

CENTROAFFINE  
HYPEROVALIODS WITH  
EINSTEIN METRIC

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We study centroaffine hyperovaloids with centroaffine Einstein metric. We prove:  
the hyperovaloid must be a hyperellipsoid.

This result answers a conjecture that was open for several decades.

# History

- **1963**

H. F. Münzner (PhD thesis FU Berlin):

**Theorem.** *Let  $x : M \rightarrow \mathbb{R}^3$  be an analytic ovaloid equipped with a centroaffine metric of constant Gauß curvature. Then  $x(M)$  is an ellipsoid.*

- **method of proof:**

index method for cubic forms

*Similar results in other relative hypersurface geometries.*

- Blaschke's **unimodular** hypersurface theory.
- **1967** R. Schneider:  
**Theorem.** *Let  $x : M \rightarrow \mathbb{R}^3$  be an ovaloid equipped with a unimodular-affine metric of constant Gauß curvature. Then  $x(M)$  is an ellipsoid.*
- **Remark.** (a) Schneider did not state this as a result, but it follows from one of his congruence theorems in his paper.  
(b) The theorem answers a conjecture of Blaschke.

- **1985** M. Kozłowski and U. Simon:
- **Theorem.** *Let  $x : M \rightarrow \mathbb{R}^{n+1}$  be a hyperovaloid equipped with a unimodular (Blaschke) metric of Einstein type. Then  $x(M)$  is a hyperellipsoid.*

**Proof.** Extension of Scheider's method applying eigenvalues of the Laplacian.

- **1998** G. Zhao
- **Theorem.** *Let  $x : M \rightarrow \mathbb{R}^{n+1}$  be a hyperovaloid equipped with a unimodular (Blaschke) metric with parallel Ricci tensor. Then  $x(M)$  is a hyperellipsoid.*

- Relative hypersurfaces with **Euclidean** normalization.
- **1972** R. Schneider:
- **Theorem.** *Let  $x : M \rightarrow \mathbb{R}^{n+1}$  be a hyperovaloid equipped with the second fundamental form as relative metric of a Euclidean normalization. If this metric is of constant curvature then  $x(M)$  is a Euclidean sphere.*
- **Remark.** Schneider's proof works under the weaker assumption that the scalar curvature of the second fundamental form metric is constant.

Proof of our **centroaffine result**:  
(any dimension, no analyticity).

**Main Theorem:** Let  $x : M \rightarrow \mathbb{R}^{n+1}$  be a hyperovaloid equipped with a centroaffine metric of Einstein type. Then  $x(M)$  is a hyperellipsoid.

## Centroaffine hypersurfaces - basics

- $M$   $n$ -dim.  $C^\infty$ -manifold  
 $x : M \rightarrow \mathbb{R}^{n+1}$   
loc. str. convex hypersurf. imm.  
position vector nowhere tangential;  
 $\{x, \xi := \pm x\}$  centroaffine hypersurf.

- **Gauß structure equations.**

$$D_X dx(Y) = dx(\nabla_X Y) + h(X, Y)\xi,$$

- $\nabla$  torsion free, Ricci-symmetric,  
called *induced connection*
- $h$  is *centroaffine metric*, Riemannian

## **Intrinsic geometry of $(M, h)$ .**

$h$  induces:

- Riemannian volume form  $\omega$ ,
- Levi-Civita  $\hat{\nabla}$
- $R$  curvature tensor,
- $Ric$  Ricci tensor,
- $\kappa$  normalized scalar curvature.

## “Extrinsic” geometry of $\{x, \pm x\}$ :

- symmetric difference tensor:

$$K_X Y := K(X, Y) := \nabla_X Y - \hat{\nabla}_X Y;$$

- cubic form

$$C(X, Y, Z) := h(K_X Y, Z) \text{ tot. symm.,}$$

$$n(n-1)J := \|K\|^2 = \|C\|^2 \text{ Pick inv.,}$$

- Tchebychev form  $T$ :

$$nT(X) := \text{trace} K_X;$$

$$\text{Tchebychev field } h(T^\#, X) := T(X).$$

- The Tchebychev form  $T$  is closed:

$$T = \frac{n+2}{2n} d \ln \rho;$$

$\rho$  is the unimodular support function

- Tchebychev operator  $\mathcal{T}$ :

$$\mathcal{T}X := \hat{\nabla}_X T^\#.$$

## **Integrability conditions for $\hat{\nabla}$ :**

$$\begin{aligned} R(X, Y)Z &= K_Y K_X Z - K_X K_Y Z \\ &\quad + (h(Y, Z)X - h(X, Z)Y), \\ (\hat{\nabla}C)(W, X, Y, Z) &= (\hat{\nabla}C)(X, W, Y, Z), \end{aligned}$$

.

## **centroaffine theorema egregium:**

$$n(n-1)\kappa = n(n-1)(J+1) - n^2 \cdot h(T^\#, T^\#).$$

**Remark.** In analogy to Bonnet's fundamental theorem in Euclidean hypersurface theory there is such a fundamental existence and uniqueness theorem in each relative hypersurface theory, with prescribed data  $h$  and  $C$  satisfying integrability conditions.

There are other versions prescribing  $h$  and a symmetric projectively flat connection.

## **Quadratics.**

The traceless part  $\tilde{C}$  of the cubic form  $C$ :

$$\begin{aligned}\tilde{C}(X, Y, Z) := & C(X, Y, Z) - \\ & \frac{n}{n+2} (T(X)h(Y, Z) + T(Y)h(Z, X) + T(Z)h(X, Y)).\end{aligned}$$

**Theorem.** Let  $x : M \rightarrow \mathbb{R}^{n+1}$  be a centroaffine hypersurface. Then  $x$  is a hyperquadric if and only if  $\tilde{C} \equiv 0$  on  $M$ .

## **Inequalities for intrinsic curvature inv.**

**Lemma** (Calabi): We have

- (i)  $\|R\|^2 \geq \frac{2}{n-1} \|Ric\|^2.$
- (ii)  $\|Ric\|^2 \geq n(n-1)^2 \kappa^2.$
- (iii)  $\|R\|^2 \geq 2n(n-1)\kappa^2.$

## Cubic and quartic form inequalities.

**Lemma.**

$$0 \leq \|\tilde{C}\|^2 = \|C\|^2 - \frac{3n^2}{n+2} \|T\|^2.$$

**Lemma.** Define  $\mathcal{U}$

totally symmetric, traceless  $(0,4)$ -tensor

$$\begin{aligned} \mathcal{U}(X, Y, Z, W) := & (\hat{\nabla}C)(X, Y, Z, W) - \\ & \frac{n}{n+4} [(\hat{\nabla}T)(X, Y)h(Z, W) + (\hat{\nabla}T)(X, Z)h(Y, W) + \\ & (\hat{\nabla}T)(X, W)h(Y, Z) + (\hat{\nabla}T)(Z, W)h(X, Y) + \\ & (\hat{\nabla}T)(Y, W)h(X, Z) + (\hat{\nabla}T)(Y, Z)h(X, W)] + \\ & \frac{n \cdot \text{trace } \mathcal{T}}{(n+2)(n+4)} \cdot [h(X, Y)h(Z, W) + h(X, Z)h(Y, W) \\ & + h(X, W)h(Y, Z)], \end{aligned}$$

then we have:

$$\|\mathcal{U}\|^2 = \|\hat{\nabla}C\|^2 - \frac{6n^2}{(n+4)} \|\mathcal{T}\|^2 + \frac{3n^2}{(n+2)(n+4)} (\text{trace } \mathcal{T})^2.$$

## Differential equations

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### **Lemma of Bochner-Lichnerowicz.**

$$\frac{1}{2}\Delta\|T\|^2 = \|\hat{\nabla}T\| + Ric(T, T) + T(\text{grad}(\text{trace } T)).$$

### **Laplacian of the Pick invariant.**

Let  $\{e_1, \dots, e_n\}$  be a local  $h$ -orthonormal frame; define

$$\alpha(X, Y) := \sum_{i,j=1}^n C(X, e_i, e_j)C(Y, e_i, e_j).$$

We have  $\text{trace } \alpha = \|C\|^2$  and

$$\begin{aligned} \frac{1}{2}\Delta\|C\|^2 &= \\ \|\hat{\nabla}C\|^2 + \|R\|^2 - 2n(n-1)\kappa + \sum_{i,j} \alpha(e_i, e_j)Ric(e_i, e_j) \\ &+ n \sum_{i,j,k} (\hat{\nabla}^2 T)(e_i, e_j, e_k)C(e_i, e_j, e_k). \end{aligned}$$

## Laplacian of $\|\tilde{C}\|^2$

Combine the preceding PDE's and  $\|\mathcal{U}\|^2$ .  
notation  $\text{div} :=$  divergence-type terms.

## Calculations.

$$\bullet \quad \hat{\nabla}_j C^{ijk} = \hat{\nabla}^k C_j^{ij} = n \hat{\nabla}^k T^i,$$

•

$$\begin{aligned} C^{ijk} \hat{\nabla}_j \hat{\nabla}_i T_k &= \hat{\nabla}_j (C^{ijk} \hat{\nabla}_i T_k) - n \|\hat{\nabla}_i T_k\|^2 \\ &= \text{div}_1 - n \|\hat{\nabla}_i T_k\|^2 \end{aligned}$$

•

$$\begin{aligned} T^i \hat{\nabla}_i (\hat{\nabla}_j T^j) &= \hat{\nabla}_i (T^i \hat{\nabla}_j T^j) - (\text{trace } \mathcal{T})^2 \\ &= \text{div}_2 - (\text{trace } \mathcal{T})^2. \quad (1) \end{aligned}$$

## PDE for centroaffine Einstein spaces

$$\begin{aligned}
\frac{1}{2} \Delta \|\tilde{C}\|^2 &= \frac{1}{2} \Delta \|C\|^2 - \frac{1}{2} \cdot \frac{3n^2}{n+2} \cdot \Delta \|T\|^2 \\
&= \|\mathcal{U}\|^2 + \|\nabla_i T_j\|^2 \left( \frac{6n^2}{n+4} - \frac{3n^2}{n+2} \right) \\
&\quad + (\text{trace } \mathcal{T})^2 \left( \frac{3n^2}{n+2} - \frac{3n^2}{(n+2)(n+4)} \right) \\
&\quad + (n-1)\kappa \cdot \left( \|C\|^2 - \frac{3n^2}{n+2} \|T\|^2 \right) + \text{div} \\
&= \|\mathcal{U}\|^2 + \|\nabla_i T_j\|^2 \cdot \frac{3n^3}{(n+2)(n+4)} \\
&\quad + (\text{trace } \mathcal{T})^2 \cdot \frac{3n^2(n+3)}{(n+2)(n+4)} \\
&\quad + (n-1)\kappa \cdot \|\tilde{C}\|^2 + \text{div}.
\end{aligned}$$

## **Hyperovaloids, centroaffine Einstein**

**Lemma.** Hyperovaloid, centroaffine Einstein, then  $\kappa \geq 1$ .

### **Proof Main Theorem.**

Apply PDE centroaffine Einstein spaces; integration gives

$$\|\tilde{C}\|^2 \equiv 0,$$

thus quadric, compactness gives hyperellipsoid.

## Complete centroaffine Einstein

**Theorem.** Let

$$x : M \rightarrow \mathbb{R}^{n+1}$$

be a locally strongly convex hypersurface equipped with a centroaffine  $C^\infty$ -metric  $h$  of Einstein type.

Assume:

$(M, h)$  is complete and there exists  
 $0 < \epsilon \in \mathbb{R}$  s. t.

$$Ric(h) > \epsilon \cdot h.$$

Then  $x(M)$  is a hyperellipsoid.