb{R}^2 \setminus \{0\}$ with Lorentzian metric $\widetilde{g} := \frac{1}{x^2+y^2}\, dx\, dy$.

- geodesic equation
$$\ddot{x} = \frac{2x\dot{x}^2}{x^2+y^2} \quad , \quad \ddot{y} = \frac{2y\dot{y}^2}{x^2+y^2}$$
- Solution for initial conditions $\gamma(0) = (1,0)$, $\dot{\gamma}(0) = \mathbf{e}_1$:
$$\gamma(t) = \left(\frac{1}{1-t}, 0\right), \quad t \in (-\infty, 1), \quad \dot{\gamma}(t) = \frac{1}{(1-t)^2}\mathbf{e}_1 \in \mathbf{Q}_0$$
- $\mu(x,y) = (2x, 2y)$ isometry of $g \rightsquigarrow$ incomplete Lorentzian metric on
$$M \ := \ \widetilde{M}/\langle\mu\rangle \ \simeq \ \mathbb{T}^2 \quad \text{non flat 2-Torus}$$

This can be generalised to higher dimensions,

$$g_{(u,v,x_1,\dots,x_n)} := 2du\left(dv - (\sin(v) - \sum_{i=1}^{n} a_i(\cos(x_i) - 1))du\right) + \sum_{i=1}^{n} dx_i^2$$

descends to an incomplete metric on $\mathbb{T}^{n+2}$.

Fact: If $(\widetilde{M}, \widetilde{g}) \to (M, g)$ is an isometric covering, then (M, g) is complete $\iff$ $(\widetilde{M}, \widetilde{g})$ is complete.

- Let (M, g) be compact with some curvature condition (e.g., flat, constant curvature, or $\nabla R = 0$).
- If (M, g) is geodesically complete, then (M, g) is covered by a model space $(\widetilde{M}, \widetilde{g})$, i.e., a (simply connected) complete manifold with the curvature condition.
- $M = \widetilde{M}/\Gamma$ for a properly discontinuous co-compact group of isometries, i.e., classification of compact manifolds is reduced to classification of such Γ's

	flat, $R=0$	constant curvature , $R = c\, g \wedge g$		locally symmetric, $\nabla R = 0$
Riemannian model	$\mathbb{R}^n$	$\mathbb{S}^n$ sphere	$\mathbb{H}^n$ hyperbolic space	$G_{ss}/H_{max\ comp}$ symmetric space
Lorentzian model	$\mathbb{R}^{1,n-1}$	$\mathbb{S}^{1,n-1}$ de Sitter	$\mathbb{H}^{1,n-1}$ anti-de Sitter	$CW^n_D = \mathrm{Osc}^{2n-2}_D/\mathbb{R}^{n-2}$, Cahen-Wallach space, $\mathfrak{osc}^{2n-2}_D = \mathbb{R} \ltimes_D \mathfrak{hei}(n)$
(M,g) compact Riemannian $\Longrightarrow$ complete and hence:				
	$\mathbb{R}^n/\Gamma$ Γ translations [Bieberbach]	$\mathbb{S}^n/\Gamma$ [Wolf]	$\mathbb{H}^n/\Gamma$ hyperbolic ...	$\Gamma\backslash G/H$ Clifford-Klein forms ...
(M,g) compact Lorentzian, complete ?				
	✓ [Carriere '89]	✓ [Klingler '96]		✓ [Schliebner & L '13]
	$\mathbb{R}^{1,n-1}/\Gamma$ Γ polycyclic [Goldman&Kamishima]	none [Calabi-Markus]	$\mathbb{H}^{1,2k-1}/\Gamma$ [Kulkarni]	CW^n_D/Γ Γ = lattice in Osc^{2n-2}_D [Kath & Olbrich]

Further results

(M, g) compact Lorentzian implies complete, if (M, g)

- is homogeneous [Marsden '72] (pseudo-Riemannian)
- has a timelike conformal vector field [Sanchez & Romero '95]
- has $n = 3$ and is locally homogeneous [Dumitrescu & Zeghib '10]
- is a pp-wave [Schliebner & L '13]

Holonomy groups

Levi-Civita connection ∇ of (M, g) defines parallel transport

- $T_{\gamma(0)}M \ni X_0 \overset{\mathcal{P}_\gamma}{\longmapsto} X(1) \in T_{\gamma(1)}M$,
  $X(t)$ solution of ODE $\nabla_{\dot\gamma(t)}X(t) = 0$ with $X(0) = X_0$.

For $p \in M^n$ we define the (connected) holonomy group

$$\mathrm{Hol}_p^0(M, g) := \left\{\mathcal{P}_\gamma \,\middle|\, \gamma(0) = \gamma(1) = p, \gamma \sim \{p\}\right\} \subset \mathbf{O}(T_pM, g_p) \simeq \mathbf{O}(r, s)$$

- For $p, q \in M$: $\mathrm{Hol}_p(M, g) \sim \mathrm{Hol}_q(M, g)$ conjugated in $\mathbf{O}(r, s)$.
- $\mathrm{Hol}_p^0(M, g) \subset \mathrm{Hol}_p(M, g)$ normal and

$$\Pi_1(M, p) \ \ni \ [\gamma] \quad \overset{\text{surjects}}{\twoheadrightarrow} \quad [\mathcal{P}_\gamma] \ \in \ \mathrm{Hol}_p(M, g)\big/\mathrm{Hol}_p^0(M, g)$$

- Ambrose-Singer holonomy theorem: $\mathfrak{hol}_p(M, g)$ is spanned by

$$\mathcal{P}_\gamma^{-1} \circ R_{\gamma(1)}(X, Y) \circ \mathcal{P}_\gamma \ \in \ \mathfrak{so}(T_pM, g_p),$$

  where $\gamma(0) = p$, $R_{\gamma(1)}$ the curvature at $\gamma(1)$, $X, Y \in T_{\gamma(1)}M$.

Decomposable holonomy

Assume that $\mathrm{Hol}_p(M,g)$ admits an invariant *non-degenerate* subspace $\mathbb{V} \subset T_pM$. Then several things happen:

- (M,g) is locally a semi-Riemannian product.
- $\mathfrak{hol}^0_p(M,g) \simeq \mathfrak{h}_1 \oplus \mathfrak{h}_2$ acting on $\mathbb{V} \oplus \mathbb{V}^\perp$ and $\mathfrak{h}_i$ are Berger algebras.
- If (M,g) is complete and simply connected, then $(M,g) = (M_1,g_1) \times (M_2,g_2)$ a global semi-Riemannian product and

$$\mathrm{Hol}(M,g) = \mathrm{Hol}(M_1,g_1) \times \mathrm{Hol}(M_2,g_2).$$

Assume that Hol is indecomposable, i.e., no non-degenerate invariant subspace. We say that (M,g) has special holonomy if $\mathfrak{hol}(M,g) \neq \mathfrak{so}(t,s)$ and indecomposable.

- For Riemannian metrics indecomposable = irreducible
- classification of irreducible connected (semi-)Riemannian holonomy groups — Berger's list ['55]:

$$\mathbf{U}(p),\ \mathbf{SU}(p),\ \mathbf{Sp}(q),\ \mathbf{Sp}(1)\times\mathbf{Sp}(q), \mathbf{G}_2,\ \mathbf{Spin}(7)$$

where $2p = n$ and $4n = q$, or isotropy of a Riemannian symmetric space.

Special Lorentzian holonomy

A Lorentzian manifold (M^{n+2}, g) has special holonomy if $\mathrm{Hol}^0 \neq \mathbf{SO}^0(1, n+1)$ and Hol^0 acts indecomposably on T_pM.

Fundamental difference to Riemannian (where we have Berger's list):

- $H \subset \mathbf{SO}^0(1, n+1)$ irreducible $\overset{\text{[DiScala,Olmos]}}{\Longrightarrow}$ $H = \mathbf{SO}^0(1, n+1)$.
- Special holonomy $\Longrightarrow$ Hol^0-invariant null line $\mathbb{L} = \mathbb{V} \cap \mathbb{V}^\perp \subset T_pM$, i.e., $\mathrm{Hol}^0 \subset \mathrm{Stab}_{\mathbf{SO}(1,n+1)}(\mathbb{L}) = \mathbf{Sim}(n) = (\mathbb{R}_{>0} \times \mathbf{SO}(n)) \ltimes \mathbb{R}^n$.
- Lie algebra

$$\mathfrak{hol} \subset \mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n = \left\{ \left(\begin{array}{ccc} a & v^\top & 0 \\ 0 & A & -v \\ 0 & 0^\top & -a \end{array} \right) \middle| \begin{array}{l} a \in \mathbb{R} \\ v \in \mathbb{R}^n, \\ A \in \mathfrak{so}(n) \end{array} \right\}.$$

There is a classification of Lorentzian special holonomy algebras:

- description of indecomposable subalgebras of $\mathfrak{so}(1, n+1)$ [Berard-Bergery & Ikemakhen '93]
- $pr_{\mathfrak{so}(n)}(\mathfrak{hol})$ is a Riemannian holonomy algebra [L '03] $\rightsquigarrow$ Berger's list
- Construction of local metrics for all possible groups [... Galaev '05]

- M admits a parallel null line bundle $\mathbb{L}$, i.e., fibres are invariant under parallel transport and $\nabla_X : \Gamma(\mathbb{L}) \to \Gamma(\mathbb{L})$

$\Rightarrow$ $\mathbb{L}^\perp$ is integrable and hence involutive yielding a foliation $\mathcal{N}$ of M into leafs of $\mathbb{L}^\perp$.

- Screen bundle over M:

$$\mathbb{S} = \mathbb{L}^\perp / \mathbb{R} \cdot \mathbb{L}, \qquad h^{\mathbb{S}}([X],[Y]) = g(X,Y), \qquad \nabla^{\mathbb{S}}_X[Y] = [\nabla_X Y],$$

  is a vector bundle with positive def. metric and compatible connection $\nabla^{\mathbb{S}}$.

- $\mathfrak{hol}(\nabla^{\mathbb{S}}) = \mathrm{pr}_{\mathfrak{so}(n)}\mathfrak{hol}(M,g)$ is a Riemannian holonomy algebra (product of groups on Berger's list)
- Fixing a time-like unit vf $T \in \Gamma(M)$, or a null vector field Z transversal to $V^\perp$ gives a canonical identification of $\mathbb{S}$ with a tangent subbundle, the screen distribution

$$\mathbb{S} = \mathbb{L}^\perp \cap Z^\perp \subset TM$$

- There are incomplete compact Lorentzian manifolds with special holonomy. E.g., the above generalisation of the Clifton-Pohl torus has holonomy $\mathbb{R} \ltimes \mathbb{R}^n$.

Definition

A Lorentzian mf is called pp-wave, if it admits a parallel null vector field V, i.e., $g(V,V)=0$ and $\nabla V=0$, and

$$\mathrm{R}(X,Y):V^{\perp}\longrightarrow \mathbb{R}\cdot V,\quad \forall\, X,Y\in TM.$$

I.e., in a basis $V,E_1,\ldots,E_n,Z$ with

$$g=\begin{pmatrix}0&0&1\\0&\mathbf{1}_n&0\\1&0&0\end{pmatrix},\quad\text{it is}\quad \mathrm{R}(X,Y)=\begin{pmatrix}0&\mathbf{r}(X,Y)&0\\0&0&\mathbf{r}^{\top}(X,Y)\\0&0&0\end{pmatrix}$$

- In coordinates $(v,x^1,\ldots,x^n,u)$:

$$g=g^H:=2\,du\,dv+2\,H(u,x^i)\,du^2+\delta_{ij}\,dx^i\,dx^j \tag{1}$$

- curvature: $\mathbf{r}(\partial_i,\partial_u)=(\partial_i\partial_j H)_{j=1}^n$, $\mathrm{Ric}=\Delta(H)\,du^2$, $\Delta=\sum_{i=1}^n\frac{\partial^2}{(\partial x^i)^2}$ flat Laplace-operator,
- standard pp-wave if defined on $\mathbb{R}^{n+2}$, screen bundle $\mathbb{S}=\mathrm{span}(\partial_i)$

- A pp-wave is a plane wave $\overset{def}{\iff}$ $\nabla_X \mathrm{R} = 0 \;\forall X \in V^\perp \iff H = S_{ij}(u)x^i x^j$.
- A locally symmetric pp-wave is called Cahen-Wallach space. This is equivalent to $H = S_{ij}x^i x^j$ with S_{ij} constant.

Theorem (Cahen & Wallach '70)

An indecomposable, locally symmetric Lorentzian manifold either has constant curvature or is a Cahen-Wallach space.

- Cahen Wallach spaces have solvable transvection group Osc_D^{2n-2}, $CW_D^n = \mathrm{Osc}_D^{2n-2}/\mathbb{R}^{n-2}$
- constant curvature: de Sitter or anti de Sitter, i.e, $\mathbf{SO}(1, n-1)/\mathbf{SO}(n-1)$ or $\mathbf{SO}(2, n-2)/\mathbf{SO}(1, n-2)$.
- neither de Sitter spaces [Calabi-Markus '62] nor even dimensional anti de Sitter spaces [Kulkarni '81] have compact quotients.
- compact quotients of Cahen-Wallach spaces: [Kath & Olbrich '15]

Some history

- Brinkmann '25 in conformal geometry: an Einstein metric g can be rescaled to another Einstein metric $e^{2\varphi} g \iff g$ is a warped product of $\mathbb{R}$ with a space of constant curvature (if $\|\nabla\varphi\| \neq 0$), or a pp-wave metric (if $\|\nabla\varphi\| \neq 0$).
- GR: wave-like solutions of the vacuum Einstein equations, $\Delta(H) = 0$. [Einstein & Rosen '35]
  propagation in direction of $V = \partial_v$, wave fronts $\{(v, u) = const\}$ with induced Riemannian metric are flat.
- Ehlers-Kundt '62 (in Dimension 4): "plane fronted with parallel rays".
  "Prove that complete, Ricci-flat pp-waves are plane waves, no matter which topology one chooses!"
- Penrose '76: Every spacetime has a *plane wave* as limit, "Penrose limit".
- pp-waves have the submaximal number of parallel spinors $\rightsquigarrow$ supergravity backgrounds [Hull '84, Figueroa O'Farrill].
- pp-waves have no scalar invariants, i.e., all functions obtained from derivatives of the metric and the curvature vanish.
- $\mathrm{Hol} \subset \mathbb{R}^n$ is abelian

The screen bundle of a pp-wave

Let (M, g) be a Lorentzian manifolds with parallel null vector field V.

Equivalences: (M, g) is a pp-wave

- $\Leftrightarrow$ the *screen bundle* $\mathbb{S} = V^{\perp}/\mathbb{R} \cdot V$ is flat,
- $\Leftrightarrow$ For each leaf N of $V^{\perp}$ the connection ∇^N induced by ∇^g is flat.
- $\Leftrightarrow$ each local screen distribution admits orthonormal sections $S_1, \dots S_n$ with $\nabla S_i = \alpha^i \otimes V$, where α^i local one-forms with $d\alpha^i|_{V^{\perp} \wedge V^{\perp}} = 0$.

Let (M, g) be a pp-wave and $\mathbb{S}$ a screen distribution (locally, globally on M, or along a leaf N of $V^{\perp}$).

- $\mathbb{S}$ is horizontal $\overset{def}{\iff}$ $[V, \mathbb{S}] \subset \mathbb{S} \iff \nabla_V S_i = 0 \iff V$ is Killing vector field for the Riemannian metric $h^{\mathbb{S}}$, defined on the leafs of $V^{\perp}$ by

$$h^{\mathbb{S}}(V, V) = 1, \quad h^{\mathbb{S}}(V, \mathbb{S}) = 0, \quad h^{\mathbb{S}}|_{\mathbb{S}\times\mathbb{S}} = g|_{\mathbb{S}\times\mathbb{S}}.$$

- $\mathbb{S}$ is horizontal and involutive (i.e., $[\mathbb{S}, \mathbb{S}] \subset \mathbb{S}$) $\iff h^{\mathbb{S}}$ is flat.

Examples of compact pp-waves

h flat metric on the torus $\mathbb{T}^n$, $H \in C^\infty(\mathbb{T}^n)$ smooth. $\mathcal{M} := \mathbb{T}^2 \times \mathbb{T}^n$ with

$$g^H = 2d\theta d\varphi + 2Hd\theta^2 + h,$$

$\Rightarrow$ complete pp-wave metric on the torus $\mathbb{T}^{n+2}$, in general not a plane wave.

Example without involutive and horizontal screen:

- Torus $\mathbb{T}^n$ with canonical 1-forms $\xi^1, \ldots, \xi^n$. Set
$$\omega = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \xi^i \wedge \xi^j \in \Omega^2(\mathbb{T}^n)$$
with constants a_{ij} such that $0 \neq [\omega] \in H^2(\mathbb{T}^n, \mathbb{Z})$.
- Let $\pi : N \to \mathbb{T}^n$ be the S^1-bundle with 1st Chern class $[\omega]$, $A \in T^*\mathcal{S} \otimes \mathrm{i}\mathbb{R}$ the S^1-connection with curvature $F := dA = -2\pi\mathrm{i}\pi^*\omega$.
- $M := N \times \mathrm{S}^1$, function $H \in C^\infty(\mathbb{T}^n)$, $\eta \in T^*\mathrm{S}^1$, $\sigma^i =$ horizontal lifts of ξ^i,
$$g = 2\,(H\eta - \mathrm{i}A) \cdot \eta + \sum_{i=1}^{n} (\sigma^i)^2 \quad = \quad \text{pp-wave metric on } M.$$
with parallel null vector field V, which is the fundamental vector field to the S^1 action on N, i.e., $A(V) = \mathrm{i}$.
- Note: (M, g) does *not* admit an involute and horizontal screen distribution.
  - Since S_i are horizontal lifts, $[V, S_i] = 0$, so $\mathbb{S} = \mathrm{span}(S_1, \ldots, S_n)$ is horizontal.
  - But $[S_i, S_j] = \mathrm{i}F(S_i, S_j) \neq 0$, so not involutive.
  - Change screen $\hat{S}_i = S_i + b_i V$, then, with $\beta = b_i \sigma^i$ we have
$$[V, \hat{S}_i] = -db_i(V)V, \quad \left[\hat{S}_i, \hat{S}_j\right] = \left(\mathrm{i}F((S_i, S_j) - d\beta(S_i, S_j) + b_{[i}db_{j]}(V)\right) V,$$
  - Hence, since $[F] \neq 0$, F is not exact, and $\hat{\mathbb{S}}$ cannot be involutive and horizontal at the same time.

The universal covering of a compact pp-wave

Theorem A [Schliebner & L '13]

A compact pp-wave is universally and isometrically covered by $\mathbb{R}^{n+2}$ with the standard pp-wave metric $\widetilde{g} = g^H = 2dudv + 2H(u, x^i)du^2 + \delta_{ij}dx^i dx^j$.

Proof — Step 1: Let $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ be the universal covering and $\eta = V^\flat$

- $\nabla V = 0 \implies d\eta = 0$ and in particular $\ker(\eta) = V^\perp$ integrable.
- M compact $\implies \exists$ complete vector field Z: $\eta(Z) \equiv 1$.
- Milnor '63:

$$\mathbb{R} \times \underset{\uparrow}{\widetilde{N}}^{\text{leaf of } V^\perp} \simeq \widetilde{M} \quad , \quad (u, p) \mapsto \underset{\uparrow}{\Phi_u^Z}^{\text{flow of } Z}(p)$$

Step 2: Fix screen $\mathbb{S} = \mathrm{span}(S_i)$ on $\widetilde{N}$, with S_i lifts from N and thus complete.

- $d\alpha^i = 0 \implies \alpha^i = db^i$
- $\hat{\mathbb{S}} = \mathrm{span}(\hat{S}_i = S_i - b_i V)$ is involutive and horizontal. Hence, $h^{\hat{\mathbb{S}}}$ is flat on $\widetilde{N}$.
- Since $\widetilde{V}$ is complete, $\widetilde{N} = \mathbb{R} \times \hat{S}$. Use this to show that $\hat{S}_i$ are complete.
- Palais '57: $\widetilde{N} = \mathbb{R}^{n+1}$.

Step 3: Then show that $\widetilde{g} = g^H$ (very technical).

Is $\left(\mathbb{R}^{n+2}, g^H = 2dudv + 2H(u,x^i)du^2 + \delta_{ij}dx^i dx^j\right)$ complete?

$$\begin{cases} \ddot{u} &= 0 & \Longrightarrow \ u(t) = at+b \\ \ddot{v} &= -2a\dot{x}^k \frac{\partial H}{\partial x^k} - a^2 \frac{\partial H}{\partial u} & \Longrightarrow \ \text{solution on } \mathbb{R} \text{ if } x^k \text{ on } \mathbb{R} \text{ def.} \\ \ddot{x}^i &= a^2 \frac{\partial H}{\partial x^i} & (*) \end{cases}$$

In general not complete, e.g., $n=1$, $H(x,u) := \frac{1}{2}x^4$: $(*) \iff \ddot{x} = 2a^2x^3$

$$\text{Solution:} \qquad x(t) = \frac{a^2}{1-t}, \quad t \in (-\infty, 1), \text{ if } a \neq 0.$$

Note: Solutions of $(*)$ defined on $\mathbb{R}$, if $\frac{\partial H}{\partial x^i}$ Lipschitz.

Lemma

If M is compact, then $\frac{\partial^2 H}{\partial x^i \partial x^j}$ is bounded, hence $\frac{\partial H}{\partial x^i}$ Lipschitz.

Proof: Define bilinear form on M, using the vector field Z,

$$Q(X,Y) := \mathrm{R}(X,Z,Z,Y)$$

and compute using $\pi : (\widetilde{M}, g^H) \to (M, g)$,

$$\underbrace{g(Q,Q)}_{\substack{\in C^\infty(M) \\ \text{bounded}}} = g^H(\pi^*Q, \pi^*Q) = \sum_{i,j=1}^n (\pi^*Q(\partial_i, \partial_j))^2 = \sum_{i,j=1}^n \widetilde{\mathrm{R}}(\partial_i, \partial_u, \partial_u, \partial_j)^2 = \sum_{i,j=1}^n \left(\frac{\partial^2 H}{\partial x^i \partial x^j}\right)^2 > 0.$$

Completeness of compact pp-waves

Theorem B [Schliebner & L '13]

A compact pp-wave is geodesically complete.

Ehlers-Kundt '62 (in Dimension 4): *"Prove that complete, Ricci-flat pp-waves are plane waves, no matter which topology one chooses!"*

- Recall: plane wave $\overset{def}{\iff} \nabla_X \mathrm{R} = 0 \ \forall X \in V^\perp \iff H(x^i, u) = A_{ij}(u) x^i x^j$.

Corollary

Every compact Ricci-flat pp-wave is a plane wave.

Proof:

- g^H Ricci-flat $\implies$ H harmonic $\implies$ $\partial_i \partial_j (H)$ harmonic.
- $\partial_i \partial_j (H)$ bounded $\implies$ $\partial_i \partial_j (H)$ constant w.r.t. x^i's, i.e., $H = A_{ij}(u) x^i x^j$.
- $(\widetilde{M}, \widetilde{g})$ and therefore (M, g) is a plane wave.

Corollary for Cahen-Wallach spaces

Corollary

Every indecomposable compact locally symmetric Lorentzian manifold is complete.

- (M, g) indecomposable $\Rightarrow$ (M, g) locally isometric to
  - constant curvature and hence complete [Klingler '96]
  - or Cahen-Wallach space $2du(dv + \mathbf{x}^\top S\mathbf{x}du) + d\mathbf{x}^\top d\mathbf{x}$.
- (M, g) compact and locally isometric to Cahen-Wallach $\Rightarrow$ (M, g) complete.
  - Difficulty: (M, g) might not have have a global parallel null vector field, just a parallel null line bundle.
  - Assume that

    $$\mathrm{Hol}(M, g) \ni h = \begin{pmatrix} a & * & * \\ 0 & A & * \\ 0 & 0 & a^{-1} \end{pmatrix},$$

    with $a \in \mathbb{R}_{\neq 0}$ and $A \in \mathbf{O}(n)$. Then $h \cdot R = R$, which implies $S = a^2 A^\top S A$. Hence $a^2 = 1$
  - Time orientable cover admits a parallel null vector field and is a compact Cahen Wallach space, which then is complete. Hence, (M, g) is complete.

Generalisation

Generalise completeness result to compact Lorentzian manifolds with special holonomy, i.e., with parallel null line:

- Counterexample in the case when (M, g) does *not* admit a parallel null vector field (generalisation of Clifton Pohl torus).

Conjecture:

A compact Lorentzian manifold with parallel null vector field is complete.

- The proof is work in progress, needs a few assumptions and still has a few gaps (see next slides).

(M, g) a compact Lorentzian manifold with parallel null vector field V

Theorem

Assume that there is a screen distribution on M such that

(H) $\mathbb{S}$ *is horizontal, and*

(I) *there is a compact leaf N of* $V^\perp$ *such that* $\mathbb{S}|_N$ *is involutive (with leaf* S*).*

Then there is a local diffeomorphism

$$\Phi : \widetilde{M} = \mathbb{R}^2 \times S \to M \quad \text{such that} \quad \widetilde{g} := \Phi^* g = 2du(dv + \mu_u) + h_u$$

where μ_u *and* h_u *are u-dependent families of* 1*-forms and metrics on* S *such that:*

(A) h_u *is complete, and*

(B) *the norm (w.r.t.* h_u*) of* μ_u *and* $\partial_u h_u$ *are bounded functions on* S.

Theorem

A Lorentzian mfd $(\widetilde{M}, \widetilde{g})$ *as above is complete if it has properties (A) and (B).*

Corollary

If (M, g) *admits a screen distribution with (H) and (I) then it is complete.*

Involutive/horizontal screens and basic cohomology

Problem: We do not we always have a screen with (H) and (I), recall the example.
When do we have an involutive or horizontal screen?

- A Riemannian flow $\mathcal{F}$ on a manifold N is the flow of a non-vanishing vector field F such that $\exists$ Riemannian metric h with $(\mathcal{L}_F h)|_{F^\perp \times F^\perp} = 0$.
- $\mathcal{F}$ is an isometric flow, if $\mathcal{F}$ is given by a Killing vf K for a metric h.
- Basic forms for $(N, \mathcal{F})$: $\Omega^p_b := \{\alpha \in \Omega^p \mid F \lrcorner \alpha = F \lrcorner d\alpha = 0\} \stackrel{d}{\to} \Omega^{p+1}_b$.
- If a flow is isometric, then the following cohomology sequence is exact

$$\ldots \to H^p_b \longrightarrow H^p_{dR} \xrightarrow{\iota_K} H^{p-1}_b \xrightarrow{dK^\flat \wedge} H^{p+1}_b \to \ldots,$$

Apply this to $N =$ leaf of $V^\perp$ and $\mathcal{F} = V$. Given $\mathbb{S}$, $\mathcal{F}$ is a Riemannian flow via $h^{\mathbb{S}}$.

- If $\mathbb{S}$ is horizontal, then V is Killing for $h^{\mathbb{S}}$ and $\mathcal{F}$ is isometric.
- $\mathcal{F}$ isometric $\stackrel{[Sergiescu\,\&\,Molino'85]}{\Longleftrightarrow}$ $H^n_b \neq 0$. Hence the existence of an horizontal screen implies that $H^n_b \neq 0$.
- If $H^1_b \neq H^1_{dR}$, then $H^n_b \neq 0$. Then the exact sequence can be used to show that there is a closed one form η with $\eta(V) \neq 0$ and hence an involutive screen.

¡Muchas Gracias!